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ASYMPTOTIC ESTIMATE OF ABSOLUTE PROJECTION CONSTANTS

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Abstract. In this note we construct a sequence of real, k-dimensional symmetric spaces Y^k satisfying

$$\liminf_k \lambda_k^S/\sqrt{k} \geq \liminf_k \lambda(Y^k, l_1)/\sqrt{k} > \max_{w\in[0,a_2]} h(w) > 1/(2-\sqrt{2/\pi}),$$

where λ_k^S is defined by (4) and

$$h(w) = a_1^2\sqrt{2/\pi} + 2a_1\sqrt{a_2^2 - w^2} + w\sqrt{a_2^2 - w^2}$$

with $a_1 = 1/(2-\sqrt{2/\pi})$ and $a_2 = 1 - a_1$. This improves the lower bound obtained in [**3**], Th. 5.3 by $\max_{w\in[0,a_2]} h(w)$.

1. Introduction. Let X be a normed space and let V be a linear subspace of X. Denote by $\mathcal{P}(X,V)$ the set of all projections from X onto V, i.e., the set of all continuous extensions of $id : V \to V$ to X. Let

$$\lambda(V,X) = \inf\{\|P\| : P \in \mathcal{P}(X,V)\}$$

and

$$\lambda(V) = \sup\{\lambda(V,X) : V \subset X, \text{ as Banach spaces}\}.$$

We call $\lambda(V,X)$ the *relative projection constant* of V in X and $\lambda(V)$ the *absolute projection constant* of V. A projection $P \in \mathcal{P}(X,V)$ is called *minimal* if $\|P\| = \lambda(V,X)$. Let us denote

$$\lambda_k = \sup\{\lambda(Y) : Y \text{ is a real, } k\text{-dimensional space}\}.$$

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It is known (see e.g [**8**]), by the compactness of the Banach–Mazur compactum and the continuity of the function $X \to \lambda(X)$, that there exists a k-dimensional, real space X^k such that

$$\lambda_k = \lambda(X^k). \tag{1}$$

Moreover, X^k, as a separable Banach space, is isometric to a subspace of l^∞ and $\lambda(X^k) = \lambda(X^k, l^\infty)$ (see e.g. [**11**]). By the Kadeč–Snobar Theorem [**7**], $\lambda_k \leq \sqrt{k}$. Moreover, the examples from [**4**] show that this estimate is asymptotically the best possible, which means that

$$\lim_k \lambda(X^k)/\sqrt{k} = 1, \tag{2}$$

where X^k is given by (1). For other related results see [**2, 4–6**].
It is worth saying that the spaces X^k defined by (1) are not symmetric. Recall that a k-dimensional real Banach space V is called *symmetric* if there is a basis $v^1, \ldots, v^k$ of V such that

$$\|\sum_{j=1}^{k} a_j v^j\| = \|\sum_{j=1}^{k} \epsilon_j a_{\sigma(j)} v^j\| \tag{3}$$

for any $a_1, \ldots, a_k \in \mathbb{R}$, $\epsilon_k \in \{-1, 1\}$ and $\sigma \in \Sigma_k$, where Σ_k denotes the set of all permutations of $\{1, \ldots, k\}$. Moreover, equality (2) does not hold in the case of symmetric spaces, which has been shown in [**6**]. It has been proven in [**6**] that

$$\limsup_k (\lambda_k^S/\sqrt{k}) < 1 - 1/900,$$

where

$$\lambda_k^S = \sup\{\lambda(Y) : Y \text{ real, } k\text{-dimensional, symmetric space}\}. \tag{4}$$

It also has been conjectured in [**6**], p. 36, that

$$\limsup_k \lambda_k^S/\sqrt{k} = 1/(2 - \sqrt{2/\pi}). \tag{5}$$

This conjecture has been partially motivated by [**5**], Prop. 2, where the existence of k-dimensional, real, symmetric spaces Y^k satisfying

$$\limsup_k \lambda(Y^k)/\sqrt{k} = 1/(2 - \sqrt{2/\pi}) \tag{6}$$

has been shown. Observe that by [**10**]

$$\lim_k \lambda(l_2^{(k)})/\sqrt{k} = \sqrt{2/\pi}. \tag{7}$$

Since

$$\sqrt{2/\pi} = 0.7979\ldots < 1/(2 - \sqrt{2/\pi}) = 0.8319\ldots,$$

the spaces Y^k have asymptotically larger absolute projection constants than the Euclidean spaces $l_2^{(k)}$. Also in [**9**] the Marcinkiewicz spaces satisfying (6) have been constructed.
Above conjecture (5) has been disproved in [**3**], Th. 5.3. Moreover, in [**1**], for $k \geq 3$, there have been constructed symmetric k-dimensional subspaces V^k of l_1 having a very simple structure such that

$$\lim_k \lambda(V^k, l_1)/\sqrt{k} = 1/(2 - \sqrt{2/\pi}).$$

The aim of this note is to show the existence of k-dimensional, real and symmetric subspaces V^k satisfying

$$\liminf_k \lambda_k^S/\sqrt{k} \geq \liminf_k \lambda(Y^k, l_1)/\sqrt{k} > \max_{w \in [0,a_2]} h(w) > 1/(2 - \sqrt{2/\pi}),$$

where $h(w) = a_1^2\sqrt{2/\pi} + 2a_1\sqrt{a_2^2 - w^2} + w\sqrt{a_2^2 - w^2}$ with $a_1 = \frac{1}{2-\sqrt{2/\pi}}$ and $a_2 = 1 - a_1$. This improves the lower bound obtained in [**3**], Th. 5.3 by $\max_{w \in [0,a_2]} h(w)$.

2. Auxiliary results. In this section we present some definitions and results which will be of use later.

Definition 2.1. Let $x \in \mathbb{R}^k$ and let Σ_k denote the set of all permutations of $\{1, \ldots, k\}$. Suppose $J : \Sigma_k \times \{-1, 1\}^k \to \{1, \ldots, 2^k k!\}$ is a fixed bijection such that $J(\mathrm{id}, (1, \ldots, 1)) = 1$. By $[[x]]$ we denote the $k \times 2^k k!$ matrix with the columns $x^{(j)}$, $j = 1, \ldots, 2^k k!$, where

$$x^{(j)} = (\epsilon \circ \sigma)(x) = (\epsilon_1 x_{\sigma(1)}, \ldots, \epsilon_k x_{\sigma(k)}).$$

Here $\epsilon \in \{-1, 1\}^k$, $\sigma \in \Sigma_k$ are so chosen that $j = J(\sigma, \epsilon)$. Observe that, for any $x \in \mathbb{R}^k$, $x^{(1)} = x$. We will refer to the matrix $[[x]]$ as *the block generated by x*.

Definition 2.2. Let $k, N \in \mathbb{N}$. Put $n = N2^k k!$. Let $x^1, \ldots, x^N \in \mathbb{R}^k$. A linear subspace $V \subset l_1^{(n)}$ is said to be *generated by* $(x^1, \ldots, x^N)$ if and only if the rows $v^1, \ldots, v^k$ of the $k \times N2^k k!$ matrix $[[x^1]], \ldots, [[x^N]]$ form a basis of V, where, for $i = 1, \ldots, N$, $[[x^i]]$ is the block generated by x^i (see Def. 2.1). It is easy to check that V is a symmetric space (see (3)) with respect to $v^1, \ldots, v^k$.

The following notation will also be used. For $x, y \in \mathbb{R}^k$, set

$$y \cdot [[x]] = \sum_{i=1}^{2^k k!} \Big(\Big|\sum_{j=1}^{k} y_j x_j^{(i)}\Big|\Big) = \sum_{(\sigma,\epsilon) \in \Sigma_k \times \{-1,1\}^k} \Big|\sum_{j=1}^{k} y_j \epsilon_j x_{\sigma(j)}\Big|.$$

Observe that for any $x, y \in \mathbb{R}^k$

$$y \cdot [[x]] = x \cdot [[y]]. \tag{8}$$

Now let $V \subset l_1^{(n)}$ be a subspace generated by $x^1, \ldots, x^N$ from $\mathbb{R}^k$. For $z \in \mathbb{R}^k$ we set

$$\|z\| = \Big\| \sum_{j=1}^{k} z_j v^j \Big\|_1,$$

where $v^1, \ldots, v^k$ is the basis of V associated with $x^1, \ldots, x^N$ by Def. (2.2). By $\|z\|_e$ we denote the Euclidean norm of z. Observe that

$$\|z\| = \sum_{j=1}^{N} z \cdot [[x^j]].$$

The main tool for our investigations will be the following theorem proved in [**3**], Th. 4.1. We present here a version of it more convenient for our purposes.

THEOREM 2.1. *Let $k, N \in \mathbb{N}$. Let $n = N2^k k!$. Consider the following extremal problem. Maximize the function $f : \mathbb{R}^{kN} \to \mathbb{R}$ defined for $x^1, \ldots, x^N \in \mathbb{R}^k$ by*

$$f(x^1, \ldots, x^N) = \sum_{j=1}^{N} \|x^j\| \tag{9}$$

under the conditions

$$\sum_{j=1}^{N} \|x^j\|_e = 1.$$

If f attains its maximum at $(y^1, \ldots, y^N)$ then the symmetric k-dimensional space Y^N generated by $y^1, \ldots, y^N$ satisfies

$$\frac{f(y^1, \ldots, y^N)}{2^k (k-1)!} = \lambda(Y^N, l_1) \le \lambda(Y^N).$$

3. Main result. In this section we show that there exist k-dimensional maximal symmetric spaces V^k satisfying

$$\liminf_k \lambda(Y^k, l_1)/\sqrt{k} > \max_{w \in [0, a_2]} h(w) > 1/(2 - \sqrt{2/\pi}), \tag{10}$$

where $h(w) = a_1^2 \sqrt{2/\pi} + 2a_1 \sqrt{a_2^2 - w^2} + w\sqrt{a_2^2 - w^2}$ with $a_1 = 1/(2 - \sqrt{2/\pi})$ and $a_2 = 1 - a_1$. In fact, we show that (10) holds true for k-dimensional maximal symmetric spaces generated by three blocks. To do this, for $k \in \mathbb{N}$, $k \ge 2$ and $a_1, l \in [0, 1]$, set

$$x^{1,k,l} = \frac{a_1(1, \ldots, 1)}{\sqrt{k}}, \quad x^{2,k,l} = la_2(c_k, d_k, \ldots, d_k) \tag{11}$$

and

$$x^{3,k,l} = (1-l)a_2(1, 0, \ldots, 0)$$

where $a_2 = 1 - a_1$ and c_k, d_k are nonnegative numbers such that

$$\sqrt{(c_k)^2 + (k-1)(d_k)^2} = 1 \tag{12}$$

and

$$\sqrt{k-1}\, d_k = w. \tag{13}$$

Here $w \in [0,1]$ is a fixed number independent of k. Notice that for any $a_1 \in [0,1]$

$$\|x^{1,k,l}\|_e + \|x^{2,k,l}\|_e + \|x^{1,k,l}\|_e = a_1 + a_2 = 1,$$

which shows that the above vectors can be used to estimate from below the function f from Theorem 2.1.
We start with

Lemma 3.1. *For any $k \in \mathbb{N}$ and $a, b \in \mathbb{R}_+$*

$$\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, b, \ldots, b) \rangle| \geq 2^k a$$

and

$$\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, a, b, \ldots, b) \rangle| \geq 2^k a,$$

where for $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$

$$\langle x, y \rangle = \sum_{j=1}^{k} x_k y_k.$$

Proof. Notice that

$$\begin{aligned}
\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, b, \ldots, b) \rangle| &= 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-1}} |a + \langle \epsilon, (b, \ldots, b) \rangle| \\
&= 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-2}} |a + b + \langle \epsilon, (b, \ldots, b) \rangle| + |a - b - \langle \epsilon, (b, \ldots, b) \rangle| \\
&\geq 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-2}} |2a| = 2^k a.
\end{aligned}$$

The second inequality can be proved in the same way. □

Lemma 3.2. *Let f^k be the function defined in Th.2.1 by (9) for $N = 3$ and $k \geq 2$. Then for any $a_1 \in [0,1]$*

$$\begin{aligned}
\frac{f^k(x^{1,k,l}, x^{2,k,l}, x^{3,k,l})}{2^k (k-1)!} &\geq g_{a_1,k,l}(w) := \frac{a_1^2 C_k}{2^{k-1}} + 2\sqrt{k} a_1 a_2 l c_k + 2\sqrt{k} a_1 a_2 (1-l) \\
&\quad + 2a_2^2 l(1-l)(c_k + (k-1)d_k) + (a_2 l)^2 (c_k^2 + (k-1)c_k d_k)
\end{aligned}$$

where

$$C_k = \sum_{l=0}^{(k-1)/2} \binom{k}{l}(k-2l)$$

for k odd,

$$C_k = \sum_{l=0}^{k/2-1} \binom{k}{l}(k-2l)$$

for k even and $a_2 = 1 - a_1$.

Proof. By (9) and (8),

$$\frac{f^k(x^{1,k,l}, x^{2,k,l}, x^{3,k,l})}{2^k(k-1)!} = \sum_{i,j=1}^{3} \frac{x^{i,k,l}[[x^{j,k,l}]]}{2^k(k-1)!}. \tag{14}$$

Note that by elementary calculations (compare with [**1**], Th. 2.8)

$$x^{1,k,l}[[x^{1,k,l}]] = (a_1^2/k)2k!C_k. \tag{15}$$

Also

$$x^{2,k,l}[[x^{2,k,l}]] \geq (a_2 l)^2 2^k (k-1)! c_k (c_k + (k-1)d_k), \tag{16}$$

$$x^{3,k,l}[[x^{3,k,l}]] = 2^k(k-1)! a_2^2 (1-l)^2, \tag{17}$$

$$x^{1,k,l}[[x^{2,k,l}]] = \frac{(a_1 a_2 l)2^k k! c_k}{\sqrt{k}}, \tag{18}$$

$$x^{1,k,l}[[x^{3,k,l}]] = \frac{(a_1 a_2 (1-l))2^k k!}{\sqrt{k}} \tag{19}$$

and

$$x^{2,k,l}[[x^{3,k,l}]] = a_2^2 (1-l) l (c_k + (k-1)d_k) 2^k (k-1)!. \tag{20}$$

To prove (16), notice that by Lemma 3.1

$$\begin{aligned} x^{2,k,l}[[x^{2,k,l}]] &= (a_2 l)^2 (k-1)! \cdot \Big(\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (c_k^2, d_k^2, \ldots, d_k^2) \rangle| \\ &\quad + (k-1) \sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (c_k d_k, c_k d_k, d_k^2, \ldots, d_k^2) \rangle| \Big) \\ &\geq (a_2 l)^2 2^k (k-1)! (c_k^2 + (k-1) c_k d_k), \end{aligned}$$

as required. The proof of equalities (17)–(20) follows by elementary calculations. Applying (14)–(20), we get the result. □

Lemma 3.3. *Let* $g : [0,1]^3 \to \mathbb{R}$ *be defined by*

$$g(a_1, w, l) = a_1^2\sqrt{2/\pi} + 2a_1a_2l\sqrt{1-w^2} + 2a_2a_1(1-l) + (a_2l)^2w\sqrt{1-w^2} + 2a_2^2w(1-l)l, \tag{21}$$

where $a_2 = 1 - a_1$. *Let, for* $k \in \mathbb{N}$, $k \geq 2$, $x^{1,k,l}, x^{2,k,l}, x^{3,k,l} \in \mathbb{R}^k$ *be the vectors associated with* w *by (11)–(13). Let* f^k *be as in Lemma 3.2. Then for any* $(a_1, l, w) \in [0,1]^3$

$$\liminf_k \frac{f^k(x^{1,k,l}, x^{2,k.l}, x^{3,k.l})}{\sqrt{k}2^k(k-1)!} \geq g(a_1, w, l).$$

Proof. By Lemma 3.2 for any $(l, w) \in [0,1]^2$

$$\begin{aligned}\frac{f^k(x^{1,k,l}, x^{2,k.l}, x^{3,k.l})}{\sqrt{k}2^k(k-1)!} &\geq g_{a_1,k,l}(w)/\sqrt{k} \\ &= \frac{a_1^2C_k}{\sqrt{k}2^{k-1}} + 2a_1a_2l\sqrt{1-w^2} + 2a_2a_1(1-l) \\ &\quad + 2a_2^2l(1-l)\frac{c_k + \sqrt{k-1}w}{\sqrt{k}} + \left(a_2^2l^2\sqrt{k-1}w\sqrt{1-w^2}\right)/\sqrt{k}.\end{aligned}$$

By [**1**], Lemma 2.3, $\frac{C_k}{2^{k-1}} = \lambda(l_2^{(k)})$. By (7), $\lim_k \frac{C_k}{2^{k-1}\sqrt{k}} = \sqrt{\frac{2}{\pi}}$. Consequently,

$$\liminf_k \frac{f^k(x^{1,k,l}, x^{2,k.l}, x^{3,k.l})}{\sqrt{k}2^k(k-1)!} \geq \lim_k g_{a_1,k,l}(w)/\sqrt{k} = g(a_1, w, l),$$

as required. □

Remark 3.1. Notice that for any $a_1 \in [0,1]$

$$g(a_1, w, 1) = a_1^2\sqrt{2/\pi} + 2a_1a_2\sqrt{1-w^2} + a_2^2w\sqrt{1-w^2}.$$

Set $a_1 = \frac{1}{2-\sqrt{2/\pi}}$. Changing variables from $w \in [0,1]$ to $a_2w \in [0, a_2]$ we get

$$\max_{w\in[0,1]} g(a_1, w, 1) = \max_{w\in[0,a_2]} h(w),$$

where

$$h(w) = a_1^2\sqrt{2/\pi} + 2a_1\sqrt{a_2^2 - w^2} + w\sqrt{a_2^2 - w^2}.$$

In [**3**], Lemma 5.2 it was shown that

$$\lim_k \frac{g_{a_1,k,1}(w)}{\sqrt{k}} \geq g(a_1, w, 1) = h(a_2w)$$

for any $w \in [0,1]$. Moreover it can be shown by elementary calculations that for $a_1 = \frac{1}{2-\sqrt{2/\pi}}$ the function $h(w)$ attains its global maximum on $[0, a_2]$ at

$$w_{\mathrm{o}} = \frac{\sqrt{a_1^2 + 2a_2^2} - a_1}{2}.$$

LEMMA 3.4. *For any $a_1 \in [0,1)$ and $w \in (0,1]$ there exists $l \in (0,1)$ such that $g(a_1, w, l) > g(a_1, w, 1)$.*

PROOF. First notice that for any $l \in [0,1]$ and $k \in \mathbb{N}$,

$$2a_1a_2lc_k + 2a_1a_2(1-l) \geq 2a_1a_2c_k = 2a_1a_2\sqrt{1-w^2}.$$

To end our proof let us consider (for fixed $a_1 \in [0,1]$ and $w \in (0,1]$) the function

$$u(l) = 2a_2^2w(1-l)l + (a_2l)^2w\sqrt{1-w^2}.$$

Notice that

$$u(l) = 2a_2^2wl + l^2a_2^2w(\sqrt{1-w^2} - 2).$$

It is easy to see that $u'(l) = 0$ if and only if $l = l_w = \frac{1}{2-\sqrt{1-w^2}}$ and that u attains its global maximum on $[0,1]$ at $l_w \in (0,1)$. By (21) and the above reasoning for any $a_1 \in [0,1]$ and $w \in (0,1]$, $g(a_1, w, l_w) > g(a_1, w, 1)$, which shows our claim. □

Now we can state the main result of this note

THEOREM 3.1. *For each $k \in \mathbb{N}$ there exist $y^{1,k}, y^{2,k}, y^{3,k} \in \mathbb{R}^k$ such that the symmetric spaces V^k generated by $y^{1,k}, y^{2,k}, y^{3,k}$ satisfy*

$$\liminf_k \left(\lambda_k^S/\sqrt{k}\right) \geq \liminf_k \left(\lambda(V^k, l_1)/\sqrt{k}\right) \geq \max_{(a,w,l)\in[0,1]^3} g(a,w,l)$$
$$> \max_{w\in[0,a_2]} h(w) > \frac{1}{2-\sqrt{2/\pi}}.$$

PROOF. We apply Th. 2.1. Let $a_1 = \frac{1}{2-\sqrt{2/\pi}}$ and $a_2 = 1 - a_1$. Fix $k \in \mathbb{N}$, $N = 3$, $w_{\mathrm{o}} = \frac{\sqrt{(a_1/a_2)^2+2} - a_1/a_2}{2}$. Let $l_{\mathrm{o}} = \frac{1}{2-\sqrt{1-w_{\mathrm{o}}^2}}$. Let $y^{1,k}$, $y^{2,k}$ and $y^{3,k}$ be the vectors maximizing the function $f = f^k$ defined by (9). Let V^k be the symmetric space generated by $y^{1,k}, y^{2,k}$ and $y^{3,k}$. By Th. 2.1 and (4),

$$\lambda_k^S \geq \lambda(V^k, l_1) = \frac{f^k(y^{1,k}, y^{2,k}, y^{3,k})}{2^k(k-1)!} \geq \frac{f^k(x^{1,k,l_{\mathrm{o}}}, x^{2,k,l_{\mathrm{o}}}, x^{3,k,l_{\mathrm{o}}})}{2^k(k-1)!},$$

where $x^{1,k,l_{\mathrm{o}}}, x^{2,k,l_{\mathrm{o}}}$ and $x^{3,k,l_{\mathrm{o}}}$ are as in Lemma 3.3. By Lemma 3.2, Lemma 3.3 and Lemma 3.4 we get the result. □

Remark 3.2. Let $a_1 = \frac{1}{2-\sqrt{2/\pi}}$ and $a_2 = 1 - a_1$. Lemma 3.3 provides the lower estimate

$$\begin{aligned}\liminf_k \left(\lambda(V^k, l_1)/\sqrt{k}\right) &\geq g(a_1, w_o, l_o) = 0.83345\dots\\ &> h(a_2 w_o) = \max_{w\in[0,a_2]} h(w) = 0.83327\dots\\ &> h(0) = (2-\sqrt{2/\pi})^{-1} = 0.8319\dots,\end{aligned}$$

where $w_o = \frac{\sqrt{(a_1/a_2)^2+2} - a_1/a_2}{2}$ and $l_o = \frac{a_2 w_o + a_1(\sqrt{1-w_o^2}-1)}{a_2 w_o (2-\sqrt{1-w_o^2})}$.

At the end of this note we show how to maximize the function $g(a, w, l)$ numerically, which will improve the numerical estimate from Remark (3.2).

Lemma 3.5. *Let g be as in Lemma 3.3. Then*

$$\max\{g(a, w, l) : (a, w, l) \in [0, 1]^3\} = \max_{x\in[0,\pi/2]} l(x),$$

where

$$\begin{aligned}l(x) = \sqrt{2/\pi}a(x)^2 &+ \frac{4}{3}a(x)(1-a(x))\big(\cos(x)+\frac{1}{2}\big)\\ &+\frac{4}{9}(1-a(x))^2\sin(x)(\cos(x)+1)\end{aligned}$$

with

$$a(x) = \frac{\sin(x)^2 - \cos(x)^2 - \cos(x)}{\sin(x)^2 - \cos(x)^2 - 3\sin(x) - \cos(x)}.$$

Proof. Setting $w = \sin(x)$, we obtain

$$\begin{aligned}g(a, w, l) = a^2\sqrt{2/\pi} &+ 2a(1-a)l\cos(x) + 2(1-a)a(1-l)\\ &+ (1-a)^2 l^2 \sin(x)\cos(x) + 2(1-a)^2(1-l)l\sin(x),\end{aligned}$$

which needs to be maximized on $0 \leq l, a \leq 1$ and $0 \leq x \leq \frac{\pi}{2}$. Taking partial derivatives with respect to x and l and setting them to zero, after elementary but tedious calculations, one can obtain $l = \frac{2}{3}$. Hence to maximize g, it is enough to maximize

$$\begin{aligned}z(w, x) = a^2\sqrt{2/\pi} &+ \frac{4}{3}a(1-a)\cos(x) + \frac{2}{3}(1-a)a\\ &+ \frac{4}{9}(1-a)^2\sin(x)\cos(x) + \frac{4}{9}(1-a)^2\sin(x),\end{aligned}$$

on $0 \leq a \leq 1$ and $0 \leq x \leq \frac{\pi}{2}$. Taking the partial derivative of z with respect to x we easily get that

$$a = a(x) = \frac{\cos(2x) + \cos(x)}{\cos(2x) + \cos(x) - 3\sin(x)}.$$

Hence our problem reduces to maximizing the function $l(x)$ on $[0, \pi/2]$, which proves our lemma. $\square$

Remark 3.3. Maximazing numerically $l(x)$ on $[0, \pi/2]$ one can get

$$\max\{l(x) : x \in [0, \pi/2]\} = 0.8337894\ldots$$

Hence by Theorem 3.1

$$\liminf_k \left(\lambda_k^S/\sqrt{k}\right) \geq \max_{(a,w,l)\in[0,1]^3} g(a, w, l) = 0.833789\ldots$$

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