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INTEGRO-DIFFERENTIAL EVOLUTION NONLOCAL PROBLEM FOR THE FIRST ORDER EQUATION (II)

CAŁKOWO-RÓŻNICZKOWE EWOLUCYJNE ZAGADNIENIE NIELOKALNE DLA RÓWNANIA PIERWSZEGO RZĘDU (II)

Abstract

The aim of this paper is to give two theorems on the existence and uniqueness of mild and classical solutions of a nonlocal semilinear integro-differential evolution Cauchy problem for the first order equation. The method of semigroups, the Banach fixed-point theorem and the Bochenek theorem are applied to prove the existence and uniqueness of the solutions of the considered problem.

Keywords: nonlocal problem, integro-differential evolution problem, abstract Cauchy problem

Streszczenie

W artykule udowodniono dwa twierdzenia o istnieniu i jednoznaczności rozwiązań całkowych i klasycznych nielokalnego semiliniowego całkowo-różniczkowego ewolucyjnego zagadnienia Cauchy'ego dla równania rzędu pierwszego. W tym celu zastosowano metodę półgrup, twierdzenie Banacha o punkcie stałym i twierdzenie Bochenka.

Słowa kluczowe: zagadnienie nielokalne, ewolucyjne zagadnienie całkowo-różniczkowe, abstrakcyjne zagadnienie Cauchy'ego

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1. Introduction

In this paper, we give two theorems on the existence and uniqueness of mild and classical solutions of semilinear integro-differential evolution nonlocal Cauchy problem for the first order equation. To achieve this, the method of semigroups, the Banach fixed point theorem and the Bochenek theorem will be used.

Let E be a real Banach space with norm $\|\cdot\|$ and let $A : E \to E$ be a closed densely defined linear operator. For the operator A, let $\mathcal{D}(A)$, $\rho(A)$ and A^* denote its domain, resolvent set and adjoint, respectively.

For the Banach space E, $\mathcal{C}(E)$ denotes the set of closed linear operators from E into itself.

We will need the class $G(\tilde{M}, \beta)$ of operators A satisfying the conditions:

There exist constants $\tilde{M} > 0$ and $\beta \in \mathbb{R}$ such that

(C_1) $A \in \mathcal{C}(E)$, $\overline{\mathcal{D}(A)} = E$ and $(\beta, +\infty) \subset \rho(-A)$,

(C_2) $\left\|(A+\xi)^{-k}\right\| \leqslant \tilde{M}(\xi - \beta)^{-k}$ for each $\xi > \beta$ and $k = 1, 2, \ldots$

It is known (see [4], p. 485 and [5], p. 20) that for $A \in G(\tilde{M}, \beta)$, there exists exactly one strongly continuous semigroup $T(t) : E \to E$ for $t \geqslant 0$ such that $-A$ is its infinitesimal generator and

$$\|T(t)\| \leqslant \tilde{M}e^{\beta t} \quad \text{for } t \geqslant 0.$$

Throughout this paper, we shall use the notation:

$$\begin{aligned} \mathcal{J} &:= [t_0, t_0 + a], \quad \text{where } t_0 \geqslant 0 \text{ and } a > 0, \\ \Delta &:= \{(t,s) : t_0 \leqslant s \leqslant t \leqslant t_0 + a\}, \\ M &:= \sup\{\|T(t)\|, t \in [0, a]\} \end{aligned}$$

and

$$X := \mathcal{C}(\mathcal{J}, E).$$

The Cauchy problem considered here is of the form:

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u(b(t))) + \int_{t_0}^{t} f_1(t, s, u(s))ds + \\ &+ \int_{t_0}^{t_0+a} f_2(t, s, u(s))ds, \ t \in (t_0, t_0 + a], \qquad (1) \\ u(t_0) + g(u) &= u_0, \qquad (2) \end{aligned}$$

where f, f_i $(i = 1, 2)$, g and b are given functions satisfying some assumptions and $u_0 \in E$.

The results obtained in the paper are a continuation of those given in [3] and they are based on those from [1] – [6].

2. The Bochenek theorem

The results of this section were obtained by J. Bochenek (see [2]).

Let us consider the Cauchy problem

$$u'(t) + Au(t) = k(t), \; t \in \mathcal{J} \setminus \{t_0\}, \tag{3}$$
$$u(t_0) = x. \tag{4}$$

A function $u : \mathcal{J} \to E$ is said to be a classical solution of problem (3)–(4) if

(i) u is continuous and continuously differentiable on $\mathcal{J} \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = k(t)$ for $t \in \mathcal{J} \setminus \{t_0\}$,

(iii) $u(t_0) = x$.

Assumption (Z). The adjoint operator A^* is densely defined in E^*, i.e. $\overline{\mathcal{D}(A^*)} = E^*$.

Theorem 2.1. *Let conditions* (C_1), (C_2) *and Assumption (Z) be satisfied. Moreover, let* $k : \mathcal{J} \to E$ *be Lipshitz continuous on* $\mathcal{J}$ *and* $x \in \mathcal{D}(A)$.

Then u *given by the formula*

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)k(s)ds, \; t \in \mathcal{J} \tag{5}$$

is the unique classical solution of the Cauchy problem (3)–(4).

3. Theorem about a mild solution

A function $u : \mathcal{J} \to E$ satisfying the integral equation

$$\begin{aligned} u(t) &= T(t - t_0)u_0 - T(t - t_0)g(u) + \int_{t_0}^{t} T(t - s)\Big(f(s, u(s)), u(b(s))) + \\ &+ \int_{t_0}^{s} f_1(s, \tau, u(\tau))d\tau + \int_{t_0}^{t_0+a} f_2(s, \tau, u(\tau))d\tau\Big)ds, \; t \in \mathcal{J} \end{aligned}$$

is said to be a mild solution of the integrodifferential evolution nonlocal Cauchy problem (1)–(2).

Arguing analogously as in [3] we can obtain, by the Banach fixed point theorem, the following theorem:

Theorem 3.1. *Assume that:*

(i) *the operator* $A : E \to E$ *satisfies conditions* (C_1) *and* (C_2),

(ii) *$f : \mathcal{J} \times E^2 \to E$ is continuous with respect to the first variable in $\mathcal{J}$, $f_i : \Delta \times E \to E$ $(i = 1, 2)$ are continuous with respect to the variables in Δ, $g : X \to E$, $b : \mathcal{J} \to \mathcal{J}$ are continuous and there exist positive constants L, L_i $(i = 1, 2)$ and K such that*

$$\|f(s, z_1, z_2) - f(s, \tilde{z}_1, \tilde{z}_2)\| \leqslant L \sum_{i=1}^{2} \|z_i - \tilde{z}_i\|$$

for $s \in \mathcal{J}$, $z_i, \tilde{z}_i \in E$ $(i = 1, 2)$,

$$\|f_i(s, \tau, z) - f_i(s, \tau, \tilde{z})\| \leqslant L_i \|z - \tilde{z}\| \ \ (i = 1, 2)$$

for $(s, \tau) \in \Delta$, $z, \tilde{z} \in E$
and

$$\|g(w) - g(\tilde{w})\| \leqslant K \|w - \tilde{w}\|_X \quad \text{for } w, \tilde{w} \in X.$$

(iii) $M[a(2L + aL_1 + aL_2) + K] < 1.$

(iv) $u_0 \in E.$

Then the integrodifferential evolution nonlocal Cauchy problem (1)–(2) *has a unique mild solution.*

4. Theorem about a classical solution

A function $u : \mathcal{J} \to E$ is said to be a classical solution of the nonlocal Cauchy problem (1)–(2) on $\mathcal{J}$ if :

(*i*) u is continuous on $\mathcal{J}$ and continuously differentiable on $\mathcal{J} \setminus \{t_0\}$,

(*ii*) $u'(t) + Au(t) = f\big(t, u(t), u(b(t))\big) + \int_{t_0}^{t} f_1(t, s, u(s))ds +$
$+ \int_{t_0}^{t_0+a} f_2(t, s, u(s))ds \quad \text{for } t \in \mathcal{J} \setminus \{t_0\}$,

(*iii*) $u(t_0) + g(u) = u_0$.

Theorem 4.1. *Assume that:*

(i) *the operator $A : E \to E$ satisfies conditions (C_1) and (C_2), and Assumption* (Z).

(ii) *$f : \mathcal{J} \times E^2 \to E$, $g : X \to E$, for any $(s, z) \in \mathcal{J} \times E$ and $i = 1, 2$ functions $f_i(s, \cdot, z) : \mathcal{J} \ni \tau \mapsto f(s, \tau, z) \in E$ are continuous, $b : \mathcal{J} \to \mathcal{J}$ is continuous on $\mathcal{J}$ and there exist positive constants C, C_i $(i = 1, 2)$ and K such that:*

$$\|f(s, z_1, z_2) - f(\tilde{s}, \tilde{z}_1, \tilde{z}_2)\| \leqslant C\Big(|s - \tilde{s}| + \sum_{i=1}^{2} \|z_i - \tilde{z}_i\| \Big)$$

for $s, \tilde{s} \in \mathcal{J}$, $z_i, \tilde{z}_i \in E$ $(i = 1, 2)$,

$$\|f_i(s, \tau, z) - f_i(\tilde{s}, \tau, \tilde{z})\| \leqslant C_i(|s - \tilde{s}| + \|z - \tilde{z}\|)$$

for (s, τ), $(\tilde{s}, \tau) \in \Delta$, $z, \tilde{z} \in E$
and

$$\|g(w) - g(\tilde{w})\| \leqslant K \|w - \tilde{w}\|_X \quad \text{for } w, \tilde{w} \in X.$$

(iii)
$$M\Big(a(2C + aC_1 + aC_2) + K\Big) < 1.$$

Then the integrodifferential evolution nonlocal Cauchy problem (1)–(2) *has a unique mild solution (which is denoted by)* u. *Moreover, if* $u_0 \in \mathcal{D}(A)$, $g(u) \in \mathcal{D}(A)$ *and if there exists a positive constant* $\mathcal{H}$ *such that*

$$\|u(b(s)) - u(b(\tilde{s}))\| \leqslant \mathcal{H} \|u(s) - u(\tilde{s})\| \quad \text{for } s, \tilde{s} \in \mathcal{J}$$

then u *is the unique classical solution of the problem* (1)–(2).

Proof. Since all the assumptions of Theorem 3.1 are satisfied, it is easy to see that problem (1)–(2) possesses a unique mild solution which according to the last assumption is denoted by u.

Now we shall show that u is the classical solution of the problem (1)–(2). To this end, observe that as in [3] u is Lipschitz continuous on $\mathcal{J}$.

The Lipschitz continuity of u on $\mathcal{J}$ combined with the Lipschitz continuity of f on $\mathcal{J} \times E^2$ and f_i $(i = 1, 2)$ with respect to the first variables imply that the function

$$\mathcal{J} \ni t \mapsto f\big(t, u(t), u(b(t))\big) + \int_{t_0}^{t} f_1(t, s, u(s))ds + \int_{t_0}^{t_0+a} f_2(t, s, u(s))ds$$

is Lipschitz continuous on $\mathcal{J}$. This property of f together with the assumptions of Theorem 4.1 imply, by Theorem 2.1 and Theorem 3.1, that the linear Cauchy problem:

$$\begin{aligned} v'(t) + Av(t) &= f\big(t, u(t), u(b(t))\big) + \int_{t_0}^{t} f_1(t, s, u(s))ds + \\ &+ \int_{t_0}^{t_0+a} f_2(t, s, u(s))ds, \ t \in \mathcal{J} \setminus \{t_0\}, \\ v(t_0) &= u_0 - g(u) \end{aligned}$$

has a unique classical solution v and it is given by

$$\begin{aligned} v(t) &= T(t - t_0)u_0 - T(t - t_0)g(u) + \int_{t_0}^{t} T(t - s)\Big(f\big(s, u(s), u(b(s))\big) + \\ &+ \int_{t_0}^{s} f_1(s, \tau, u(\tau))d\tau + \int_{t_0}^{t_0+a} f_2(s, \tau, u(\tau))d\tau\Big)ds = u(t), \ t \in \mathcal{J}. \end{aligned}$$

Consequently, u is the unique classical solution of the integrodifferential evolution Cauchy problem (1)–(2) and, therefore, the proof of Theorem 4.1 is complete.

□

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