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APPROXIMATION THEOREMS FOR SZÁSZ-MIRAKJAN--DURRMEYER TYPE OPERATORS

TWIERDZENIA APROKSYMACYJNE DLA OPERATORÓW TYPU SZÁSZA-MIRAKJANA-DURRMEYERA

A b s t r a c t

In this paper we study an integral modification of Szász-Mirakjan type operators. The modification will be called Szász-Mirakjan-Durrmeyer type operators as in many papers examining this type of operators. We give direct approximation theorems for these operators using the modulus of continuity and the modulus of smoothness for functions belonging to exponential weighted spaces.

Keywords: *linear positive operators*, *Bessel function*, *modulus of continuity*, *degree of approximation*

S t r e s z c z e n i e

W artykule badamy całkową modyfikację operatorów typu Szásza-Mirakjana. Tę modyfikację będziemy nazywać operatorami typu Szász-Mirakjan-Durrmeyera, jak to się czyni w wielu pracach badających tego typu operatory. Podajemy twierdzenia aproksymacyjne wykorzystujące moduł ciągłości i moduł gładkości dla funkcji z wykładniczych przestrzeni wagowych.

Słowa kluczowe: *dodatni operator liniowy*, *funkcja Bessela*, *moduł ciągłości*, *rząd aproksymacji*

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1. Introduction

In paper [6] we investigated operators of the Szász-Mirakjan type defined as follows

$$L_n^{\nu}(f;x)=\begin{cases}\sum_{k=0}^{\infty} p_{n,k}^{\nu}(x) f\left(\frac{2k}{n+q}\right) & x>0;\\ f(0) & x=0\end{cases} \tag{1}$$

where the coefficients

$$p_{n,k}^{\nu}(x)=\frac{1}{I_{\nu}(nx)}\frac{x^{2k+\nu}}{2^{2k+\nu}\,\mathrm{k}!\,\Gamma(k+\nu+1)} \tag{2}$$

Γ is the gamma function and I_ν the modified Bessel function of the first kind defined by the formula ([15], p. 77)

$$I_{\nu}(z)=\sum_{k=0}^{\infty}\frac{z^{2k+\nu}}{2^{2k+\nu}\,\mathrm{k}!\,\Gamma(k+\nu+1)}$$

This means that we replaced the coefficients of well-known Szász-Mirakjan operators by some terms involving the modified Bessel function I_ν.

We studied the approximation properties of these operators in exponential weight spaces

$$E_q=\{f\in C(\mathbb{R}_0): w_q f \quad \text{is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where $C(\mathbb{R}_0)$ denotes the space of all real-valued function continuous on $\mathbb{R}_0=[0;\infty)$ and w_q is the exponential weight function defined as follows

$$w_q(x)=e^{-qx},\quad q\in\mathbb{R}_0 \tag{2}$$

for $x\in\mathbb{R}_0$.

In the spaces we introduced the weighted norm

$$\left\|f_q\right\|=\sup\left\{w_q(x)\left|f(x)\right|: x\in\mathbb{R}_0\right\} \tag{3}$$

and we established ([6], Theorem 2.1) that operators L_n^{ν} are linear, positive, bounded and transform the space E_q into E_q.

In this paper we introduce an integral modification of (1)

$$\tilde{L}_n^{\nu}(f;x)=\begin{cases}\sum_{k=0}^{\infty} p_{n,k}^{\nu}(x)\int_0^{\infty}\tilde{g}_{n,k}^{\nu}(t)dt, & x>0;\\ f(0), & x=0\end{cases} \tag{4}$$

where the coefficients $p_{n,k}^{\nu}$ are defined above and

$$\tilde{g}_{n,k}^{\nu}(t)=\frac{n+q}{\Gamma(2k+\nu+1)}e^{-(n+q)t}((n+q)t)^{2k+\nu}$$

The idea of integral modifications of this kind of operators comes from J.L. Durrmeyer ([2]) who introduced the integral modification of the genuine Bernstein operators. Later on new modifications of other classical operators appeared, for example, M.M. Derriennic ([3]), S.M. Mazhar and V. Totik ([11]), A. Sahai and G. Prasad ([13]), M. Heilmann ([5]). Now the operators are still under consideration [1, 4, 7–10, 12, 14].

The note was inspired by the above results which investigate approximation problems for integral operators and it is a natural continuation of the author's results from paper [7].

Among other things, in the paper we shall prove the theorems giving the degree of approximation of functions from E_q by operators $\tilde{L}_n^\nu$. We will estimate the error of approximation using the weighted modulus of continuity of the first and the second order defined as follows

$$\omega_1(f, E_q; t) = \sup\left\{\left\|\Delta_h f\right\|_q : h \in [0, t]\right\}, \quad t > 0 \tag{5}$$

and

$$\omega_2(f, E_q; t) = \sup\left\{\left\|\Delta_h^2 f\right\|_q : h \in [0, t]\right\}, \quad t > 0$$

respectively, where

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

for $x, h \in \mathbb{R}_0$.

It is worth mentioning that Bessel functions are the most important special functions which play a pivotal role in mathematical physics, for example: signal processing, heat conduction, diffusion problems. We hope that the operators examined will have applications to these areas of study.

Remark 1.1

In the paper we shall denote by $M(p, t)$ suitable positive constants depending on the parameters indicated p, t.

2. Auxiliary results

Let us denote

$$e_r(t) = t^r,\ f_r(t) = e_r(t)e^{qt},\ \phi_{x,r}(t) = (t-x)^r,\ \psi_{x,r}(t) = \phi_{x,r}(t)e^{qt}$$

for $r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $q, x \in \mathbb{R}_0$.

In this section we shall recall preliminary results which are immediately obtained from papers [6, 7] and definition (4).

Remark 2.1

For all $\nu \in \mathbb{R}_0$ and $n, r \in \mathbb{N}$ it holds

$$\tilde{L}_n^\nu(e_0; 0) = 1, \quad \tilde{L}_n^\nu(f_0; 0) = 1$$

$$\tilde{L}_n^\nu(e_r;0)=\tilde{L}_n^\nu(\phi_{0,r};0)=\tilde{L}_n^\nu(\psi_{0,r};0)=\tilde{L}_n^\nu(f_r;0)=0$$

Lemma 2.1 ([6], Lemma 2.1)

For each $\nu\in\mathbb{R}_0$ *there exists a positive constant* $M(\nu)$ *such that for all* $n\in\mathbb{N}$ *and* $x\in\mathbb{R}_0$ *we have*

$$\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right|\le M(\nu),\quad nx\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right|\le M(\nu)$$

By elementary calculations and Lemma 2.2. ([6]) we get

Lemma 2.2

For each $n\in\mathbb{N}$, $\nu,q\in\mathbb{R}_0$ and $x\in\mathbb{R}_0$

$$\tilde{L}_n^\nu(e_0;x)=L_n^\nu(e_0;x)=1,\quad \tilde{L}_n^\nu(e_1;x)=L_n^\nu(e_1;x)+\frac{\nu+1}{n+q}=\frac{n}{n+q}\left(\frac{xI_{\nu+1}(nx)}{I_\nu(nx)}+\frac{\nu+1}{n}\right),$$

$$\tilde{L}_n^\nu(e_2;x)=L_n^\nu(e_2;x)+\frac{2\nu+3}{n+q}L_n^\nu(e_1;x)+\frac{(\nu+1)(\nu+2)}{(n+q)^2}$$

$$=\left(\frac{n}{n+q}\right)^2\left(\frac{x^2I_{\nu+2}(nx)}{I_\nu(nx)}+\frac{(2\nu+5)}{n}\frac{xI_{\nu+1}(nx)}{I_\nu(nx)}+\frac{(\nu+1)(\nu+2)}{n^2}\right),$$

$$\tilde{L}_n^\nu(\phi_{x,1};x)=L_n^\nu(\phi_{x,1};x)+\frac{\nu+1}{n+q}=x\left(\frac{n}{n+q}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right)+\frac{\nu+1}{n+q},$$

$$\tilde{L}_n^\nu(\phi_{x,2};x)=L_n^\nu(\phi_{x,2};x)+\frac{2\nu+3}{n+q}L_n^\nu(\phi_{x,1};x)+\frac{x}{n+q}\frac{(\nu+1)(\nu+2)}{(n+q)^2}$$

$$=x^2\left(\left(\frac{n}{n+q}\right)^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)}-\frac{2n}{n+q}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}+1\right)$$

$$+\frac{2(\nu+1)x}{n+q}\left(\frac{n}{n+q}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right)+\frac{3nx}{(n+q)^2}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}$$

$$+\frac{(\nu+1)(\nu+2)}{(n+q)^2}.$$

By Lemmas 2.2 and 2.5 [7] we get

Lemma 2.3 ([7], Lemma 2.6)

For all $\nu,q\in\mathbb{R}_0$ *there exists a positive constant* $M(\nu,q)$ *such that for each* $n\in\mathbb{N}$ *we have*

$$\left\|\tilde{L}_n^\nu(f_0;\cdot)\right\|_q\le M(\nu,q).$$

An obvious consequence of the above lemma and definition (4) is

Theorem 2.1 ([7], Theorem 2.1)

For all $\nu, q \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu, q)$ *such that for each* $n \in \mathbb{N}$ *and* $f \in E_q$ *we have*

$$\left\| \tilde{L}_n^{\nu}(f;\cdot) \right\|_q \le M(\nu,q) \|f\|_q .$$

Note that in the case of the integral modification of our operators we also have the endomorphism E_q into E_q. This is a better result than the one in [8], Theorem 3.1.

Applying Lemma 2.1 and Lemma 2.2 we immediately obtain

Lemma 2.5 ([7], Lemma 3.1)

For all $\nu, q \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu, q)$ *such that for each* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_0$ *we have*

$$\left| \tilde{L}_n^{\nu}(\phi_{x,2};x) \right| \le M(\nu,q) \frac{x(x+1)}{n}.$$

Lemma 2.6 ([7], Lemma 3.3)

For all $\nu, q \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu, q)$ *such that for each* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_0$ *we have*

$$w_q(x) \left| \tilde{L}_n^{\nu}(\psi_{x,2};x) \right| \le M(\nu,q) \frac{x(x+1)}{n}.$$

3. Degree of approximation

The following theorems estimate a weighted error of approximation for functions belonging to the space $E_q^k = \{f \in E_q : f', f'', \ldots, f^k \in E_q\}$ for $k = 1, 2$.

The proofs of the theorems are analogous to the proofs which are known from the literature but we enclose them for the completeness of the paper.

Remark 3.1

Note that for $x = 0$ in the following lemmas and theorems we get the assertion using Remark 2.1.

Theorem 3.1

For all $\nu, q \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu, q)$ *such that for all* $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ *and* $f \in E_q^1$ *we have*

$$w_q(x) \left| \tilde{L}_n^{\nu}(f;x) - f(x) \right| \le M(\nu,q) \|f'\|_q \left(\frac{x(x+1)}{n} \right)^{1/2} .$$

Proof. Let $x > 0$. For $f \in E_q^1$ we have

$$f(t) - f(x) = \int_x^t f'(u)du$$

for $t > 0$. By Lemma 2.2 we have $\tilde{L}_n^{\nu}(e_0;x)=1,$ hence we can write

$$\tilde{L}_n^{\nu}(f;x)-f(x)=\tilde{L}_n^{\nu}\left(\int_x^t f'(u)du;x\right),$$

using the linearity of $\tilde{L}_n^{\nu}$.

Note that

$$\left|\int_x^t f'(u)du\right|\le \|f'\|_q \left|\int_x^t e^{qu}du\right|\le \|f'\|_q\,(e^{qt}+e^{qx})|t-x|.$$

Therefore, we have

$$w_q\left|\tilde{L}_n^{\nu}(f;x)-f(x)\right|\le w_q\,\|f'\|_q\,\tilde{L}_n^{\nu}\left(\left|\psi_{x,1}\right|;x\right)+\|f'\|_q\,\tilde{L}_n^{\nu}\left(\left|\phi_{x,1}\right|;x\right). \qquad (7)$$

If we apply the Cauchy-Schwarz inequality and Lemma 2.2 we get

$$\tilde{L}_n^{\nu}\left(\left|\phi_{x,1}\right|;x\right)\le\left(\tilde{L}_n^{\nu}\left(\left|\phi_{x,2}\right|;x\right)\right)^{1/2},$$

$$\tilde{L}_n^{\nu}\left(\left|\psi_{x,1}\right|;x\right)\le\left(\tilde{L}_n^{\nu}\left(\left|\psi_{x,2}\right|;x\right)\right)^{1/2}(\tilde{L}_n^{\nu}(f_0;x))^{1/2}.$$

Now we can use Lemma 2.3, 2.5 and 2.6 to estimate (7)

$$w_q\left|\tilde{L}_n^{\nu}(f;x)-f(x)\right|\le M(\nu,q)\|f'\|_q\left(\frac{x(x+1)}{n}\right)^{1/2}$$

for $x > 0$ and $n \in \mathbb{N}$.

Theorem 3.2

For all $\nu, q \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu, q)$ *such that for all* $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ *and* $f \in E_q$ *we have*

$$w_q(x)\left|\tilde{L}_n^{\nu}(f;x)-f(x)\right|\le M(\nu,q)\omega_1\left(f,E_q;\left(\frac{x(x+1)}{n}\right)^{1/2}\right).$$

Proof. Let $x > 0$. As always we denote by f_h the Steklov function of f, this means

$$f_h(x)=\frac{1}{h}\int_0^h f(x+t)dt$$

for $h > 0$. Note that

$$f_h(x)-f(x)=\frac{1}{h}\int_0^h f(x+t)-f(x)dt,$$

$$f_h'(x)=\frac{1}{h}(f(x+h)-f(x))$$

for $h > 0$. Therefore, we immediately conclude that $f_h, f_h' \in E_q$ because $f \in E_q$ and we have the following estimations

$$\|f_h - f\|_q \le \omega_1(f, E_q; h) \tag{8}$$

$$\|f_h'\|_q \le \frac{1}{h}\omega_1(f, E_q; h) \tag{9}$$

for $h > 0$. By the linearity of the operators $\tilde{L}_n^{\nu}$ we get the inequality

$$\begin{aligned} w_q(x)\left|\tilde{L}_n^{\nu}(f;x) - f(x)\right| &\le w_q(x)\left|\tilde{L}_n^{\nu}(f - f_h; x\right| + w_q(x)\left|\tilde{L}_n^{\nu}(f_h;x) - f_h(x)\right| \\ &+ w_q(x)\left|f_h(x) - f(x)\right| \end{aligned}$$

Taking into account the boundedness of the operators $\tilde{L}_n^{\nu}$ and (8) we obtain

$$w_q(x)\left|\tilde{L}_n^{\nu}(f - f_h; x\right| \le M(\nu, q)\|f_h - f\|_q \le M(\nu, q)\omega_1(f, E_q; h)$$

for $x, h > 0$. From Theorem 3.1 and (9) we have

$$\begin{aligned} w_q(x)\left|\tilde{L}_n^{\nu}(f_h;x) - f_h(x)\right| &\le M(\nu, q)\|f_h'\|_q \left(\frac{x(x+1)}{n}\right)^{1/2} \\ &\le M(\nu, q)\frac{1}{h}\omega_1(f, E_q; h)\left(\frac{x(x+1)}{n}\right)^{1/2} \end{aligned}$$

for $x, h > 0$.

By the definition of the norm $\|\cdot\|_q$ and (8) we get

$$w_q(x)\left|(f_h(x) - f(x)\right| \le \|f_h - f\|_q \le \omega_1(f, E_q; h)$$

for $x, h > 0$.

Using above inequalities we estimate the expression

$$w_q(x)\left|\tilde{L}_n^{\nu}(f;x) - f(x)\right| \le \omega_1(f, E_q; h)\left(M(\nu, q) + \frac{M(\nu, q)}{h}\left(\frac{x(x+1)}{n}\right)^{1/2} + 1\right).$$

Now substituting $h = \left(\frac{x(x+1)}{n}\right)^{1/2}$ we get the assertion of our theorem.

Theorem 3.2 implies the following corollary.

Corollary 3.3

If $\nu, q \in \mathbb{R}_0$ and $f \in E_q$ then for all $x \in \mathbb{R}_0$

$$\lim_{n\to\infty}\{\tilde{L}_n^{\nu}(f;x) - f(x)\} = 0.$$

Moreover, the above convergence is uniform on every compact subset of the interval $[0;\infty)$.

Remark 3.4

We can obtain the above convergence in a different way, see Theorem 3.1 ([7]).

To estimate the error of approximation by the second order modulus of smoothness (5) we define the following linear operators

$$\tilde{H}_n^{\nu}(f;x)=\tilde{L}_n^{\nu}(f;x)-f(\tilde{L}_n^{\nu}(e_1;x))+f(x) \qquad (10)$$

for $\nu, q\in\mathbb{R}_0, f\in E_q$ and $x\in\mathbb{R}_0$.

Note that the operators preserve linear functions, namely

$$\tilde{H}_n^{\nu}(\phi_{x,1};x)=0. \qquad (11)$$

Lemma 3.5

For all $\nu, q\in R_0$ *there exists a positive constant* $M(\nu,q)$ *such that for all* $n\in N$, $x\in R_0$ *and* $g\in E_q^2$ *we have*

$$w_q\left|\tilde{H}_n^{\nu}(g;x)-g(x)\right|\le M(\nu,q)\|g''\|_q\frac{x(x+1)}{n}.$$

Proof. Let $x>0$ be fixed. By the Taylor formula we can write

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t(t-u)g''(u)du$$

for $t>0$. Now applying linearity of $\tilde{H}_n^{\nu}$ and (11) we derive

$$\left|\tilde{H}_n^{\nu}(g;x)-g(x)\right|=\left|\tilde{H}_n^{\nu}(g(t)-g(x);x)\right|=\left|\tilde{H}_n^{\nu}\left(\int_x^t(t-u)g''(u)du;x\right)\right|. \qquad (12)$$

Further, the definition of $\tilde{H}_n^{\nu}$ implies

$$\tilde{H}_n^{\nu}\left(\int_x^t(t-u)g''(u)du;x\right)=\tilde{L}_n^{\nu}\left(\int_x^t(t-u)g''(u)du;x\right)$$
$$-\int_x^{\tilde{L}_n^{\nu}(t;x)}(\tilde{L}_n^{\nu}(t;x)-u)g''(u)du.$$

Estimating (12) we can write

$$\left|\tilde{H}_n^{\nu}(g;x)-g(x)\right|\le\tilde{L}_n^{\nu}\left(\left|\int_x^t(t-u)g''(u)du\right|;x\right)+\left|\int_x^{\tilde{L}_n^{\nu}}(\tilde{L}_n^{\nu}(t;x)-u)g''(u)du\right|.$$

Note that

$$\left|\int_x^t(t-u)g''(u)du\right|\le\frac{1}{2}\|g''\|_q(t-x)^2(e^{qx}+e^{qt})$$

and

$$\left|\int_x^{\tilde{L}_n^\nu(e_1;x)} (\tilde{L}_n^\nu(e_1;x)-u)g''(u)du\right| \le \frac{1}{2}\|g''\|_q (\tilde{L}_n^\nu(e_1;x)-x)^2 (e^{qx}+e^{q\tilde{L}_n^\nu(e_1;x)})$$

$$\le \frac{1}{2}\|g''\|_q (\tilde{L}_n^\nu(\phi_{x,1};x))^2 e^{qx}(1+e^{q\tilde{L}_n^\nu(\phi_{x,1};x)}).$$

Now we can observe that the expression $e^{q\tilde{L}_n^\nu(\phi_{x,1};x)}$ is bounded. We immediately obtain it from Lemma 2.2 and 2.1 as follows

$$e^{q\tilde{L}_n^\nu(\phi_{x,1};x)} = e^{qx\left(\frac{n}{n+q}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right)} e^{q\frac{\nu+1}{n+q}} \le e^{nx\left(\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right)} e^{q\frac{\nu+1}{1+q}} \le M(\nu).$$

Therefore, we have

$$w_q(x)\left|\tilde{H}_n^\nu(g;x)-g(x)\right|$$

$$\le \frac{1}{2}\|g''\|_q \tilde{L}_n^\nu(\phi_{x,2};x) + \frac{1}{2}\|g''\|_q w_q(x)\tilde{L}_n^\nu(\psi_{x,2};x)$$

$$+\frac{1}{2}M(\nu)\|g''\|_q (\tilde{L}_n^\nu(\phi_{x,1};x))^2.$$

Applying the Cauchy-Schwarz inequality to the term $\tilde{L}_n^\nu(\phi_{x,1};x)$ and Lemmas 2.5, 2.6 we get the desired estimation.

Theorem 3.6

For all $\nu, q \in R_0$ *there exists a positive constant* $M(\nu, q)$ *such that for all* $n \in N$, $x \in R_0$ *and* $f \in E_q$ *we have*

$$w_q\left|\tilde{L}_n^\nu(f;x)-f(x)\right| \le M(\nu,q)\omega_2\left(f, E_q; \left(\frac{x(x+1)}{n}\right)^{1/2}\right) + \omega_1\left(f, E_q; \left|\tilde{L}_n^\nu(\phi_{x,1};x)\right|\right).$$

Proof. Let $x > 0$ and $\overline{f}_h$ be the second order Steklov mean of $f \in E_q$, i.e.

$$\overline{f}_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} \{2f(x+s+t)-f(x+2(s+t))\}dsdt, \quad h,x>0$$

Note that

$$f(x)-\overline{f}_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} \Delta_{s+t}^2 f(x)dsdt.$$

By definition (6) we get the following estimation

$$\left\|f-\overline{f}_h\right\|_q \le \omega_2(f, E_q; h)$$

and since

$$\overline{f}_h''(x) = \frac{1}{h^2}(8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x))$$

we have

$$\left\|\bar{f}_h''\right\|_q \le \frac{9}{h^2}\omega_2(f,E_q;h).$$

The above inequalities imply that the Steklov mean $\bar{f}_h$ and $\bar{f}_h''$ belong to E_q.

Moreover, by the linearity of $\tilde{L}_n^\nu$, $\tilde{H}_n^\nu$ and the connection (10) we can write

$$\begin{aligned}&\left|\tilde{L}_n^\nu(f;x)-f(x)\right|\\ &\qquad\le\left|\tilde{H}_n^\nu(f-\bar{f}_h;x)\right|+\left|(f(x)-\bar{f}_h(x)\right|+\left|\tilde{H}_n^\nu(\bar{f}_h;x)-\bar{f}_h(x)\right|\\ &\qquad+\left|f(\tilde{L}_n^\nu(e_1;x)-f(x)\right|.\end{aligned}$$

By the above, the boundedness of the operators $\tilde{H}_n^\nu$ and Lemma 3.5 we conclude that

$$\begin{aligned}&w_q(x)\left|\tilde{L}_n^\nu(f;x)-f(x)\right|\\ &\qquad\le w_q(x)\left|\tilde{H}_n^\nu(f-\bar{f}_h;x)\right|+w_q(x)\left|f(x)-\bar{f}_h(x)\right|\\ &\qquad+w_q(x)\left|\tilde{H}_n^\nu(\bar{f}_h;x)-\bar{f}_h(x)\right|+w_q(x)\left|f(\tilde{L}_n^\nu(e_1;x)-f(x)\right|\\ &\qquad\le M(\nu,q)\left\|f-\bar{f}_h\right\|_q+\left\|f-\bar{f}_h\right\|_q+M(\nu,q)\left\|\bar{f}_h''\right\|_q\frac{x(x+1)}{n}\\ &\qquad+w_q(x)\left|f(\tilde{L}_n^\nu(\phi_{x,a};x)+x)-f(x)\right|\\ &\qquad\le M(\nu,q)\omega_2(f,E_q;h)\left(1+\frac{1}{h^2}\frac{x(x+1)}{n}\right)+\omega_1\left(f,E_q;\left|\tilde{L}_n^\nu(\phi_{x,1};x)\right|\right).\end{aligned}$$

where $\tilde{L}_n^\nu(\phi_{x,1};x)=x\left(\frac{n}{n+q}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right)+\frac{\nu+1}{n+q}$. Substituting $h=\left(\frac{x(x+1)}{n}\right)^{1/2}$ we get the estimation in the theses of Theorem 3.6.

The above theorem shows that one can estimate the weighted error of approximation for positive linear operators reproducing constant functions by the sum of two moduli of continuity.

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