**TECHNICAL TRANSACTIONS | CZASOPISMO TECHNICZNE**
MECHANICS | MECHANIKA
4-M/2016

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# VARIATIONAL ITERATION TECHNIQUE FOR SOLVING HIGHER ORDER BOUNDARY VALUE PROBLEM WITH ADDITIONAL BOUNDARY CONDITION

# ITERACYJNA TECHNIKA WARIACYJNA ZASTOSOWANA DO ZAGADNIENIA BRZEGOWEGO WYŻSZEGO RZĘDU Z DODATKOWYM WARUNKIEM BRZEGOWYM

A b s t r a c t

This paper treats a variational iteration technique, which is based on variational iteration method, for solving linear and non – linear two – point boundary value problems in the form of a fourth – order differential equation and five boundary conditions. The solution of this problem is possible only when the considered equation includes an unknown parameter. The presented method has been illustrated with a numerical example.

*Keywords: boundary value problem, variational iteration technique, approximate solution, system of integral equations*

S t r e s z c z e n i e

W artykule przedstawiono iteracyjną technikę wariacyjną opartą na iteracyjnej metodzie wariacyjnej, zastosowaną do rozwiązywania zarówno liniowego, jak i nieliniowego dwupunktowego zagadnienia brzegowego składającego się z równania różniczkowego czwartego rzędu oraz pięciu warunków brzegowych. Rozwiązanie tak postawionego problemu jest możliwe tylko wtedy, gdy rozpatrywane równanie zawiera nieznany parametr. Prezentowaną metodę zilustrowano przykładem obliczeniowym.

*Słowa kluczowe: zagadnienie brzegowe, iteracyjna technika wariacyjna, rozwiązanie przybliżone, układ równań całkowych*

**DOI: 10.4467/2353737XCT.16.226.5975**

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## 1. Introduction

Boundary value problems (BVPs) play an important role in many fields, e.g. in mathematical modeling of viscoelastic and inelastic flows, physical and engineering sciences. There are different methods of solving these problems, e.g. the spectral Galerkin method [1], Adomian decomposition method [2], shooting method [3] or method with the use of B-spline functions [4]. In [5], the homotopy perturbation method was described, which is very efficient in finding analytical solutions. The variational iteration method (VIM) was developed by He [6], but it should be mentioned that this method was first considered by Inokuti and his colleagues [7]. It has proved to solve effectively and accurately a large class of differential equations with approximate solutions, which converge rapidly to accurate solutions. In [8], some useful iteration formulas about VIM were summarised. This method requires the determination of a convergent series by means of a correct functional. Describing the variational iteration technique (VIT), which is based on VIM [9], the authors [10] showed that higher-order BVPs are equivalent to the system of integral equations by using a suitable transformation. Next, this system can be solved efficiently by means of the VIM. This technique may be considered as an alternative method for solving BVPs.

In this paper the VIT [10] will be considered, but it will be modified and applied for solving non-standard fourth-order non-linear BVPs, where the number of boundary conditions exceeds the order of the differential equation. An example is given to illustrate this method.

## 2. Basic Concept of Variational Iteration Method

In order to present the VIM, we consider the following general differential equation:

$$Lu(x) + Nu(x) = g(x) \tag{1}$$

where $L$ denotes a linear operator, $N$ is a non-linear operator, $u(x)$ is a sought function and $g(x)$ is a given continuous function. We can construct a correct functional of the form [6, 7]:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(t) + N\tilde{u}_n(t) - g(t))\,dt \tag{2}$$

where $\lambda$ denotes a general Lagrange multiplier, which can be identified optimally by the variational theory, the subscripts $n$ denotes the $n$-th approximation, $\tilde{u}_n$ is a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

The successive approximations $u(x)$, $n \geq 0$ of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x)$, which is an initial approximation with possible unknowns. In [8] authors wrote, that the main feature of this method is that the initial solution can be chosen with some unknown parameters in the form of the searched solution. Consequently, the solution may be obtained by using:

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{3}$$

By applying the imposed boundary conditions, we obtain values of unknown parameters, which are included in a function $u_0(x)$, subsequently in $u_n(x)$ and consequently a solution in terms of convergent series.

### 3. Variational Iteration Technique – BVP with additional boundary condition

We consider the non-standard BVP, which consists of fourth–order differential equation:

$$u^{(\mathrm{IV})}(x) + f(x,u,u',u'',u''',p_1) = 0 \tag{4}$$

and five boundary conditions:

$$u(a) = u_a, \quad u'(a) = u_{1a}, \quad u''(a) = u_{2a}, \quad u(b) = u_b, \quad u'(b) = u_{1b} \tag{5}$$

A solution to equation (4) can fulfill five boundary conditions only when this equation contains one unknown parameter $p_1$. By means of the following transformation:

$$u_1 = u, \quad u_2 = u', \quad u_3 = u'', \quad u_4 = u''' \tag{6}$$

we can convert the BVP (4) and (5) to an initial value problem (IVP), which consists of a system of four first-order differential equations:

$$\frac{du_1}{dx} = u_2(x), \quad \frac{du_2}{dx} = u_3(x), \quad \frac{du_3}{dx} = u_4(x), \quad \frac{du_4}{dx} = f(x,u_1,u_2,u_3,p_1) \tag{7}$$

and initial conditions, that include a subsequent unknown parameter $p_2$:

$$u_1(0) = u_a, \quad u_2(0) = u_{1a}, \quad u_3(0) = u_{2a}, \quad u_4(0) = p_2 \tag{8}$$

According to the correct functional (2), we can rewrite system (7) as a system of four integral equations:

$$\begin{cases} u_1^{(k+1)}(x) = u_1^{(k)}(x) + \lambda \int_0^x u_2^{(k)}(t)\,dt \\ u_2^{(k+1)}(x) = u_2^{(k)}(x) + \lambda \int_0^x u_3^{(k)}(t)\,dt \\ u_4^{(k+1)}(x) = u_4^{(k)}(x) + \lambda \int_0^x f(t,u_1^{(k)}(t),u_2^{(k)}(t),u_3^{(k)}(t),p_1)\,dt \end{cases} \tag{9}$$

If we start with the initial approximations:

$$u_1^{(0)}(x) = u_a, \quad u_2^{(0)}(x) = u_{1a}, \quad u_3^{(0)}(x) = u_{2a}, \quad u_4^{(0)}(x) = p_2 \tag{10}$$

we obtain a subsequent approximations of $u_i(x)$, $i = 1\ldots4$ and finally the solution:

$$u(x) = \lim_{n\to\infty} u^{(n)}(x) \tag{11}$$

By applying the boundary conditions (5) at the right side of the domain, we obtain values of the unknown parameters $p_1$, $p_2$ and consequently, a solution in terms of convergent series.

### 4. Numerical Example

To apply VIT to BVP with additional boundary condition, an example will be presented that consists of the fourth–order differential equation which contains unknown $p_1$:

$$u^{(\mathrm{IV})}(x) = u^2(x) - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + p_1 x - 48 \tag{12}$$

and five boundary conditions:

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 4, \quad u(1) = 1, \quad u'(1) = 1 \tag{13}$$

We know ([10]) that for $p_1 = 120$, equation (12) has the following exact solution:

$$u(x) = x^5 - 2x^4 + 2x^2 \tag{14}$$

which fulfills all boundary conditions (13). Taking into consideration (6), we rewrite the above BVP as a system of four first-order differential equations:

$$\begin{cases} \dfrac{du_1}{dx} = u_2(x), \\ \dfrac{du_2}{dx} = u_3(x), \\ \dfrac{du_3}{dx} = u_4(x), \\ \dfrac{du_4}{dx} = u_1^2(x) - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + p_1 x - 48 \end{cases} \tag{15}$$

with initial conditions, which include a second unknown parameter $p_2$:

$$u_1(0) = 0, \quad u_2(0) = 0, \quad u_3(0) = 4, \quad u_4(0) = p_2 \tag{16}$$

By means of (2) we can rewrite the system of differential equations (15) as a system of four integral equations with $\lambda_i = 1$, $i = 1 \ldots 4$. The initial approximations are based on the initial conditions (16).

$$\begin{cases} u_1^{(k+1)}(x) = 0 + \displaystyle\int_0^x u_2^{(k)}(t)dt, \\ u_2^{(k+1)}(x) = 0 + \displaystyle\int_0^x u_3^{(k)}(t)dt, \\ u_3^{(k+1)}(x) = 4 + \displaystyle\int_0^x u_4^{(k)}(t)dt, \\ u_4^{(k+1)}(x) = p_2 + \displaystyle\int_0^x ((u_1^{(k)}(t))^2 - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + p_1 t - 48)dt \end{cases} \tag{17}$$

The *Maple*™ program with accuracy *Digits* = 20 was used to solve this non-standard problem. Calculations have been completed after 12 approximations. Using the boundary conditions at $x = 1$, we obtain:

$$p_1 = 120.0000000048, \quad p_2 = 2.08 \cdot 10^{-10} \tag{18}$$

and the series solution is given by:

$$u(x) = 2x^2 - 3.46 \cdot 10^{-11} x^3 - 2x^4 + x^5 + \ldots - 5.61 \cdot 10^{-36} x^{67} \tag{19}$$

Table 1 exhibits the exact solutions (14), the errors obtained by using the modified VIT and iterative shooting method (ISM) used to the BVP with additional boundary conditions, which was described in [11].

Table 1

**Error estimates**

| $x$ | $u_{exact}(x)$ | Errors* (VIT) | Errors* (ISM) |
|---|---|---|---|
| 0.0 | 0.00000 | 0.0000 | 0.0000 |
| 0.1 | 0.01981 | $3.4279\ 10^{-14}$ | $9.8256\ 10^{-15}$ |
| 0.2 | 0.07712 | $2.6465\ 10^{-13}$ | $8.0465\ 10^{-14}$ |
| 0.3 | 0.16623 | $8.3881\ 10^{-13}$ | $8.4766\ 10^{-14}$ |
| 0.4 | 0.27904 | $1.7599\ 10^{-12}$ | $3.4477\ 10^{-14}$ |
| 0.5 | 0.40625 | $1.4173\ 10^{-12}$ | $1.0776\ 10^{-13}$ |
| 0.6 | 0.53856 | $2.3855\ 10^{-11}$ | $1.0341\ 10^{-13}$ |
| 0.7 | 0.66787 | $2.9626\ 10^{-10}$ | $4.0618\ 10^{-14}$ |
| 0.8 | 0.78848 | $2.2899\ 10^{-9}$ | $3.4080\ 10^{-14}$ |
| 0.9 | 0.89829 | $1.3479\ 10^{-8}$ | $5.9746\ 10^{-14}$ |
| 1.0 | 1.00000 | $6.4569\ 10^{-8}$ | $1.6930\ 10^{-13}$ |

* Error = abs (exact solution – series solution (VIT) or discreet solution (ISM))

## 5. Summary

There are many methods of solving the BVP, but most of them concern the standard BVP, where the order of equation and number of boundary conditions are the same. Taking into consideration methods for the non-standard BVP, the ISM [11], and the VIT described in this article, we can conclude that the differential equation must contain unknown components. Their number must correspond to the number of excessive boundary conditions. The values of these parameters can be calculated by applying additional boundary conditions. Due to the modification of the VIT, there is no need to do huge and time consuming computational work that is available in the ISM. Taking the VIT into consideration, we obtain a solution in the terms of convergent series with easy computable components. Higher accuracy can be obtained by increasing the expansion order in series solution by means of a larger number of iterations.

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