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A FAMILY OF GOODNESS-OF-FIT TESTS FOR THE CAUCHY DISTRIBUTION

RODZINA TESTÓW ZGODNOŚCI Z ROZKŁADEM CAUCHY'EGO

Abstract

A new family of goodness-of-fit test for the Cauchy distribution is proposed in the paper. Every member of this family is affine invariant and consistent against any non Cauchy distribution. Results of the Monte Carlo simulations performed to verify finite sample behaviour of the new tests are presented.

Keywords: Cauchy distribution, goodness-of-fit test, empirical characteristic function

Streszczenie

W artykule zaproponowano nową rodzinę testów zgodności z rozkładem Cauchy'ego. Każdy test z tej rodziny jest afinicznie niezmienniczy i zgodny przeciwko każdej alternatywie nie będącej rozkładem Cauchy'ego. Zaprezentowano także wyniki symulacji numerycznych przeprowadzonych w celu zbadania zachowania nowych testów dla skończonych prób.

Słowa kluczowe: Rozkład Cauchy'ego, test zgodności, empiryczna funkcja charakterystyczna

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1. Introduction

Let X_1, X_2, ... be a sequence of i.i.d. random variables with distribution function F. We consider the problem of testing the hypothesis:

$$H_0 : F \in \mathcal{F}$$

against

$$H_1 : F \notin \mathcal{F}$$

where $\mathcal{F}$ is the family of the Cauchy distributions, i.e.

$$\mathcal{F} = \left\{ F : F(x) = F_0\left((x-m)/\sigma\right), (m,\sigma)' \in \mathbb{R} \times (0,\infty) \right\}$$

with

$$F_0(x) = 1/2 + \pi^{-1} \arctan x.$$

The location parameter m is the median and the scale parameter σ represents half of the interquartile range in this case. In recent years there were several papers devoted to this problem. Gürtler and Henze [5] and Matsui and Takemura [9] considered the test statistics of the form

(1) $$D_n = n \int_{\mathbb{R}} \left| \varphi_n(t) - \varphi_0(t) \right|^2 w(t)\,dt,$$

where ϕ_n is the empirical characteristic function

$$\varphi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \exp\left(itY_j\right)$$

of the standardized with suitable estimators data

(2) $$Y_j = \left(X_j - \hat{m}_n\right) / \hat{\sigma}_n,\ j = 1, \ldots, n,$$

$\varphi_0(t) = \exp(-|t|)$ is the theoretical characteristic function of the standard Cauchy distribution and $w(t)$ is the weight function. The weight $w(t) = \exp(-\lambda|t|)$ considered in [5, 9] results with simply and closed form of the test statistics, namely

$$D_{n,\lambda} = \frac{2}{n} \sum_{j,k=1}^{n} \frac{\lambda}{\lambda^2 + \left(Y_j - Y_k\right)^2} - 4 \sum_{j=1}^{n} \frac{1+\lambda}{(1+\lambda)^2 + Y_j^2} + \frac{2n}{2+\lambda}.$$

Gürtler and Henze [5] showed that with the sample median and half of the interquartile range as the estimators of location and scale, respectively, the test based on $D_{n,\lambda}$ is consistent against each alternative distribution having a unique median and unique upper and lower quartiles. In this paper we propose another test statistics of the form (1). Since the most important properties of a distribution are determined by the behaviour of the characteristic function in a neighbourhood of zero, especially in the case of heavily tailed distribution like the Cauchy one, one should use the weight putting more mass around zero. For this reason we will use unbounded in zero weight function. With the weight

$$w(t) = \exp(-\lambda|t|)/|t|^{\gamma},$$

where $\lambda > 0$ and $\gamma \in (0; 1)$, the test statistic again has closed form, namely

$$D_{n,\lambda,\gamma} = 2\Gamma(1-\gamma)\Big(n(2+\lambda)^{\gamma-1} - 2\sum_{j=1}^{n}\left((1+\lambda)^2 + Y_j^2\right)^{(\gamma-1)/2}\cos\left((\gamma-1)\arctan\left(\frac{Y_j}{1+\lambda}\right)\right)$$
$$+\frac{1}{n}\sum_{j,k=1}^{n}\left(\lambda^2 + (Y_j - Y_k)^2\right)^{(\gamma-1)/2}\cos\left((\gamma-1)\arctan\left(\frac{Y_j - Y_k}{\lambda}\right)\right)\Big).$$

Since the family of the Cauchy distributions is closed with respect to the affine transformations one is interested in affine invariant test. To obtain an affine invariant test statistic of the form (1) it is enough to standardize the sample with equivariant estimators in (2), i.e. estimators $\hat{m}_n = \hat{m}_n(X_1,...,X_n)$ and $\hat{\sigma}_n = \hat{\sigma}_n(X_1,...,X_n)$ such that for every $a > 0$ and $b \in \mathbb{R}$ we have

$$\hat{m}_n(aX_1 + b,\ldots,aX_n + b) = a\hat{m}_n(X_1,\ldots,X_n) + b$$

and

$$\hat{\sigma}_n(aX_1 + b,\ldots,aX_n + b) = a\hat{\sigma}_n(X_1,\ldots,X_n).$$

The previous authors considered the sample median and half of the interquartile range [5], the maximum likelihood estimators (MLE) and the EISE estimators [9]. Since the use of EISE do not improve the power of the test complicating the calculations at the same time we do not consider these estimators in this paper.

The paper is organized as follows. Section 2 contains a review of properties of estimators proposed in [10] by Pudełko. In Sections 3 and 4 there are main results of the paper, e.i. theorems concerning the weak convergence of $D_{n,\lambda,\gamma}$ when the sample comes from the Cauchy distribution, its limit distribution and the consistency of corresponding test against each non-Cauchy alternative distribution. Section 5 presents the results of the numerical simulations performed to verify the finite sample behaviour of the new test.

2. Estimators of the parameters of the Cauchy distribution

The choice of the parameters used to standardize the data in (2) is very important to the performance of the test. In this paper besides MLE and order estimators (sample median and half of the quartile range) we will use estimators proposed by Pudełko in [10]. These estimators are defined as argument $\hat{\theta}_{n,\alpha} = (\hat{m}_n, \hat{\sigma}_n)$ minimizing[1] the distance

$$\delta_\alpha(\theta) = \int_{\mathbb{R}} \frac{|\varphi_n(t) - \varphi_\theta(t)|^2}{|t|^{1+\alpha}}\mathrm{d}t,$$

[1] Estimators defined as argument minimizing

$$\int_{\mathbb{R}} \left|\varphi_n(t) - \varphi_\theta(t)\right|^2 w(t)dt$$

were proposed independently by [6] and [11] but these authors considered bounded weight function w.

where φ_n is the empirical characteristic function, $\varphi_\theta(t) = e^{itm-\sigma|t|}$ is the characteristic function of the Cauchy distribution with the median m and the interquartile range 2σ. In [10] was showed that $\hat{\theta}_{n,\alpha}$ may be equivalently defined by

$$\hat{\theta}_{n,\alpha} := \arg\min_{\theta\in\Theta} 2\Gamma(-\alpha)\sigma^{\alpha}\left(2^{\alpha-1} - \frac{1}{n}\sum_{j=1}^{n}\cos^{-\alpha} Z_j \cos(\alpha Z_j)\right),$$

where Γ is the Gamma function and $Z_j = \arctan((X_j - m)/\sigma)$: As it was showed in [10] the family of the above estimators can be continuously closed by taking for $\alpha = 0$ the ML estimators

$$\hat{\theta}_{n,0} = \arg\max_{\theta\in\Theta}\left(\log\sigma - \frac{1}{n}\sum_{j=1}^{n}\log\left(\sigma^2 + (X_j - m)^2\right)\right).$$

Estimators $\hat{\theta}_{n,\alpha}$ are affine equivariant, strongly consistent, asymptotically normally distributed with the covariance matrix

$$\Sigma(\theta_0) = \frac{2\sigma_0^2}{(\alpha-1)^2}\left(\frac{1}{(3-2\alpha)B(2-\alpha,2-\alpha)} - 1\right)I_2,$$

where B is the Beta function and I_2 is the 2×2 identity matrix and have the following Bahadur representation

$$\begin{aligned}\sqrt{n}\hat{m}_{n,\alpha} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} l_1(X_j) + o_P(1),\\ \sqrt{n}(\hat{\sigma}_{n,\alpha} - \sigma_0) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} l_2(X_j) + o_P(1),\end{aligned} \tag{3}$$

with

$$\begin{aligned} l_1(x) &= \frac{2^{2-\alpha}}{1-\alpha}\cos^{1-\alpha} z \sin((1-\alpha)z),\\ l_2(x) &= \frac{2^{2-\alpha}}{1-\alpha}\left(2^{\alpha-1} - \cos^{1-\alpha} z\cos((1-\alpha)z)\right).\end{aligned}$$

3. Asymptotic behaviour of the proposed test statistic

The following useful representation of $D_{n,\lambda,\gamma}$ can be obtained by straightforward algebra

$$D_{n,\lambda,\gamma} = \int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}\hat{\sigma}_n^{1-\gamma}}{|t|^{\gamma}}\mathrm{d}t, \tag{4}$$

where

$$\hat{Z}_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\cos(tX_j) + \sin(tX_j) - e^{-\hat{\sigma}_n|t|}(\cos(t\hat{m}_n) + \sin(t\hat{m}_n))\right). \tag{5}$$

We will consider $\hat{Z}_n(t)$ as a random element in the Frechet space $C(\mathbb{R})$ of continuous functions on the real line endowed with the metric

(6) $$\rho(f,g)=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{\rho_j(f,g)}{1+\rho_j(f,g)},$$

where $\rho_j(f; g) = \sup_{|t|\le j} |f(t) - g(t)|$.

Now we formulate the following theorem on the convergence of the process $\hat{Z}_n$.

Theorem 1. Let X_1 X_2, ... *be a sequence of independent, identically, Cauchy distributed random variables. Then there exists a centered Gaussian process Z in* $C(\mathbb{R})$ *such that*

$$\hat{Z}_n \xrightarrow{d} Z \text{ in } C(\mathbb{R}),$$

where "$\xrightarrow{d}$" *denotes weak convergence. If* $\hat{m}_n$ *and* $\hat{\sigma}_n$ *are the sample median and half of the interquartile range, respectively, than the covariance kernel of Z is*

$$K(s,t)=e^{-|t-s|}+\frac{\pi}{2}e^{-|t|-|s|}\left(\frac{\pi}{2}st+\frac{\pi}{2}|st|+|s|+|t|-1\right)$$
$$-e^{-|t|}(tJ_1(s)+2|t|J_2(s))-e^{-|s|}(sJ_1(t)+2|s|J_2(t)),$$

for all $t, s \in \mathbb{R}$, *where*

$$J_1(s)=\int_0^{\infty}\frac{\sin(sx)}{1+x^2}dx, \qquad J_2(s)=\int_0^{1}\frac{\cos(sx)}{1+x^2}dx.$$

For the estimators $\hat{\theta}_{n,\alpha}$ *we have*

(7) $$K_\alpha(s,t)=\begin{cases} e^{-|t-s|}-e^{-|t|-|s|}\left[1-\frac{4st}{(\alpha-1)^2}\left(\frac{1}{(3-2\alpha)B(2-\alpha,2-\alpha)}-1\right)\right. \\ \left.+\frac{2}{\alpha-1}\left(|t|e^{2|s|}\frac{\Gamma(2-\alpha,2|s|)}{\Gamma(2-\alpha)}+|s|e^{2|t|}\frac{\Gamma(2-\alpha,2|t|)}{\Gamma(2-\alpha)}-|s|-|t|\right)\right] & \text{if } t\cdot s>0 \\ 0 & \text{if } t\cdot s<0. \end{cases}$$

In particular, for the maximum likelihood estimators we have

(8) $$K_0(s,t)=e^{-|t-s|}-\left(1+2(st+|st|)\right)e^{-|t|-|s|}.$$

Proof. In the case of the MLE, the sample median and half of the interquartile range this theorem was proved in [5, 9] respectively. Here we prove the case of estimators $\hat{\theta}_{n,\alpha}$. Let $S \subset \mathbb{R}$. By $C(S)$ we denote the space of real-valued continuous function on S with the supremum norm. Using the Theorem of Csörgő and the notation therein (Section 3. of [3]) we will show that $Z_n(t)$ is weakly convergent in $C(S)$ to the zero mean Gaussian process with the covariance kernel $K_\alpha(\cdot,\cdot)$. Assumptions (i)*, (ii)* and (vi) do not depend on the choice of the estimators and were verified in [5].

Assumption (iv) is a consequence of the Bahadur representation of the estimators $\hat{\theta}_{n,\alpha}$ presented in the previous section. In order to verify the Assumption (v) we estimate

$$\sup_{|x|\leqslant u}\left(\left\|l\left(x,\theta_0\right)\right\|+\left\|D_x l\left(x,\theta_0\right)\right\|\right)$$

$$\leq \sup_{|x|\leqslant u}\left(\frac{2^{2-\alpha}}{1-\alpha}\max(|\cos^{1-\alpha} z\sin(z(1-\alpha))|,|2^{\alpha-1}-\cos^{1-\alpha} z\cos(z(1-\alpha))|\right.$$

$$\left.+2^{2-\alpha}\max(|\cos^{2-\alpha} z\sin(z(2-\alpha))|,|\cos^{2-\alpha} z\cos(z(2-\alpha))|)\right)$$

$$\leqslant\frac{2^{2-\alpha}}{1-\alpha}(2^{\alpha-1}+1)+2^{2-\alpha}<\infty.$$

Hence, $\hat{Z}_n$ converges weakly in $C(S)$ to the zero mean Gaussian process with the covariance kernel of the form

$$K(s,t)=e^{-|s-t|}-e^{-|s|-|t|}+H(s,\theta_0)^T E(l(X_1)l(X_1)^T)H(t,\theta_0)$$

$$-\langle H(t,\theta_0),\int_{\mathbb{R}}k(x,s)l(x)dF_0(x)\rangle-\langle H(s,\theta_0),\int_{\mathbb{R}}k(x,t)l(x)dF_0(x)\rangle,$$

where

$$H(t,\theta)=\int_{\mathbb{R}}k(x,t)d\nabla_\theta F(x,\theta).$$

By the direct calculation we have

$$H(t,\theta_0)=(te^{-|t|},-|t|e^{-|t|})^T.$$

Let us now calculate next components of $K(s,t)$.

$$E(l(X_1)l(X_1)^T)=E(\Lambda^{-1}\psi(X_1,\theta_0)(\Lambda^{-1}\psi(X_1,\theta_0))^T)=\Lambda^{-1}C\Lambda^{-1}$$

$$=\Sigma(\theta_0)=\frac{2}{(\alpha-1)^2}\left(\frac{1}{(3-2\alpha)B(2-\alpha,2-\alpha)}-1\right)I_2,$$

thus,

$$H(s,\theta_0)^T E(l(X_1)l(X_1)^T)H(t,\theta_0)$$

$$=\frac{2}{(\alpha-1)^2}\left(\frac{1}{(3-2\alpha)B(2-\alpha,2-\alpha)}-1\right)e^{-|t|-|s|}(st+|st|).$$

$$\int_{\mathbb{R}}k(x,s)l_1(x)dF_0(x)=\frac{2^{2-\alpha}}{\pi(1-\alpha)}\int_{\mathbb{R}}(\cos(sx)+\sin(sx))\cos^{1-\alpha} z\sin(z(1-\alpha))\frac{dx}{1+x^2}$$

$$=\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha} z\sin(z(1-\alpha))\sin(s\tan z)dz,$$

$$\int_{\mathbb{R}}k(x,s)l_2(x)dF_0(x)=\frac{2}{\pi(1-\alpha)}\int_{\mathbb{R}}(\cos(sx)+\sin(sx))\frac{dx}{1+x^2}$$

$$-\frac{2^{2-\alpha}}{\pi(1-\alpha)}\int_{\mathbb{R}}(\cos(sx)+\sin(sx))\cos^{1-\alpha} z\sin(z(1-\alpha))\frac{dx}{1+x^2}$$

$$=\frac{4}{\pi(1-\alpha)}\int_0^{\pi/2}\cos(s\tan z)dz-\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha}z\cos(z(1-\alpha))\cos(s\tan z)dz$$
$$=\frac{2}{(1-\alpha)}e^{-|s|}-\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha}z\cos(z(1-\alpha))\cos(s\tan z)dz,$$

(comp. [4] formula 3.723.2).
Hence,

$$\langle H(t,\theta_0),\int_{\mathbb{R}}k(x,s)l(x)dF_0(x)\rangle$$
$$=te^{-|t|}\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha}z\sin(z(1-\alpha))\sin(s\tan z)dz-|t|e^{-|t|}\frac{2}{(1-\alpha)}e^{-|s|}$$
$$+|t|e^{-|t|}\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha}z\cos(z(1-\alpha))\cos(s\tan z)dz$$
$$=|t|e^{-|t|}\frac{2^{3-\alpha}}{\pi(1-\alpha)}\int_0^{\pi/2}\cos^{1-\alpha}z\cos\big(z(1-\alpha)\mp s\tan z\big)dz-\frac{2|t|e^{-|s|-|t|}}{(1-\alpha)}.$$

In the above integral there is minus when $s\cdot t>0$ and plus in another case.

In the case of $s\cdot t>0$ using the formulas 3.718.6 and 9.224 of [4] we have

$$\langle H(t,\theta_0),\int_{\mathbb{R}}k(x,s)l(x)dF_0(x)\rangle$$
$$=\frac{2|t|e^{-|t|}|s|^{(1-\alpha)/2}}{1-\alpha}\frac{W_{(1-\alpha)/2,(2-\alpha)/2}(2|s|)}{\Gamma(2-\alpha)}-\frac{2|t|e^{-|s|-|t|}}{(1-\alpha)}$$
$$=\frac{2|t|e^{-|t|+|s|}}{1-\alpha}\frac{\Gamma(2-\alpha,2|s|)}{\Gamma(2-\alpha)}-\frac{2|t|e^{-|s|-|t|}}{(1-\alpha)}$$
$$=\frac{2|t|e^{-|t|-|s|}}{1-\alpha}\left(e^{2|s|}\frac{\Gamma(2-\alpha,2|s|)}{\Gamma(2-\alpha)}-1\right),$$

where W is the Whittaker function, and $\Gamma(\cdot,\cdot)$ denotes the incomplete Gamma function. In the second case ($s\cdot t<0$) by formula 3.718.5 of Gradshteyn, Ryzhik [4] we obtain

$$\langle H(t,\theta_0),\int_{\mathbb{R}}k(x,s)l(x)dF_0(x)\rangle=0.$$

Thus $K_\alpha(\cdot,\cdot)$ is of the form (7).
Since the convergence of $\hat{Z}_n$ in $C(S)$ was showed for any compact set $S\subset\mathbb{R}$; $\hat{Z}_n$ converges to Z also in the Frechet space $C(\mathbb{R})$ with the metric ρ (comp. [8], p. 62).

Now we present the theorem on the convergence of the test statistic $D_{n,\lambda,\gamma}$. □

Theorem 2. *Under the assumptions of Theorem 1 we have*

(9) $$D_{n,\lambda,\gamma}=\int_{\mathbb{R}}\frac{\hat{Z}_n^2(t)\hat{\sigma}_n^{1-\gamma}e^{-\hat{\sigma}_n\lambda|t|}}{|t|^\gamma}\mathrm{d}t\xrightarrow{d}D_{\lambda,\gamma}:=\int_{\mathbb{R}}\frac{Z^2(t)e^{-\lambda|t|}}{|t|^\gamma}\mathrm{d}t.$$

Proof. Since

$$\int_{\mathbb{R}} K(t,t)\frac{e^{-\lambda|t|}}{|t|^{\gamma}}dt<\infty$$

and

$$\int_{\mathbb{R}} K_{\alpha}(t,t)\frac{e^{-\lambda|t|}}{|t|^{\gamma}}dt<\infty,$$

by the Tonelli Theorem we have

$$ED_{\lambda,\gamma}=\int_{\mathbb{R}} Z^2(t)\frac{e^{-\lambda|t|}}{|t|^{\gamma}}dt<\infty.$$

Thus, $D_{\lambda,\gamma}$ is finite with probability 1. By the following Taylor expansion

$$F(x,\hat\theta_n)-F(x,\theta_0)=\langle\hat\theta_n-\theta_0,\nabla_\theta F(x,\theta_n^*)\rangle,$$

where $\left|\theta_n^*-\theta_0\right|\le\left|\hat\theta_n-\theta_0\right|\to 0$ with probability 1, $\hat Z_n$ has the form

$$\begin{aligned}\hat Z_n(t)&=\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F_n(x)-F(x,\hat\theta_n))\\
&=\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F_n(x)-F(x,\theta_0))+\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F(x,\theta_0)-F(x,\hat\theta_n))\\
&=\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F_n(x)-F(x,\theta_0))-\langle\sqrt{n}(\hat\theta_n-\theta_0),\int_{\mathbb{R}} k(x,t)d\nabla_\theta F(x,\theta_n^*)\rangle\\
&=\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F_n(x)-F(x,\theta_0))-\langle\frac{1}{\sqrt{n}}\sum_{j=1}^{n} l(X_j),H(t,\theta_0)\rangle\\
&\quad+\langle\frac{1}{\sqrt{n}}\sum_{j=1}^{n} l(X_j),H(t,\theta_0)\rangle-\langle\sqrt{n}(\hat\theta_n-\theta_0),H(t,\theta_n^*)\rangle\\
&=Z_n^*(t)+\langle\sqrt{n}(\hat\theta_n-\theta_0),H(t,\theta_0)-H(t,\theta_n^*)\rangle\\
&\quad+\langle\frac{1}{\sqrt{n}}\sum_{j=1}^{n} l(X_j)-\sqrt{n}(\hat\theta_n-\theta_0),H(t,\theta_0)\rangle\\
&=Z_n^*(t)+\langle\sqrt{n}(\hat\theta_n-\theta_0),H(t,\theta_0)-H(t,\theta_n^*)\rangle+\langle o_P(1),H(t,\theta_0)\rangle,\end{aligned}$$

where Z_n^* is the following process

$$\begin{aligned}Z_n^*(t)&:=\int_{\mathbb{R}} k(x,t)d\sqrt{n}(F_n(x)-F_0(x))-\left\langle\frac{1}{\sqrt{n}}\sum_{j=1}^{n} l(X_j),H(t,\theta_0)\right\rangle\\
&=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\cos(tX_j)+\sin(tX_j)-e^{-|t|}-te^{-|t|}l_1(X_j)+|t|e^{-|t|}l_2(X_j)\right).\end{aligned}$$

By straightforward calculations it is easy to show that the process Z_n^* has zero mean function and the same covariance kernel as the process Z and that Z_n^* converges weakly to Z in $C(S)$. Since this convergence takes place for any compact set $S\subset\mathbb{R}$, Z_n^* converge weakly do Z in the Frechet space $C(\mathbb{R})$.

Further in this proof the following convergences will be needed

(10) $$\int_{\mathbb{R}} (\hat{Z}_n(t) - Z_n^*(t))^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \xrightarrow{P} 0,$$

(11) $$\int_{\mathbb{R}} (\hat{Z}_n(t) - Z_n^*(t))^2 \left(\frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} - \frac{e^{-\lambda|t|}}{|t|^\gamma} \right) dt \xrightarrow{P} 0$$

and

(12) $$\int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt - \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \xrightarrow{P} 0.$$

In order to obtain (10) we calculate

$$\begin{aligned}
&\int_{\mathbb{R}} (\hat{Z}_n(t) - Z_n^*(t))^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \\
&= \int_{\mathbb{R}} (\langle \sqrt{n}(\hat{\theta}_n - \theta_0), H(t,\theta_0) - H(t,\theta_n^*) \rangle + \langle o_P(1), H(t,\theta_0) \rangle)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \\
&= \sum_{i,j=1}^{2} \tau_{ni}\tau_{nj} \int_{\mathbb{R}} (H_i(t,\theta_0) - H_i(t,\theta_n^*)(H_j(t,\theta_0) - H_j(t,\theta_n^*) \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \\
&\quad + 2\sum_{i,j=1}^{2} \tau_{ni} o_P(1) \int_{\mathbb{R}} (H_i(t,\theta_0) - H_i(t,\theta_n^*) H_j(t,\theta_0) \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \\
&\quad + \sum_{i,j=1}^{2} o_P(1) \int_{\mathbb{R}} H_i(t,\theta_0) H_j(t,\theta_0) \frac{e^{-\lambda|t|}}{|t|^\gamma} dt,
\end{aligned}$$

where $\tau_{n1} = \sqrt{n}\hat{m}_n$ and $\tau_{n2} = \sqrt{n}(\hat{\sigma}_n - 1)$. H_i are bounded and continuous on the set $S \times \Theta_0$, where $S \subset \mathbb{R}$ is any compact set and Θ_0 is closure of certain neighborhood of θ_0, the sequences τ_{n1} and τ_{n2} are tight and $H_i(t, \theta_0) - H_i(t; \theta_n^*)$ converge to 0 with probability 1 for $i = 1, 2$; thus, in the consequence, we obtain (10).

Convergence (11) can be obtained analogously.

Using the Taylor expansion

$$e^{-\lambda\hat{\sigma}_n|t|} = e^{-\lambda|t|} - \lambda|t|e^{-\lambda|t|\delta_n}(\hat{\sigma}_n - 1),$$

where $|\delta_n - 1| \leq |\hat{\sigma}_n - 1|$ and the Schwarz inequality we estimate

$$\left| \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt - \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \right|$$

(13) $$\leqslant \lambda|\hat{\sigma}_n - 1| \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|\delta_n}}{|t|^{\gamma-1}} dt$$

$$\leqslant \lambda|\hat{\sigma}_n - 1| \left(\int_{\mathbb{R}} Z_n^*(t)^4 e^{-\lambda|t|} dt \right)^{1/2} \left(\int_{\mathbb{R}} \frac{e^{-\lambda|t|(2\delta_n - 1)}}{|t|^{2\gamma-2}} dt \right)^{1/2}.$$

As it was showed in Gürtler i Henze [5] the sequence

$$\left(\int_{\mathbb{R}} Z_n^*(t)^4 e^{-\lambda|t|} dt \right)^{1/2}$$

is tight. Since $\hat{\sigma}_n \to 1$ with probability 1, the last integral in (13) converge with probability 1 to $2\Gamma(3-2\gamma)/\lambda^{3-2\gamma}$ and in the consequence we obtain (12).

Convergence

$$\int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \xrightarrow{d} \int_{\mathbb{R}} Z(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \tag{14}$$

can be proved analogously to Henze and Wagner ([7], proof of 2.17, pp. 10-12) By (10) and (11) we have

$$\left| \left(\int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \right)^{1/2} - \left(\int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \right)^{1/2} \right|$$

$$\leqslant \left(\int_{\mathbb{R}} (\hat{Z}_n(t) - Z_n^*(t))^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \right)^{1/2} \xrightarrow{P} 0,$$

thus

$$\left(\int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \right)^{1/2} = \left(\int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \right)^{1/2} + o_P(1),$$

and in consequence we have

$$\int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt = \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt + o_P(1). \tag{15}$$

Finally, applying (12), (14), (15) and the Slutsky Lemma we obtain

$$\int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt = \left(\int_{\mathbb{R}} Z_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt - \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\sigma_n|t|}}{|t|^\gamma} dt \right)$$

$$+ \left(\int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt - \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \right)$$

$$+ \int_{\mathbb{R}} Z_n^*(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt \xrightarrow{d} \int_{\mathbb{R}} Z(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt,$$

hence

$$D_{n,\lambda,\gamma} = \hat{\sigma}_n^{1-\gamma} \int_{\mathbb{R}} \hat{Z}_n(t)^2 \frac{e^{-\lambda\hat{\sigma}_n|t|}}{|t|^\gamma} dt \xrightarrow{d} \int_{\mathbb{R}} Z(t)^2 \frac{e^{-\lambda|t|}}{|t|^\gamma} dt = D_{\lambda,\gamma}. \qquad \square$$

The covariance kernel of the process Z determine an integral operator on the space $L^2(\mathbb{R})$

$$\mathcal{K}: L^2(\mathbb{R}) \ni f \to \int_{\mathbb{R}} K_\alpha(s,t) f(t) \frac{e^{-\lambda(|s|+|t|)/2}}{(st)^{\gamma/2}} dt \in L^2(\mathbb{R}).$$

Theorem 1(iii) of Buescu ([1]) guarantees that the kernel of this operator has the representation as absolutely and uniformly convergent series

$$K_\alpha(s,t) \frac{e^{-\lambda(|s|+|t|)/2}}{(st)^{\gamma/2}} = \sum_{j=1}^{\infty} \eta_j \phi_j(s) \phi_j(t), \tag{16}$$

where η_j are eigenvalues of the operator $\mathcal{K}$ ordered nonincreasingly ($1 \geq 2 \geq \ldots \geq 0$), and ϕ_j are the corresponding eigenfunctions. Let us define the following stochastic process

(17) $$Y(t) = \sum_{j=1}^{\infty} \sqrt{\eta_j}\, \phi_j(t) N_j,$$

where N_1, N_2, is a sequence of independent random variables distributed according to the standard normal low. Since the series (17) is convergent in mean, Y is centered Gaussian process with the covariance function (16), there is the covariance function of the process

$$Z(t)\frac{e^{-\lambda|t|/2}}{t^{\gamma/2}}.$$

Taking into account orthonormality of the eigenfunctions we obtain

$$D_{\lambda,\gamma} = \int_{\mathbb{R}} Z(t)^2 \frac{e^{-\lambda|t|}}{|t|^{\gamma}} dt$$

$$\stackrel{\mathcal{L}}{=} \int_{\mathbb{R}} Y(t)^2\, dt = \int_{\mathbb{R}} \left(\sum_{j=1}^{\infty} \sqrt{\eta_j}\, \phi_j(t) N_j \right)^2 dt = \sum_{j=1}^{\infty} \eta_j N_j^2,$$

where $\stackrel{\mathcal{L}}{=}$ denote equality of probability laws. Hence the limit distribution of statistics $D_{n,\lambda,\gamma}$ is the same as the distribution of $\sum_{j=1}^{\infty} \eta_j N_j^2$.

4. Consistency

In order to obtain consistent goodness-of-fit test for the Cauchy distribution the following procedure can be applied: first we estimate the parameters and then we compute the test statistics and compare its value with critical value for fixed signicance level. In [10] it was showed that estimators $\hat{\theta}_{n,\alpha}$ cannot be computed if, and only if, $\#\{k : X_k = m\}/n \geq 2^{\alpha-1}$ for some m. For Cauchy distributed samples the probability of such event is equal to 0. Thus, in that case the hypothesis H_0 should be rejected.

The following theorem guarantees consistency of the test based on the statistic $D_{n,\lambda,\gamma}$ against any non Cauchy alternative. Let us stress that this theorem does not impose any restrictions on the alternative distribution. Theorem 2.3. of Gürtler and Henze [5] can be proved analogously. In this way one can obtain consistency of test considered by Gürtler and Henze [5] against any non Cauchy alternative without assumptions on uniqueness of the median and interquartile range.

Theorem 3. *Let X_1, X_2, ... be a sequence of independent, identically distributed random variables with common characteristic function φ, $\hat{m}_n$ and $\hat{\sigma}_n$ be any earlier considered estimators. Than*

(18) $$\liminf_{n\to 0} \frac{1}{n} D_{n,\lambda,\gamma} \geqslant \inf_{(m,\sigma)\in\Theta} \int_{\mathbb{R}} \left| e^{-itm/\sigma} \varphi(t/\sigma) - e^{-|t|} \right|^2 \frac{e^{-\lambda|t|}}{|t|^{\gamma}} dt$$

with probability 1.

Let us notice that right-handside of (18) is equal to 0 if, and only if, φ is characteristic function of the Cauchy distribution.

Proof. For positive constants T and K we will denote

$$R_{\sigma,K} := [-T/\sigma, T/\sigma] \cap [-K, K].$$

Using substitution $s = t/\hat{\sigma}_n$ and applying the Minkowski inequality we estimate as follows

$$\begin{aligned}
\frac{1}{n} D_{n,\lambda,\gamma} &\geqslant \int_{-T}^{T} \left| \frac{1}{n} \sum_{j=1}^{n} e^{it(X_j - \hat{m}_n)/\sigma_n} - e^{-|t|} \right|^2 \frac{e^{-\lambda|t|}}{|t|^{\gamma}} dt \\
&= \int_{-T/\hat{\sigma}_n}^{T/\hat{\sigma}_n} \left| e^{-is\hat{m}_n} \frac{1}{n} \sum_{j=1}^{n} e^{isX_j} - e^{-\hat{\sigma}_n |s|} \right|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \\
&\geqslant \int_{R_{\hat{\sigma}_n,K}} \left| e^{-is\hat{m}_n} \frac{1}{n} \sum_{j=1}^{n} e^{isX_j} - e^{-\hat{\sigma}_n |s|} \right|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \\
&\geqslant \left(\left(\int_{R_{\hat{\sigma}_n,K}} \left| e^{-is\hat{m}_n} \frac{1}{n} \sum_{j=1}^{n} e^{isX_j} - e^{-\hat{m}_n |s|} \varphi(s) \right|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \right)^{1/2} \right. \\
&\qquad \left. \left(\int_{R_{\hat{\sigma}_n,K}} \left| e^{-\hat{m}_n |s|} \varphi(s) - e^{-\hat{\sigma}_n |s|} \right|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \right)^{1/2} \right)^2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{n} D_{n,\lambda,\gamma} &\geqslant \left(\left(\int_{R_{\hat{\sigma}_n,K}} |\varphi_n(s) - \varphi(s)|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \right)^{1/2} \right. \\
&\qquad \left. \left(\int_{R_{\hat{\sigma}_n,K}} \left| e^{-\hat{m}_n |s|} \varphi(s) - e^{-\hat{\sigma}_n |s|} \right|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \right)^{1/2} \right)^2 .
\end{aligned} \tag{19}$$

The first integral in the last inequality can be estimated in the following way

$$\begin{aligned}
&\int_{R_{\hat{\sigma}_n,K}} |\varphi_n(s) - \varphi(s)|^2 \frac{\hat{\sigma}_n^{1-\gamma} e^{-\lambda \hat{\sigma}_n |s|}}{|s|^{\gamma}} ds \\
&\leqslant 2 \sup_{s \in R_{\hat{\sigma}_n,K}} |\varphi_n(s) - \varphi(s)| \hat{\sigma}_n^{1-\gamma} \int_0^{\min(T/\hat{\sigma}_n, K)} \frac{1}{|s|^{\gamma}} ds \\
&\leqslant \frac{2T^{1-\gamma}}{1-\gamma} \sup_{s \in R_{\hat{\sigma}_n,K}} |\varphi_n(s) - \varphi(s)|.
\end{aligned}$$

Since the empirical characteristic function converges uniformly on any compact set with probability 1 to characteristic function (comp. Csöorgő [2], Theorem. 2.1) the first integral on the right-handsize of (19) converges to 0 with probability 1. Therefore we obtain

$$\liminf_{n\to 0}\frac{1}{n}D_{n,\lambda,\gamma}\geqslant \inf_{(m,\sigma)\in\Theta}\int_{R_{\sigma,K}}\left|e^{-ism}\varphi(s)-e^{-\sigma|s|}\right|^2\frac{\sigma^{1-\gamma}e^{-\lambda\sigma|s|}}{|s|^\gamma}ds$$

$$=\inf_{(m,\sigma)\in\Theta}\int_{[-T,T]\cap[-\sigma K,\sigma K]}\left|e^{-itm/\sigma}\varphi(t/\sigma)-e^{-|t|}\right|^2\frac{e^{-\lambda|t|}}{|t|^\gamma}dt.$$

Letting K and T to infinity completes the proof. □

5. Simulation results

In this section we present results of the numerical simulations performed to verify the finite sample behaviour of the new test. Since the results of simulations presented in Gürtler and Henze [5] show the advantage of the test based on the statistic $D_{n,\lambda,0}$, with the sample median and half of the interquartile range as estimators over other tests, we compare the behaviour of the new test with the test considered by Gürtler and Henze.

Table 1 presents critical values for four dierent estimators and different values of parameters γ and λ estimated from 10 000 samples (for n = 20 and n = 50) drawn from the standard Cauchy distribution.

Table 2 and 3 present estimated powers of considered test for some alternatives for n = 20 and n = 50; respectively. In these tables $N(0, 1)$ denotes standard normal distribution, t_n denotes Student distribution with n degrees of freedom, $Log(0, 1)$ denotes logistic distribution, $U(0, 1)$ denotes uniform distribution over the interval (0, 1), $La(0, 1)$ denotes the standard Laplace distribution, χ^2_n denotes chi-square distribution with n degrees of freedom, G and B denotes Gamma and Beta distribution, respectively.

From this tables we draw a conclusion that using $\gamma > 0$ in the statistic $D_{n,\lambda,\gamma}$ does not influent signicantly on the test, while applying the new estimators $\hat{\theta}_{n,\alpha}$ considerably improves the power of the test.

Table 1

Critical values on the significance level α = 0.1

		n = 20				n = 50			
		$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$
$\gamma = 0$	$\lambda = 1$	1.253	1.102	1.304	1.531	1.330	1.103	1.304	1.521
	$\lambda = 2.5$	0.321	0.292	0.259	0.291	0.352	0.288	0.265	0.298
	$\lambda = 5$	0.116	0.118	0.083	0.081	0.127	0.116	0.083	0.081
$\gamma = 0.1$	$\lambda = 1$	1.264	1.099	1.289	1.488	1.334	1.111	1.282	1.497
	$\lambda = 2$	0.480	0.428	0.411	0.469	0.519	0.424	0.413	0.473
$\gamma = 0.5$	$\lambda = 1$	1.394	1.256	1.300	1.468	1.485	1.246	1.307	1.483

Table 2

Estimated powers for n = 20 on the significance level α = 0.1

Alternative distribution	$\gamma = 0$											
	$\lambda = 1$				$\lambda = 2.5$				$\lambda = 5$			
	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$
$N(0,1)$	27	36	50	54	41	28	45	56	49	17	8	6
t_2	8	11	17	18	10	7	11	14	12	4	3	2
t_5	16	23	34	37	25	16	26	33	30	10	4	3
t_{10}	21	28	42	45	33	21	35	44	40	13	6	4
$Log(0,1)$	19	26	39	42	29	19	32	40	35	12	5	4
$U(0,1)$	76	83	91	92	87	79	93	96	89	61	46	38
$La(0,1)$	10	15	21	23	15	10	17	22	18	6	3	3
χ^2_{10}	33	42	56	60	46	34	49	60	51	26	14	10
$G(2,1)$	48	50	67	70	54	42	57	69	54	38	24	18
$B(2.5,1.5)$	55	63	76	78	68	57	76	85	72	42	26	20
	$\gamma = 0.1$								$\gamma = 0.5$			
	$\lambda = 1$				$\lambda = 2$				$\lambda = 1$			
	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$
$N(0,1)$	27	36	50	55	32	32	48	55	38	31	48	57
t_2	9	12	16	19	9	8	13	17	10	7	12	15
t_5	17	23	33	38	19	19	30	36	22	17	29	36
t_{10}	21	29	42	47	25	24	39	46	30	23	39	47
$Log(0,1)$	19	26	38	42	22	21	35	41	26	20	35	42
$U(0,1)$	76	84	91	93	81	81	91	94	84	81	93	96
$La(0,1)$	11	14	21	23	12	11	19	22	14	10	18	22
χ^2_{10}	32	42	55	61	37	38	53	61	42	37	54	64
$G(2,1)$	51	51	66	72	53	47	66	72	53	45	64	72
$B(2.5,1.5)$	53	63	76	80	59	59	76	81	64	59	78	85

Table 3

Estimated powers for n = 50 on the significance level α = 0.1

Alternative distribution	γ = 0											
	λ = 1				λ = 2.5				λ = 5			
	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$
$N(0,1)$	79	88	96	98	94	96	98	99	97	97	98	98
t_2	20	25	38	43	29	25	33	40	37	27	22	22
t_5	50	62	81	86	73	73	83	90	85	79	76	76
t_{10}	66	77	91	94	86	88	94	97	93	92	92	93
$Log(0,1)$	58	72	88	92	82	83	91	96	91	89	88	89
$U(0,1)$	100	100	100	100	100	100	100	100	100	100	100	100
$La(0,1)$	22	32	50	58	43	42	59	70	58	47	51	56
χ^2_{10}	89	92	98	99	97	99	100	100	99	98	99	99
$G(2,1)$	98	98	99	100	99	99	100	100	99	99	99	100
$B(2.5,1.5)$	99	99	100	100	100	100	100	100	100	100	100	100
	γ = 0.1								γ = 0.5			
	λ = 1				λ = 2				λ = 1			
	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$	$\hat{\theta}_M$	$\hat{\theta}_{ML}$	$\hat{\theta}_{.6}$	$\hat{\theta}_{.8}$
$N(0,1)$	82	89	96	98	91	94	98	99	93	96	98	98
t_2	19	24	38	43	24	25	36	43	27	25	36	43
t_5	53	63	81	86	66	70	83	88	70	72	84	90
t_{10}	68	78	91	94	80	84	93	96	84	87	94	97
$Log(0,1)$	62	72	88	92	75	79	90	94	79	82	92	96
$U(0,1)$	100	100	100	100	100	100	100	100	100	100	100	100
$La(0,1)$	24	33	51	59	34	38	54	63	39	40	59	69
χ^2_{10}	91	94	98	99	96	96	99	100	99	99	100	100
$G(2,1)$	98	98	100	100	98	99	100	100	99	99	100	100
$B(2.5,1.5)$	99	100	100	100	100	100	100	100	100	100	100	100

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