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THE TOPOLOGY ON THE SPACE $\delta\mathcal{E}_\chi$

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Abstract. In this paper, we construct a locally convex topology on the vector space $\delta\mathcal{E}_\chi$. We also prove that with this topology it is a non-separable and non-reflexive Fréchet space.

1. Introduction. Let Ω be a hyperconvex domain in $\mathbb{C}^n$; by $\mathrm{PSH}^-(\Omega)$ we denote the set of negative plurisubharmonic (psh) functions on Ω. We denote by $\mathcal{H} = \mathcal{H}(\Omega)$ any subclass of the functions in $\mathrm{PSH}^-(\Omega)$. Set $\delta\mathcal{H} = \delta\mathcal{H}(\Omega) = \mathcal{H} - \mathcal{H}$, that is the set of the functions $u \in \mathrm{L}^1_{loc}(\Omega)$ which can be written as $u = v - w$, where $v, w \in \mathcal{H}$. If $\mathcal{H}$ is a convex cone in $\mathrm{PSH}^-(\Omega)$ then $\delta\mathcal{H}$ is a vector space. Let us recall the topology on the space $\delta\mathcal{H}$ when $\mathcal{H}$ is a special subclass of negative plurisubharmonic functions.

In [**7**], Cegrell has introduced and studied some energy classes, especially two classes $\mathcal{F}$ and $\mathcal{E}$. He shows that $\mathcal{E}$ is the largest subclass on which the Monge–Ampère operator is well defined and is continuous under decreasing sequences of negative plurisubharmonic functions (see Theorem 4.5 in [**7**]). In [**10**], Cegrell and Wiklund have introduced and investigated the vector space $\delta\mathcal{F}$ equipped with the Monge–Ampère norm. They have shown that the space $\delta\mathcal{F}$ is a non-separable Banach space and the topological dual space of $\delta\mathcal{F}$ can be written as $(\delta\mathcal{F})' = \mathcal{F}' - \mathcal{F}' = \delta\mathcal{F}'$. Since the function in $\delta\mathcal{E}$ belongs to $\delta\mathcal{F}$ on every relative compact K in Ω then the topology on $\delta\mathcal{E}$ can be induced by the family of semi-norms $\|.\|_K, K \Subset \Omega$, where $\|.\|$ is norm on $\delta\mathcal{F}$ (see [**15**]). Moreover, in [**15**], the authors have shown that the topological space $\delta\mathcal{E}$ is a non-separable and non-reflexive Fréchet space.

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In [**6**], Cegrell has introduced the class $\mathcal{E}_p$ (for $p > 0$) of all negative plurisubharmonic functions with well-defined and finite, pluricomplex p-energy. The space $\delta\mathcal{E}_p$ was studied by Åhag and Czyż in [**2**]. For each $u \in \delta\mathcal{E}_p$ they define

$$\|u\|_p = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_p}} \left(\int_\Omega \big(-(u_1+u_2)\big)^p \big(dd^c(u_1+u_2)\big)^n \right)^{\frac{1}{n+p}}.$$

This is a quasi-norm on $\delta\mathcal{E}_p$ and moreover $(\delta\mathcal{E}_p, \|.\|_p)$ is a quasi-Banach space for $p \neq 1$ and $(\delta\mathcal{E}_1, \|.\|_1)$ is a Banach space (see Theorem 4.7 in [**2**]).

The weighted energy class $\mathcal{E}_\chi$ that is a generalization of the class $\mathcal{E}_p$ has been introduced and studied by Benelkourchi, Guedj and Zeriahi in [**4**] and [**5**]. The aim of the present paper is to construct and investigate a locally convex topology on $\delta\mathcal{E}_\chi$ and to show that $\delta\mathcal{E}_\chi$ is a non-separable and non-reflexive Fréchet space with this topology.

2. Preliminaries. In this section, we recall some definitions and properties of psh function classes, as well as elements of pluripotential theory that will be used throughout this paper. They all may be found in [**1, 3–7, 12, 16, 21**].

2.1. Unless otherwise specified, Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$, and $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ is a decreasing function.

2.2. The following energy classes of psh functions were introduced and investigated by Cegrell in [**6**] and [**7**]:

$$\mathcal{E}_0 = \left\{ \varphi \in \mathrm{PSH}(\Omega) \cap \mathrm{L}^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < +\infty \right\}$$

$$\mathcal{F} = \left\{ \varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi, \sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty \right\}$$

$$\mathcal{E} = \Big\{ \varphi \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighbourhood } \omega \ni z_0,$$
$$\mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } \omega, \sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty \Big\}.$$

It is clear that $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{E}$, and by [**7**], they are convex cones and satisfy the max property (see Theorem 2.1).

2.3. The following weighted energy classes of psh functions were introduced and investigated in [**4, 5**] by Benelkourchi, Guedj and Zeriahi.

$$\mathcal{E}_\chi = \left\{ \varphi \in \mathrm{PSH}^-(\Omega) : \exists \;\; \mathcal{E}_0 \ni \varphi_j \searrow \varphi, \sup_{j \geqslant 1} \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty \right\}.$$

Note that from the proofs of Theorems 1.4 and 1.5 in [**1**], it follows that if $\varphi \in \mathcal{E}_\chi$ then $\lim\limits_{z\to\xi}\varphi(z) = 0$ for all $\xi \in \partial\Omega$. In [**14**], the authors have shown that if $\chi \not\equiv 0$ then $\mathcal{E}_\chi \subset \mathcal{E}$.

THEOREM 2.1. (see [**16**]). *Suppose the function χ is such that $\chi(2t) \leqslant c\chi(t)$ with some $c > 1$. Then $\mathcal{E}_\chi$ has the following properties:*

i. *$\mathcal{E}_\chi$ is a convex cone, i.e. if $\varphi, \psi \in \mathcal{E}_\chi$ and $a, b \geqslant 0$ then $a\varphi + b\psi \in \mathcal{E}_\chi$;*

ii. *$\mathcal{E}_\chi$ satisfies the max property, i.e. if $\varphi \in \mathcal{E}_\chi$ and $\psi \in PSH^-(\Omega)$ then $\max(\varphi, \psi) \in \mathcal{E}_\chi$.*

2.4. For every hyperconvex domain $D \Subset \Omega$ and $\varphi \in \mathrm{PSH}^-(\Omega)$ we set

$$h^{\varphi}_{D,\Omega} = \sup\{u \in \mathrm{PSH}^-(\Omega) : u \leqslant \varphi \ \text{ on } \ D\}.$$

Then the function $h^{\varphi}_{D,\Omega}$ is the largest negative plurisubharmonic function equal to φ on D. And by using the arguments similar to those in [**19**] we get

$$\operatorname{supp}(dd^c h^{\varphi}_{D,\Omega})^n \subset \overline{D} \Subset \Omega.$$

The following class was introduced and studied in [**16**] by Hai, Hiep and Quy

$$\mathcal{E}_{\chi,loc} = \{\varphi \in \mathrm{PSH}^-(\Omega) : h^{\varphi}_{D,\Omega} \in \mathcal{E}_\chi(\Omega), \ \forall\ D \Subset \Omega\}.$$

THEOREM 2.2. (see [**16**]). *The class $\mathcal{E}_{\chi,loc}$ has the following properties:*

i. *The class $\mathcal{E}_{\chi,loc}$ can be described as follows*

$$\begin{aligned}\mathcal{E}_{\chi,loc} = \{\varphi \in PSH^-(\Omega) : &\ \textit{for all hyperconvex domains } D \Subset \Omega,\\ &\exists\psi \in \mathcal{E}_\chi(\Omega) : \psi = \varphi \textit{ on } D\};\end{aligned}$$

ii. *The class $\mathcal{E}_{\chi,loc}$ has the local property. Here, a class $\mathcal{K}(\Omega) \subset PSH^-(\Omega)$ is said to have the local property if $\varphi \in \mathcal{K}(\Omega)$ implies $\varphi \in \mathcal{K}(D)$ for all hyperconvex domains $D \Subset \Omega$ and if $\varphi \in PSH^-(\Omega)$, $\varphi|_{\Omega_i} \in \mathcal{K}(\Omega_i)$, $\forall\ i \in I$ with $\Omega = \bigcup\limits_{i\in I}\Omega_i$, implies $\varphi \in \mathcal{K}(\Omega)$.*

2.5. Now we shall introduce the space $\delta\mathcal{E}_\chi$ and give some necessary elements that will be used to construct the topology on this space. We set

$$\begin{aligned}\delta\mathcal{E}_\chi &= \mathcal{E}_\chi - \mathcal{E}_\chi\\ &= \{u \in \mathrm{L}^1_{loc}(\Omega) : \exists v, w \in \mathcal{E}_\chi, u = v - w\}.\end{aligned}$$

If the function χ is such that $\chi(2t) \leqslant c\chi(t)$ with some $c > 1$ then the class $\mathcal{E}_\chi$ is a convex cone and so $\delta\mathcal{E}_\chi$ is a vector space.
For $u \in \mathcal{E}_\chi$ we set

$$e_\chi(u) = \int_\Omega \chi(u)(dd^c u)^n.$$

For each $m \in \mathbb{N}$ we set

$$U_m = \{u = v - w : v, w \in \mathcal{E}_\chi, e_\chi(v) < \frac{1}{m}, e_\chi(w) < \frac{1}{m}\}.$$

In the next section, we are going to study some properties of the family of subsets $U_m, m \in \mathbb{N}$. Then we will construct the topology on the vector space $\delta\mathcal{E}_\chi$ generated by that family.

2.6. We recall here the definition of capacity in the sense of Bedford and Taylor (see [**3**] for further information).

$$cap(E) = \sup\left\{\int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega), -1 \leqslant u \leqslant 0\right\},$$

for every Borel set E in Ω. It is proved in [**3**] that

$$cap(E) = \int_E (dd^c h^*_{E,\Omega})^n,$$

where $h^*_{E,\Omega}$ is the upper regularization of the relative extremal function $h_{E,\Omega}$ for E (relative to Ω), i.e.,

$$h_{E,\Omega} = \sup\{u \in \mathrm{PSH}^-(\Omega) : u \leqslant -1 \text{ on } E\}.$$

2.7. We recall the notion of the convergence in capacity. Let Ω be a domain in $\mathbb{C}^n$ and u_j $(j = 1, 2, 3, ...)$ and u be psh on Ω. We say that the sequence $\{u_j\}$ is convergent to u in capacity if for each $\epsilon > 0$ we have

$$\lim_{j\to+\infty} cap(\{z \in K : |u_j(z) - u(z)| > \epsilon\}) = 0, \text{ for all } K \Subset \Omega.$$

3. The topology on the space $\delta\mathcal{E}_\chi$. In this section, we will prove that the vector space $\delta\mathcal{E}_\chi$ is a locally convex topological space; moreover it is a Fréchet space. First, we need some lemmas.

Lemma 3.1. *The set U_m is a balanced subset in the vector space $\delta\mathcal{E}_\chi$, i.e. $\forall u \in U_m$ we have $au \in U_m, \forall |a| \leqslant 1$.*

Proof. Given $v \in \mathcal{E}_\chi$ and $0 \leqslant a \leqslant 1$. We have

$$\begin{aligned} e_\chi(av) &= \int_\Omega \chi(av)(dd^c(av))^n = a^n \int_\Omega \chi(av)(dd^c v)^n \\ &\leqslant \int_\Omega \chi(v)(dd^c v)^n = e_\chi(v). \end{aligned}$$

From this we infer that U_m is a balanced set. □

Lemma 3.2. *The set U_m is an absorbing subset in the vector space $\delta\mathcal{E}_\chi$, i.e. $\forall u \in \delta\mathcal{E}_\chi, \exists \epsilon > 0 : au \in U_m, \forall |a| < \epsilon$.*

PROOF. First, for any $t \in \mathbb{R}^-$ we have

$$\chi(2^n t) \leqslant c\chi(2^{n-1}t) \leqslant ... \leqslant c^n \chi(t).$$

For $a \geqslant 1$ we have

$$\begin{aligned}\chi(at) &= \chi(2^{log_2 a} t) \leqslant \chi(2^{[log_2 a]+1} t)\\ &\leqslant c^{[log_2 a]+1} \chi(t) \leqslant c^{log_2 a+1}\chi(t)\\ &= c.c^{log_2 a}\chi(t) = c.a^{log_2 c}\chi(t)\cdot\end{aligned}$$

Now, given $v \in \mathcal{E}_\chi$ and $a \geqslant 1$, from the result above we have

$$\begin{aligned}e_\chi(av) &= \int_\Omega \chi(av)(dd^c(av))^n = a^n \int_\Omega \chi(av)(dd^c v)^n\\ &\leqslant c.a^{log_2 c+n} \int_\Omega \chi(v)(dd^c v)^n = c.a^{log_2 c+n} e_\chi(v)\cdot\end{aligned}$$

From this result we imply that U_m is a absorbing set. $\square$

PROPOSITION 3.3. *In the class $\mathcal{E}_\chi$, the following estimates hold.*

i. *If $\varphi, \psi \in \mathcal{E}_\chi$ then*

$$e_\chi(\varphi + \psi) \leqslant 2^{2n} c^2 \left[e_\chi(\varphi) + e_\chi(\psi)\right]. \tag{3.1}$$

ii. *If $\varphi, \psi \in \mathcal{E}_\chi$ are such that $\varphi \geqslant \psi$ then*

$$e_\chi(\varphi) \leqslant 2^n c e_\chi(\psi). \tag{3.2}$$

PROOF. By the definition of the class $\mathcal{E}_\chi$ it is enough to prove the proposition when $\varphi, \psi \in \mathcal{E}_0$.
i. First, as in the proof of Proposition 3.4 in [**9**], we have

$$\int_{\{\varphi<-t\}} (dd^c\varphi)^n \leqslant t^n cap(\{\varphi < -t\}), \forall \varphi \in \mathcal{E}_0 \tag{3.3}$$

$$t^n cap(\{\varphi < -2t\}) \leqslant \int_{\{\varphi<-t\}} (dd^c\varphi)^n, \forall t > 1, \forall \varphi \in \mathcal{E}_0. \tag{3.4}$$

We have

$$\begin{aligned}e_\chi(\varphi) &= \int_\Omega \chi(\varphi)(dd^c\varphi)^n\\ &= \int_0^{+\infty} -\chi'(-t) \int_{\{\varphi<-t\}} (dd^c\varphi)^n dt + \chi(0) \int_\Omega (dd^c\varphi)^n.\end{aligned}$$

We set

$$e_0(\varphi) = \int_\Omega (dd^c\varphi)^n.$$

By Lemma 2.5 in [**10**], we have

$$e_0(\varphi+\psi)^{\frac{1}{n}} \leqslant e_0(\varphi)^{\frac{1}{n}} + e_0(\psi)^{\frac{1}{n}}.$$

We set

$$\tilde{e}_\chi(\varphi) = \int_0^{+\infty} -\chi'(-t)t^n cap(\{\varphi < -t\})dt.$$

Applying formula (3.3) we infer

$$e_\chi(\varphi) \leqslant \tilde{e}_\chi(\varphi) + \chi(0)e_0(\varphi), \forall \varphi \in \mathcal{E}_0. \tag{3.5}$$

Applying formula (3.4) we have

$$\begin{aligned}
\tilde{e}_\chi(\varphi) &\leqslant \int_0^{+\infty} -\chi'(-t)2^n \int_{\{\varphi<-\frac{t}{2}\}} (dd^c\varphi)^n dt \\
&= 2^n \int_0^{+\infty} -\chi'(-t) \int_{\{\varphi<-\frac{t}{2}\}} (dd^c\varphi)^n dt \ \text{ (set } \hat{\chi}(t) = \chi(2t)) \\
&= 2^{n-1} \int_0^{+\infty} -\hat{\chi}'(-\frac{t}{2}) \int_{\{\varphi<-\frac{t}{2}\}} (dd^c\varphi)^n dt \\
&= 2^n \int_0^{+\infty} -\hat{\chi}'(-t) \int_{\{\varphi<-t\}} (dd^c\varphi)^n dt.
\end{aligned}$$

So we imply

$$\begin{aligned}
\tilde{e}_\chi(\varphi) + \chi(0)e_0(\varphi) &\leqslant 2^n \int_0^{+\infty} -\hat{\chi}'(-t) \int_{\{\varphi<-t\}} (dd^c\varphi)^n dt + \hat{\chi}(0)e_0(\varphi) \\
&\leqslant 2^n e_{\hat{\chi}}(\varphi) = 2^n \int_\Omega \chi(2\varphi)(dd^c\varphi)^n \\
&\leqslant 2^n c \int_\Omega \chi(\varphi)(dd^c\varphi)^n = 2^n c e_\chi(\varphi).
\end{aligned}$$

Therefore, we have

$$\tilde{e}_\chi(\varphi) + \chi(0)e_0(\varphi) \leqslant 2^n c e_\chi(\varphi). \tag{3.6}$$

For every $a \in [0,1]$ we have

$$\begin{aligned}
\tilde{e}_\chi((1-a)\varphi + a\psi) &= \int_0^{+\infty} -\chi'(-t)t^n cap(\{(1-a)\varphi + a\psi < -t\})dt \\
&\leqslant \int_0^{+\infty} -\chi'(-t)t^n cap(\{\varphi < -t\} \cup \{\psi < -t\})dt \\
&\leqslant \int_0^{+\infty} -\chi'(-t)t^n [cap(\{\varphi < -t\}) + cap(\{\psi < -t\})]dt \\
&= \tilde{e}_\chi(\varphi) + \tilde{e}_\chi(\psi).
\end{aligned}$$

The following inequalities are straightforward

$$e_0(\varphi+\psi) \leqslant \big(e_0(\varphi)^{\frac{1}{n}} + e_0(\psi)^{\frac{1}{n}}\big)^n \leqslant 2^{n-1}(e_0(\varphi)+e_0(\psi)),$$
$$e_0\left(\frac{\varphi+\psi}{2}\right) \leqslant \frac{1}{2}[e_0(\varphi)+e_0(\psi)].$$

Using the results above we obtain the following estimates

$$\begin{aligned}
e_\chi(\varphi+\psi) &= e_\chi\left(2\cdot\frac{\varphi+\psi}{2}\right) \leqslant 2^n c e_\chi\left(\frac{\varphi+\psi}{2}\right)\\
&\leqslant 2^n c\left[\tilde{e}_\chi(\frac{\varphi+\psi}{2}) + \chi(0)e_0(\frac{\varphi+\psi}{2})\right]\\
&\leqslant 2^n c\left[\tilde{e}_\chi(\varphi)+\tilde{e}_\chi(\psi)+\frac{\chi(0)}{2}(e_0(\varphi)+e_0(\psi))\right]\\
&\leqslant 2^n c\left[\tilde{e}_\chi(\varphi)+\chi(0)e_0(\varphi)+\tilde{e}_\chi(\psi)+\chi(0)e_0(\psi)\right]\\
&\leqslant 2^n c\left[2^n c e_\chi(\varphi)+2^n c e_\chi(\psi)\right]\\
&= 2^{2n}c^2[e_\chi(\varphi)+e_\chi(\psi)].
\end{aligned}$$

So **i.** is proved.
ii. It is a consequence of (3.5) and (3.6). □

Remark 3.4. It follows from Proposition 3.3 that for every U_m $(m \geqslant 1)$ we can find k (by (3.1) we can choose $k = m([2^{2n}c^2]+1)$) such that the convex hull of the set U_k is contained in U_m.

Theorem 3.5. *The vector space $\delta\mathcal{E}_\chi$ is a Fréchet space.*

Proof. It follows from Lemma 3.1, Lemma 3.2 and Remark 3.4 that the family $\mathcal{A}$ of convex hulls of sets $U_m, m \geqslant 1$ is a family of absorbing, balanced, convex sets in the vector space $\delta\mathcal{E}_\chi$. So there is a locally convex topology on this space such that the family $\mathcal{A}$ becomes a neighbourhood basis of origin. It remains to show completeness.
Suppose $\{u_m\}$ is a Cauchy sequence in the space $\delta\mathcal{E}_\chi$. Then for every $m \geqslant 1$ we can find j_m such that $u_j - u_k \in U_{(2^{2n+1}c^2)^m}, \forall j,k \geqslant j_m$. We can choose the sequence $\{j_m\}$ such that $j_{m+1} > j_m, \forall m > 0$. We have

$$u_{j_m} = u_{j_1} + (u_{j_2} - u_{j_1}) + ... + (u_{j_m} - u_{j_{m-1}}).$$

Since $u_{j_k} \in \delta\mathcal{E}_\chi, \forall k = 1,..,m$ we can write $u_{j_k} - u_{j_{k-1}} = v_k - w_k$, where $v_k, w_k \in \mathcal{E}_\chi$ such that $e_\chi(v_k) < \frac{1}{(2^{2n+1}c^2)^{k-1}}, e_\chi(w_k) < \frac{1}{(2^{2n+1}c^2)^{k-1}}$.
So we have

$$\begin{aligned}
u_{j_m} &= u_{j_1} + (v_2 - w_2) + ... + (v_m - w_m)\\
&= u_{j_1} + (v_2 + ... + v_m) - (w_2 + ... + w_m).
\end{aligned}$$

By applying (3.1) repeatedly we arrive that

$$e_\chi\bigg(\sum_{k=2}^m v_k\bigg) \leqslant 2^{2n}c^2 e_\chi(v_2) + (2^{2n}c^2)^2 e_\chi(v_3) + ... + (2^{2n}c^2)^{m-1} e_\chi(v_m)$$
$$\leqslant \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{m-1}} < +\infty.$$

So the sequence $\{\sum_{k=2}^m v_k\}_{m>0}$ is a decreasing sequence with bounded total χ-energy, and in the same way the sequence $\{\sum_{k=2}^m w_k\}_{m>0}$ is. These infer that the subsequence $\{u_{j_m}\}$ is convergent in $\delta\mathcal{E}_\chi$ and therefore $\{u_m\}$ is a convergent Cauchy sequence. □

4. On the convergence in the space $\delta\mathcal{E}_\chi$. We will show a generalization of the Theorem 4.1 in [**15**] and Theorem 3.2 in [**11**].

THEOREM 4.1. *Let a sequence $\{u_j\}_{j\geqslant 1} \subset \delta\mathcal{E}_\chi$. If the sequence $\{u_j\}$ converges to a function u in $\delta\mathcal{E}_\chi$ as j tends to $+\infty$ then $\{u_j\}$ converges to u in capacity.*

PROOF. Without loss of the generality we can assume that $u = 0$. By assumptions we have

$$\forall U_m, \exists j_0 \geqslant 1: \ u_j \in U_m, \forall j \geqslant i_0.$$

We write $u_j = v_j - w_j$, where $v_j, w_j \in \mathcal{E}_\chi$ are such that $e_\chi(v_j) \to 0$ and $e_\chi(w_j) \to 0$ as $j \to +\infty$. We set

$$\chi_n(t) = \int_t^0 dt_1 \int_{t_1}^0 ... \int_{t_{n-1}}^0 \chi(t_n)dt_n.$$

Given $\epsilon > 0$ and $K \Subset \Omega$. For any $\psi \in \mathrm{PSH}(\Omega), -1 \leqslant \psi \leqslant 0$, by Theorem 4.4 in [**14**], we have

$$\int_{\{|v_j|>\epsilon\}\cap K} (dd^c\psi)^n \leqslant \frac{1}{\chi_n(-\epsilon)} \int_\Omega \chi_n(v_j)(dd^c\psi)^n \leqslant \frac{e_\chi(v_j)}{\chi_n(-\epsilon)}.$$

Therefore we get

$$cap(\{|v_j| > \epsilon\} \cap K) \leqslant \frac{e_\chi(v_j)}{\chi_n(-\epsilon)} \to 0,$$

as $j \to +\infty$. And similarly

$$cap(\{|w_j| > \epsilon\} \cap K) \leqslant \frac{e_\chi(w_j)}{\chi_n(-\epsilon)} \to 0,$$

as $j \to +\infty$. Hence

$$cap(\{|u_j| > \epsilon\} \cap K) \leqslant cap(\{|v_j| > \frac{\epsilon}{2}\} \cap K) + cap(\{|w_j| > \frac{\epsilon}{2}\} \cap K) \to 0,$$

as $j \to +\infty$ and the proof is complete. □

5. The Monge–Ampère operator on the space $\delta\mathcal{E}_\chi$. In [**10**], the authors have extended the Monge–Ampère operator from the class $\mathcal{F}$ to $\delta\mathcal{F}$. Next, in [**15**], the Monge–Ampère operator has been extended from the class $\mathcal{E}$ to $\delta\mathcal{E}$. Here, we note that if $\chi \not\equiv 0$ then $\mathcal{E}_\chi \subset \mathcal{E}$ (see [**14**] for further information) so $\delta\mathcal{E}_\chi \subset \delta\mathcal{E}$. Therefore, the Monge–Ampère operator is well defined on the space $\delta\mathcal{E}_\chi$, but for the convenience we will recall here the extension in [**15**]. First, we need the following lemma (see Lemma 5.1 in [**15**]).

LEMMA 5.1. *Let $u_1^j, u_2^j, v_1^j, v_2^j \in \mathcal{E}$ be such that $u_1^j - u_2^j = v_1^j - v_2^j, 1 \leqslant j \leqslant n$. Then*

$$(dd^c u_1^1 - dd^c u_2^1) \wedge ... \wedge (dd^c u_1^n - dd^c u_2^n) = (dd^c v_1^1 - dd^c v_2^1) \wedge ... \wedge (dd^c v_1^n - dd^c v_2^n)\cdot$$

It follows from the Lemma above that we can extend the Monge–Ampère operator $(dd^c.)^n$ from the class $\mathcal{E}$ to $\delta\mathcal{E}$ as follows. Let $u \in \delta\mathcal{E}$ and $K \Subset \Omega$. Then there exist $u_1, u_2 \in \mathcal{F}$ such that $u = u_1 - u_2$ on K. We set

$$(dd^c u)^n \mid_K = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (dd^c u_2)^i \wedge (dd^c u_1)^{n-i} \mid_K \cdot$$

The following lemma gives us a relation between the convergence of the functions on the space $\delta\mathcal{E}_\chi$ and the convergence of the operator $(dd^c\cdot)^n$ on this space.

THEOREM 5.2. *Assume that the sequence $\{u_j\} \subset \delta\mathcal{E}_\chi$ is such that $u_j \to u \in \delta\mathcal{E}_\chi$ in the topology of the space $\delta\mathcal{E}_\chi$ as $j \to +\infty$. Then $\|(dd^c u_j)^n - (dd^c u)^n\|(D) \to 0$ as $j \to +\infty, \forall D \Subset \Omega$.*

First, we need the following lemma

LEMMA 5.3. *Let $\{u_j\} \subset \mathcal{E}_\chi$ be such that $e_\chi(u_j) \to 0$ then $(dd^c u_j)^n \to 0$ weakly, as $j \to +\infty$.*

PROOF. Without loss of the generality, we can assume that $u_j \in \mathcal{E}_0$. By Kolodziej's theorem ([**20**]) there exists $v_j \in \mathcal{E}_0$ such that

$$(dd^c v_j)^n = 1_{\{u_j \leqslant -1\}} (dd^c u_j)^n,$$

where 1_E is the characteristic function of E. We have

$$(dd^c[\max(u_j, -1) + v_j])^n \geqslant (dd^c \max(u_j, -1))^n + (dd^c v_j)^n \geqslant (dd^c u_j)^n.$$

By the comparison principle, we get $u_j \geqslant \max(u_j, -1) + v_j$. This implies that $h_{D,\Omega}^{u_j} \geqslant h_{D,\Omega}^{\max(u_j,-1)} + v_j$ for all $D \Subset \Omega$. By Corollary 5.6 in [**7**] we have

$$\begin{aligned}\int\limits_D (dd^c u_j)^n &\leqslant \int\limits_\Omega (dd^c h_{D,\Omega}^{u_j})^n \\ &\leqslant \int\limits_\Omega \left(dd^c \left(h_{D,\Omega}^{\max(u_j,-1)} + v_j\right)\right)^n \\ &= \sum_{k=0}^n \binom{k}{n} \int\limits_\Omega \left(dd^c h_{D,\Omega}^{\max(u_j,-1)}\right)^k \wedge (dd^c v_j)^{n-k} \\ &\leqslant \sum_{k=0}^n \binom{k}{n} \left[\int\limits_\Omega \left(dd^c h_{D,\Omega}^{\max(u_j,-1)}\right)^n\right]^{\frac{k}{n}} \left[\int\limits_\Omega (dd^c v_j)^n\right]^{\frac{n-k}{n}}.\end{aligned}$$

We have

$$\int\limits_\Omega (dd^c v_j)^n = \int\limits_{\{u_j \leqslant -1\}} (dd^c u_j)^n \leqslant \frac{1}{\chi(-1)} \int\limits_{\{u_j \leqslant -1\}} \chi(u_j)(dd^c u_j)^n \leqslant \frac{e_\chi(u_j)}{\chi(-1)} \to 0,$$

as $j \to \infty$. By Theorem 4.1 and Main theorem in [**8**] we get

$$\int\limits_\Omega \left(dd^c h_{D,\Omega}^{\max(u_j,-1)}\right)^n = \int\limits_{\bar{D}} \left(dd^c h_{D,\Omega}^{\max(u_j,-1)}\right)^n \to 0,$$

as $j \to \infty$. Combining these and the inequalities above we get $\int\limits_D (dd^c u_j)^n \to 0$, as $j \to \infty$, for all $D \Subset \Omega$. $\square$

Proof of Theorem 5.2. Since $u_j - u \to 0$ as $j \to +\infty$ in $\delta\mathcal{E}_\chi$ then there exist $v_j, w_j, v, w \in \mathcal{E}_\chi$ such that $u_j - u = v_j - w_j$ and $e_\chi(v_j) \to 0, e_\chi(w_j) \to 0$ as $j \to +\infty$ and $u = v - w$. We have

$$\|(dd^c u_j)^n - (dd^c u)^n\|(D) \leqslant C_n \sum_{k=0}^{n-1} \int\limits_D (dd^c(v+w))^k \wedge (dd^c(v_j + w_j))^{n-k},$$

where C_n is a constant. It is enough to prove that

$$\int\limits_D (dd^c\varphi)^k \wedge (dd^c\varphi_j)^{n-k} \to 0 \text{ as } j \to +\infty,\ \forall k = 0, ..., n-1,$$

where $\varphi = v + w$ and $\varphi_j = v_j + w_j$. Indeed, let $D \Subset K \Subset \Omega$. By Corollary 5.6 in [**7**], Proposition 3.3 and Lemma 5.3, we have

$$\begin{aligned}\int\limits_D (dd^c\varphi)^k \wedge (dd^c\varphi_j)^{n-k} &\leqslant \int\limits_\Omega (dd^c h^{\varphi}_{K,\Omega})^k \wedge (dd^c h^{\varphi_j}_{K,\Omega})^{n-k} \\ &\leqslant \left[\int\limits_\Omega (dd^c h^{\varphi}_{K,\Omega})^n\right]^{\frac{k}{n}} \left[\int_\Omega (dd^c h^{\varphi_j}_{K,\Omega})^n\right]^{\frac{n-k}{n}} \\ &\to 0 \text{ as } j \to +\infty.\end{aligned}$$

□

Theorem 5.4. *The Fréchet space $\delta\mathcal{E}_\chi$ is not separable and not reflexive.*

Proof. The idea of proof is taken from [**15**].
Let $z_0 \in \Omega$, then there exists a number $r_0 > 0$ such that

$$\mathbb{B}(z_0, 3r_0) = \{z \in \mathbb{C}^n : \|z - z_0\| < 3r_0\} \subset \Omega.$$

For each $r \in (0, r_0)$, we denote by h_r the relative extremal function of the ball $\mathbb{B}(z_0, r)$ and Ω, i.e. h_r is defined by

$$h_r = h_{\mathbb{B}(z_0,r),\Omega} = \sup\{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi \leqslant -1 \text{ on } \mathbb{B}(z_0, r)\}.$$

Since $h_r \in \mathcal{E}_0$ then $h_r \in \delta\mathcal{E}_\chi$ and we also have

$$\operatorname{supp}(dd^c h_r)^n \subset \mathbb{S}(z_0, r) = \{z \in \mathbb{C}^n : \|z - z_0\| = r\}.$$

First, we prove that $\delta\mathcal{E}_\chi$ is not separable. Suppose that the space $\delta\mathcal{E}_\chi$ is separable, i.e. there exists a sequence $\{u_j\}_{j=1,2,...}$ that is dense in $\delta\mathcal{E}_\chi$. Note that the set

$$A_j = \{r \in (0, r_0) : (dd^c u_j)^n(\mathbb{S}(z_0, r)) \neq 0\}$$

is countable and so $\cup_{j=1}^\infty A_j$ is, and therefore there exists $t \in (0, r_0) \setminus \cup_{j=1}^\infty A_j$. This means that $(dd^c u_j)^n(\mathbb{S}(z_0, t)) = 0$, for each $j = 1, 2, ...$ On the other hand, since the sequence $\{u_j\}$ is dense in $\delta\mathcal{E}_\chi$ and $h_t \in \mathcal{E}_\chi$, then there exists a subsequence $\{u_{k_j}\}_{j=1,2,...}$ such that $u_{k_j} \to h_t$ as $j \to +\infty$. By Theorem 5.2 we have

$$\|(dd^c u_{k_j})^n - (dd^c h_t)^n\|(\mathbb{S}(z_0, t)) \to 0 \text{ as } j \to +\infty.$$

From this we imply

$$|(dd^c u_{k_j})^n(\mathbb{S}(z_0, t)) - (dd^c h_t)^n(\mathbb{S}(z_0, t))| \to 0 \text{ as } j \to +\infty.$$

Since $(dd^c u_j)^n(\mathbb{S}(z_0, t)) = 0$, for each $j = 1, 2, ...$, then $(dd^c h_t)^n(\mathbb{S}(z_0, t)) = 0$. Moreover, since $\operatorname{supp}(dd^c h_t)^n \subset \mathbb{S}(z_0, t)$, we obtain $(dd^c h_t)^n = 0$. By the comparison theorem we have $h_t = 0$. This is a contradiction.

Now, we prove that $\delta\mathcal{E}_\chi$ is not reflexive. Assume that the space $\delta\mathcal{E}_\chi$ is reflexive. We have

$$\begin{aligned} e_\chi(h_r) &= \int\limits_\Omega \chi(h_r)(dd^c h_r)^n \\ &\leqslant \chi(-1)\int\limits_\Omega (dd^c h_r)^n \\ &= \chi(-1)cap(\mathbb{B}(z_0, r)) \\ &\leqslant \chi(-1)cap(\mathbb{B}(z_0, 2r_0)), \end{aligned}$$

for all $r \in (0, 2r_0)$. This implies that the set $\{h_{r_0(1+\frac{1}{j})}\}_{j\geqslant 1}$ is bounded in $\delta\mathcal{E}_\chi$. Then there exists a subsequence $\left\{u_j = h_{r_0(1+\frac{1}{m_j})}\right\}_{j\geqslant 1} \subset \left\{h_{r_0(1+\frac{1}{j})}\right\}_{j\geqslant 1}$ which is weakly convergent to u. Since a convex set in a locally convex space has the same closure in the original and weak topologies then we can find a sequence of convex combinations $\left\{v_k = \sum\limits_{l=1}^{s_k} a_{kl}u_{l+k}\right\}_{k\geqslant 1}$ (where $a_{kl} \geqslant 0$ and $\sum\limits_{l=1}^{s_k} a_{kl} = 1$) such that $v_k \to u$ in $\delta\mathcal{E}_\chi$. By Theorem 4.1, we get $v_k \to u$ in capacity. Moreover, since $v_k \to h_{r_0}$ pointwise, we get $u = h_{r_0}$. By Theorem 5.2, we get

$$\|(dd^c v_k)^n - (dd^c h_{r_0})^n\|(S(z_0, r_0)) \to 0.$$

On the other hand, we have $v_k = -1$ on $\mathbb{B}(z_0, r_0(1 + \frac{1}{m_{k+s_k}}))$. Therefore $(dd^c h_{r_0})^n(\mathbb{S}(z_0, r_0)) = 0$. Moreover, since $\text{supp}(dd^c h_{r_0})^n \subset \mathbb{S}(z_0, r_0)$, we infer that $(dd^c h_{r_0})^n = 0$. By the comparison theorem we have $h_{r_0} = 0$. This is a contradiction. $\square$

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