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Comparison of results of stress cycle counting by the direct spectral method and the Fu-Cebon method for bi-modal type stress processes

Porównanie rezultatów zliczania cykli naprężeń metodą spektralną bezpośrednią i metodą Fu-Cebona dla przebiegów o charakterze dwumodalnym

Abstract
This article compares the results of a high cycle fatigue analysis obtained through the application of the two methods dedicated to analysis of response of the structure of the bi-modal type – the direct spectral method and the Fu-Cebon one. the compared parameter is the lifetime for an assumed material S–N curve and stress spectrum defined in an article by Fu & Cebon.

Keywords: fatigue analysis, stress cycle counting, spectral method, bi-modal process

Streszczenie
W artykule dokonano porównania rezultatów wysokocyklowych analiz zmęczeniowych wykonanych przy zastosowaniu dwu metod dedykowanych dla przypadków odpowiedzi struktury o charakterze dwumodalnym: metody spektralnej bezpośredniej i metody Fu-Cebona. Porównywaną wielkością był czas życia konstrukcji przy założonej postaci krzywej S–N materiału i zmienności naprężeń o charakterystyce wykorzystywanej w artykule Fu i Cebona.

Słowa kluczowe: analiza zmęczeniowa, zliczanie cykli naprężeń, metoda spektralna, przebiegi dwumodalne

1. Introduction

In some cases the spectrum of vibrations of engineering structures have the bi-modal type. The example is the suspension system analysed by T.-T.Fu & D. Cebon [5].

Structural vibrations are the reason of changing deformation of material in time. the process is described by variable in time strain tensor components. The variation of strains accounts for material damage in a relatively large number of cycles for the elastic type of deformations. Changing in time strain tensor components are correlate with variable in time stress tensor components. In the paper, the high-cycle fatigue analysis based on the stress formulation is considered.

In the case of nonstationary or multimodal vibrations, the problem of cycle identification and cycle counting is crucial when conducting fatigue analysis. In literature, the following attempts at solving the problem are discussed: direct method (harmonic process, quasi-harmonic process) [4]; bi-modal dedicated methods [1, 2, 5, 14]; stress cycle counting method (e.g. the 'rain-flow' method) [4, 6, 11]; spectral methods (using the power spectral density function) [12, 13]; kurtosis analysis [15]. The most commonly used method is the 'rain-flow' method.

In their previous papers, the authors proposed the original method of stress cycle counting dedicated for the bi-modal type spectrum of stress – this is known as the spectral direct method [7–10]. T.-T. Fu & D. Cebon proposed in their article [5] a method of fatigue analysis that was also dedicated to the bi-modal process. The method proposed by T.-T.Fu & D. Cebon is based on using of the power spectra density description of the process and leads to determination of the probability density of the stress ranges. The method proposed by M. Kozień & D. Smolarski is a type of the time-domain method and is an alternative method of conducting fatigue analysis in cases of bi-modal stress history.

The aim of this paper is to compare results obtained by application of the two alternative but not being against methods: the spectral direct one and the Fu-Cebon one. The compared parameter is the lifetime of element for assumed material S–N curve and stress spectrum defined in article of T.-T.Fu & D. Cebon [5].

2. Basis of the spectral method

The determined bi-modal stress history can be theoretically defined with the following analytical form:

$$\sigma(t) = A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2) \quad (1)$$

where:
A_1, A_2 – stress amplitudes of the harmonic components,
ω_1, ω_2 – angular frequencies of the harmonic components,
φ_1, φ_2 – phases of the harmonic components.

With such a formulation, the component frequencies f_1 and f_2 and corresponding periods T_1 and T_2 can be obtained using equations (2) and (3).

$$f_1 = \frac{\omega_1}{2\pi}, \quad f_2 = \frac{\omega_2}{2\pi} \tag{2}$$

$$T_1 = \frac{1}{f_1} = \frac{2\pi}{\omega_1}, \quad T_2 = \frac{1}{f_2} = \frac{2\pi}{\omega_2} \tag{3}$$

The basic assumptions and manner of application of the direct spectral method for bi-modal waveforms can be described as follows:

- For a given time-domain stress waveform, a spectrum is obtained. Let us assume that from this spectrum, the two frequencies f_1 and f_2 ($f_1 < f_2$) can be distinguished. They are characterized by periods T_1 and T_2, and amplitudes A_1 and A_2, respectively. The phase shift is not required and is therefore omitted in this method ($\varphi_1 = \varphi_2 = 0$). However, the proposed algorithm allows for the phase shift to be included in the calculation ($j_1 \neq 0$ or $j_2 \neq 0$) if necessary. The waveforms are analysed under the assumption that at the initial time, both of the harmonic components are of maximal (positive) value, which means that the stress value at time $t = 0$ is equal to $A_1 + A_2$.
- Period T_1 is the base period for the stress signal.
- Based on the values of periods T_1 and T_2, the so-called block of stress is determined. The block length (time range) T_B depends on the ratio T_1/T_2. It is the smallest integer number of period T_1, for which the ratio T_B/T_2 is an integer. In practical applications, this condition is nearly satisfied hence, assuming the value of T_B, is an arbitrary decision. The value of T_B depends on the precision of the determination of T_1 and T_2, usually by identification of frequencies f_1 and f_2.
- The primary stress cycle, which is the only cycle present within the block, has a stress amplitude of $A_1 + A_2$ and, if not stated otherwise (e.g. constant value present in FTT, static assembly stress or thermal stress), the effective mean stress is zero. This assumption is the basis for calculating the equivalent completely reversed uniaxial stress, e.g. Morrow's type (4) [4].
- The amplitudes of secondary stress cycles vary depending on the value of A_2 and the leading waveform of frequency f_1. Some of the identified cycles are not taken into account if they do not have a full stress-cycle form. For the high difference between frequencies f_1 and f_2, the amplitudes of the secondary cycles are approximately equal to A_2. The acquired data forms the basis of the calculation of the equivalent completely reversed uniaxial stress (4) [4].
- The obtained data which describes the identified stress cycles for a given waveform forms the basis of the fatigue analysis using the chosen stress cumulation hypothesis, e.g. Palmgreen-Miner's (5) [4].

In the analysis, the following parameters are used: effective mean stress (Sines stress) [4]; effective stress amplitude (according to the von Mises equivalent stress) [4]; equivalent completely reversed uniaxial stress (Morrow stress) (3) [4].

$$\sigma_{EQV} = \begin{cases} \dfrac{\sigma_a}{1-\dfrac{\sigma_m}{\sigma_u}} & \text{for } \sigma_m > 0 \\ \sigma_a & \text{for } \sigma_m \le 0 \end{cases} \tag{4}$$

$$B \cdot \sum_{i=1}^{N} \frac{N_i}{\left(N_f\right)_i} = 1 \tag{5}$$

$$T = B \cdot T_B \tag{6}$$

where:
σ_m – effective mean stress,
σ_a – effective stress amplitude,
σ_{EQV} – equivalent completely reversed uniaxial stress,
σ_u – ultimate stress,
N – total number of cycles identified in a block,
N_i – number of cycles with amplitude σ_i identified in a block,
$(N_f)_i$ – number of cycles to damage for stress with amplitude σ_i (S–N curve);
B – number of blocks;
T_B – time length of block;
T – estimated lifetime.

3. Estimation of values of amplitudes based on PSD function

The PSD function is calculated on the basis of a given signal $x(t)$ defined in the time domain. The two-sided PSD function $S_x(t)$ is defined as a Fourier transform of the autocorrelation function (7) and (8).

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)\,x(t+\tau)\,dt \tag{7}$$

$$S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau)\,e^{-i2\pi f\tau}\,d\tau \tag{8}$$

Because $S_x(t)$ is an even function of frequency (9), the commonly used the single-sided (or one-sided) power spectral density $G_x(f)$ (10) is defined and used.

$$S_x(-f) = S_x(f) \tag{9}$$

$$G_x(f) = \begin{cases} 0 \text{ for } f < 0 \\ S_x(f) \text{ for } f = 0 \\ 2S_x(f) \text{ for } f > 0 \end{cases} \tag{10}$$

Sometimes, the following problem arises: what are the averaged values of the signal when only the PSD function is known? The root mean square x_{RMS} value for a specific frequency range $f \in [f_1, f_2]$ can be calculated from its single-sided PSD function in the form (11) [3]. The amplitude of the signal, which is understood as the peak value x_{PEAK} of the harmonic signal, can be found based on formula (12).

$$x_{RMS} = \sqrt{\int_{f_1}^{f_2} G_x(f)\,df} \tag{11}$$

$$x_{PEAK} = \sqrt{2}\,x_{RMS} \tag{12}$$

4. Comparison of life time determined by the direct spectral and the Fu-Cebon methods

4.1. Analysed case

Let us consider after T.-T.Fu & D. Cebon's [5] fatigue analysis of the trailing arm of the car suspension system – this form is shown in Fig. 1.

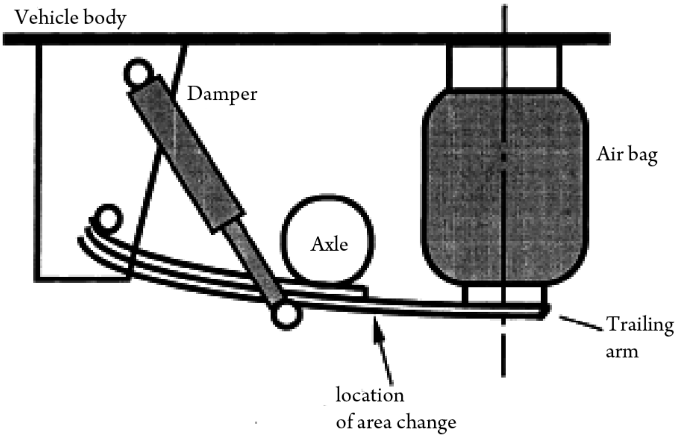

Fig. 1. A car suspension system [5]

The dominant stress component has the spectrum of the bi-modal type in steady-state case of its vibration, therefore the analysis takes a uniaxial form from the point of view of stress. The stress spectrum is defined as the single-sided power spectral density function shown in Fig. 2.

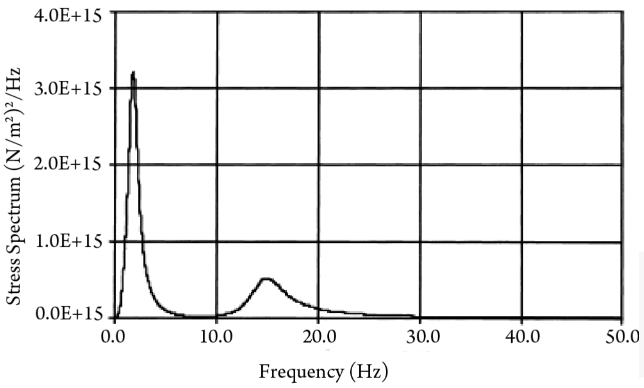

Fig. 2. One-sided power spectral density function of stress [5]

Based on the form of the one-sided power spectral density function of stress, it is possible to determine the probability function of the stress amplitudes using different methods presented in literature, e.g. pure bi-modal, Sakai, rain-flow, Rayleigh, Dirlik, Fu-Cebon [5]. With the application of the Fu-Cebon [5] method, the probability function of peak stress (stress amplitudes) can be visualised in the form shown in Fig. 3.

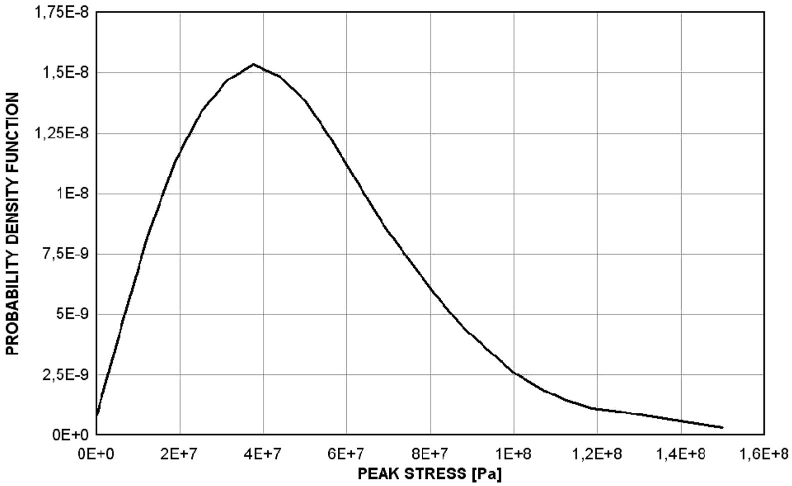

Fig. 3. Probability density function of peak stress [5]

Comparison was done for material the fatigue properties of which are described in the Wohler S–N curve in the form (13), where m = 8, and fatigue limit for $2.0{\cdot}10^6$ cycles is

100 MPa, hence $a = 2.0 \cdot 10^{22}$, where N_f is number of cycles for high-cycle fatigue damage and σ is stress amplitude for the completely reversed stress cycle. the ultimate stress σ_u is 400 MPa.

$$N_f = A\sigma^{-m} \tag{13}$$

4.2. Estimation of lifetime based on the Fu-Cebon method

The bi-modal idealised stress history considered by T.-T.Fu & D. Cebon [5] has the form (1) with assumption that $\varphi_1 = \varphi_2 = 0$. Moreover, it is assumed that the components are reasonably widely spaced in frequency, i.e. $\omega_1/\omega_2 > 4$. By applying the rain-flow counting method, the two distinct amplitude cycles are identified. the first with an amplitude $\sigma_1 = A_1 + A_2$ and $n_1 = \omega_1 T/2\pi$ cycles in T seconds and the second with an amplitude $\sigma_2 = A_2$ and $n_2 - n_1 = (\omega_1 - \omega_1)T/2\pi$ cycles. According to the Palmgren-Miner hypothesis, the total damage D is equal (14).

$$D = \frac{n_1}{N_1} + \frac{n_2 - n_1}{N_2} \tag{14}$$

where:
N_1 and N_2 – the cycles to failure at stresses σ_1 and σ_2, which follow the material S–N curve.

It is then assumed that the two components of equation (1) are narrow processes and their distributions of peaks $P_1(\sigma)$ and $P_2(\sigma)$ follow Rayleigh distributions. After suitable manipulations, the fatigue life estimated by this approach is [5]:

$$T = \frac{2\pi C}{\omega_1 \int_0^{\infty} \frac{P_1(\sigma)}{\sigma^{1/b}} d\sigma + (\omega_1 - \omega_2) \int_0^{\infty} \frac{P_2(\sigma)}{\sigma^{1/b}} d\sigma} \tag{15}$$

where:
$C = a$ and $1/b = -m$ for the material S–N curve equation (13).

Estimation of lifetime T_{FC} is done based on probability density function of peak stress determined with application of the Fu-Cebon formula (Fig. 3). The value of lifetime for a given material (the S–N curve) can be estimated from the Dirlik formula (16), where M^+ is the expected number of peaks in unit of time and $p(\sigma)$ is the probability density functions of the stress ranges [12].

$$T_{FC} = \frac{1}{M^+ \int_0^{+\infty} \frac{p(\Delta\sigma)}{N_f(\Delta\sigma)} d(\Delta\sigma)} \tag{16}$$

For the analysed case the obtained lifetime is $T_{FC} = 3.27 \cdot 10^9$ cycles.

4.3. Estimation of lifetime based on the direct spectral method

The power spectra density function of stresses (Fig. 2), show two components in frequency domain: f_1 = 2 Hz and f_2 = 15 Hz. The periods corresponding with this frequencies are respectively $T_1 = 0.5$ sec (the basic period), $T_2 = 0.0(6)$ sec (the secondary period). Therefore, the ratio is $T_1/T_2 = 7.5$ – this means that one stress block has total length (time duration) equal to $T_B = 2T_1 = 1$s, $T_2 = 1$ sec. The values of amplitudes A_1 and A_2 were identified based on the stress PSD function (Fig. 2) with the application of formulas (11) and (12). The values of integral existing in (11), were calculated as areas of triangles with length 0.3125 Hz and heights respectively: $3.22 \cdot 10^{15}$ Pa^2/Hz for f_1 = 2 Hz and $1.26 \cdot 10^{14}$ Pa^2/Hz for f_2 = 15 Hz. This means that the very narrow frequency range connected with discretisation of the PSD function in the frequency domain was taken into account. the values of the identified stress amplitudes are approximately equal to: A_1 = 32 MPa, A_2 = 6 MPa.

Stress cycle identification for a given block depends on the assumed values of the phase angles φ_1 and φ_2. For uniaxial stress, there is no reason for different values of angles; however, due to the method of cycle identification in the spectral direct method, the values of quantitative parameters describing the secondary cycles depend upon the chosen value of phase angles $\varphi_1 = \varphi_2$. The two cases were analysed: $\varphi_1 = \varphi_2 = 0$ and $\varphi_1 = \varphi_2 = \pi/2$. Reconstructed stress function in time domain, which were basis for cycle identification, is shown in Fig. 4 for one block. Hence, the amplitudes of the main cycle (32 MPa) and the secondary cycles (6 MPa) and then application of the direct spectral method algorithm leads to identification the given in Table 1 and Table 2 groups of cycles for one block with length 1 sec.

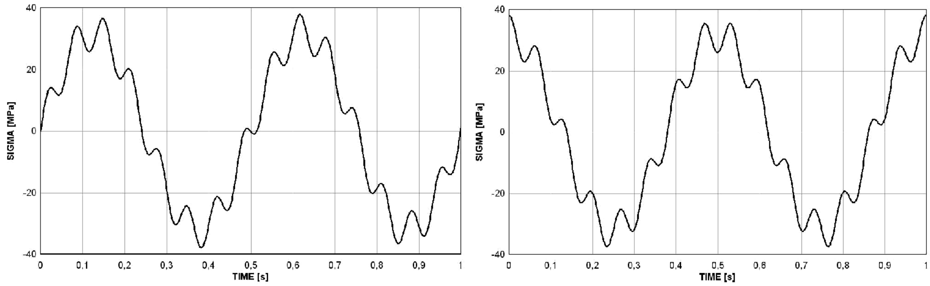

Fig. 4. Reconstructed variation of stress as functions of time: for $\varphi_1 = \varphi_2 = 0$ (left), for $\varphi_1 = \varphi_2 = \pi/2$ (right)

Even though there are differences in the time form of reconstructed stress functions, the number of identified stress cycles, especially secondary cycles, were the same in all cases (the one main cycle and the seventeen secondary cycles). After application of the Palmgren-Miner rule of stress cycle cumulation for a given material, the estimated lifetime was $T_{DS} = 4.6 \cdot 10^9$ cycles for both cases. This result is not far from that which was previously obtained after application of the Fu-Cebon method – $T_{FC} = 3.27 \cdot 10^9$ cycles.

Table 1. Identified cycles of one block for the case $\varphi_1 = \varphi_2 = 0$

NO.	σ_{MAX} [MPa]	σ_{MIN} [MPa]	$\bar{\sigma}_m$ [MPa]	$\bar{\sigma}_a$ [MPa]	σ_{EQV} [MPa]	NUMBER	TYPE
1	38.0	–38.0	0	38.0	38.0	1	main
2	14.1	11.7	12.9	1.2	1.2	1	secondary
3	34.0	25.8	29.9	4.1	4.2	1	secondary
4	36.6	25.8	31.2	5.4	5.5	1	secondary
5	20.3	17.1	18.7	1.6	1.6	1	secondary
6	–5.7	–7.5	0	0.9	0.9	1	secondary
7	–24.3	–30.4	0	3.1	3.1	1	secondary
8	–24.3	–37.8	0	6.8	6.8	1	secondary
9	–21.3	–25.8	0	2.3	2.3	1	secondary
10	0.9	–0.9	0	0.9	0.9	1	secondary
11	25.8	21.3	23.6	2.3	2.3	1	secondary
12	37.8	24.3	31.1	6.8	6.9	1	secondary
13	30.4	24.3	27.4	3.1	3.1	1	secondary
14	7,5	5.7	6.7	1.0	1.0	1	secondary
15	–17.0	–20.3	0	1.7	1.7	1	secondary
16	–25.8	–36.5	0	5.4	5.4	1	secondary
17	–25.8	–34.1	0	4.2	4.2	1	secondary
18	–11.8	–14.1	0	1.2	1.2	1	secondary

Table 2. Identified cycles of one block for the case $\varphi_1 = \varphi_2 = \pi/2$

NO.	σ_{MAX} [MPa]	σ_{MIN} [MPa]	$\bar{\sigma}_m$ [MPa]	$\bar{\sigma}_a$ [MPa]	σ_{EQV} [MPa]	NUMBER	TYPE
1	38.0	–38.0	0	38.0	38.0	1	main
2	38.0	23.0	30.5	7.5	7.6	1	secondary
3	28.2	22.9	25.6	2.7	2.7	2	secondary
4	4.2	2.4	3.3	0.9	0.9	2	secondary
5	–19.3	–23.0	0	1.9	1.9	2	secondary
6	–25.3	–37.3	0	6.0	6.0	2	secondary
7	–25.3	–32.4	0	3.6	3.6	2	secondary
8	–8.8	–10.9	0	1.1	1.1	2	secondary
9	17.2	14.5	15.9	1.4	1.4	2	secondary
10	35.4	26.0	30.7	4.7	4.8	2	secondary

4.4. Detailed conclusions of comparison

The analysis makes it possible to formulate the following detailed conclusions:

- Application of the spectral direct method gives lower errors if the spectral characteristics of the stress variation in time are given in form of the amplitude-frequency characteristics then the power spectra density function, because there is no need to transform from frequency-domain to time-domain. This is observation in the case of the spectral method, but not in comparison with the Fu-Cebon method.
- When leakage exists for spectral functions of stress (amplitude-frequency function or PSD one) around modal frequencies – this is often observed for realistic cases – the final results of cycle identification by the spectra direct method strongly depend on the chosen limits of integration in formula (8). For the discussed analysis, the integrations were performed for a very narrow frequency range. If the range is wider, the analysis would by much conservative.
- If the spectral characteristics of the stress variation in time are given only in the form of amplitude-frequency function or the power spectra density function, the reconstruction of the function is not unique due to a lack of information about values of the phase angles φ_1 and φ_2. For detailed reconstruction, the values of the angles must be known; however, if there is no information about these values, the method can be applied. If there is no other reasons, usually it is assumed that $\varphi_1 = \varphi_2 = 0$. For a uniaxial stress case, the condition $\varphi_1 = \varphi_2$ is always valid.

5. Conclusions

The fundamental conclusion resulting from the analysis is that the lifetime estimation of the analysed case is almost the same when identified using each method. This can be interpreted as verification of the possibility of using the spectral direct method for the fatigue analysis of the bi-modal process.

Moreover, the analyses make it possible to formulate the following conclusions:

- The bi-modal process exists in practical engineering applications.
- The methods based on the integration of the probability density of the stress ranges may not take the sum of modes into account.
- The results of analyses used spectral direct method applied to bi-modal type processes defined by power spectral density function strongly depend on the arbitrary chosen range of integration of the PSD for identified frequencies.
- The Fu-Cebon method based on the analysis of the PSD function can be applied in a natural way for the results experimentally obtained when the characteristics take into account random description. The spectral direct method can be applied in a natural way for the results of the dynamical response of a structure obtained from computer simulation based on the assumption of deterministic analysis.

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