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NOTES ON NO-ARBITRAGE CRITERIA

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Abstract. We consider the closedness of the modified set of hedgeable claims and new conditions for the absence of arbitrage connected with it in the classical Dalang–Morton–Willinger Theorem.

1. Introduction and the model. For the standard discrete-time finite horizon model of security market, the Dalang–Morton–Willinger Theorem asserts that there is no arbitrage if and only if the price process is a martingale with respect to an equivalent probability measure. This remarkable result is sometimes referred to as the (First) Fundamental Theorem of Asset Pricing (FTAP). The theorem has been investigated in many works and additional conditions have been proposed. Going further in this direction we consider new conditions in which it is enough that the terminal profit of the portfolio is a sum of certain characteristic functions. As we see in the example, this condition is not equivalent to the absence of arbitrage. The main reason is that the set of hedgeable claims considered by us might not be convex. In general, if we want the equivalence it is essential to assume this convexity. Now we introduce the model as it was done in [**4**].

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a finite discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)_{t=0}^T$ be an d-dimensional process adapted to $\mathbb{F}$. Put

$$R_T := \{\xi\colon\ \xi = H \cdot S_T, H \in \mathcal{P}\},$$

where $\mathcal{P}$ is the set of all predictable d-dimensional processes (i.e. H_t is $\mathcal{F}_{t-1}$-measurable) and

$$H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t, \qquad \Delta S_t := S_t - S_{t-1}.$$

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H is very often called a (portfolio) strategy and $H \cdot S$ a value process. We use notations $L^0(\mathbb{R}^d)$ for the set of d-dimensional ($\mathcal{F}$-measurable) random vectors and $L^0(\mathbb{R}^d, \mathcal{F}_t)$ for the set of d-dimensional random vectors which are $\mathcal{F}_t$-measurable. Furthermore, we denote by L^0, $L^0(\mathcal{F}_t)$ random variables which are $\mathcal{F}$-measurable and $\mathcal{F}_t$-measurable, respectively.

Put $A_T := R_T - \mathcal{L}_+$ where $\mathcal{L}_+ : \{\lambda I_A \colon\ \lambda > 0, A \in \mathcal{F}\}$ and let $\overline{A}_T$ be the closure of A_T in probability.

2. Main result. Now we formulate the main result of the paper.

THEOREM 1. *Assume that A_T is convex. Then the following conditions are equivalent:*

(a) $(R_T - L^0_+) \cap L^0_+ = \{0\}$*;*

(b) $(R_T - L^0_+) \cap L^0_+ = \{0\}$ *and* $A_T = \overline{A}_T$*;*

(c) $A_T \cap \mathcal{L}_+ = \{0\}$ *and* $A_T = \overline{A}_T$*;*

(d) $\overline{A}_T \cap \mathcal{L}_+ = \{0\}$*;*

(e) *there is a probability $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that S is a $\tilde{P}$-martingale.*

REMARK 1. To be precise, we need to know A_T to be a convex cone (i.e. $0 \in A_T$ and for all $\lambda, \mu > 0$ and $x, y \in A_T$ we have that $\lambda x + \mu y \in A_T$). It is obvious that A_T is always a cone but usually is not convex. Moreover, it is clear that if A_T is convex then the conditions from the above theorem are equivalent to some different ones (see e.g. [**3**] and [**4**]) due to conditions (a), (e). Especially these conditions are equivalent to:

(f) $R_T \cap L_0^+ = \{0\}$;

(g) $\{\eta \Delta S_t \colon \eta \in L^0(\mathcal{F}_{t-1})\} \cap L_0^+ = \{0\}$ for all $t \le T$;

where L^0_+ is the set of non-negative random variables ($\mathcal{F}$-measurable). The set $R_T - L^0_+$ can be interpreted as the set of hedgeable claims. Conditions (a) and (f) are often called the no arbitrage (NA) property of the model whereas condition (g) is the NA property for one-step model.

REMARK 2. The NA property for the class of all strategies (conditions (a),(f)) is equivalent to the NA property in the narrower class of bounded strategies H (see [**3**, Remark, p. 73]).

REMARK 3. Notice that if A_T is convex then $\overline{A}_T \overline{R_T - L^0_+}$ and the implications (d) $\Rightarrow$ (e), (e) $\Rightarrow$ (a) follow from the Dalang–Morton–Willinger Theorem (see e.g. [**3, 4**]).

PROOF. As $A_T \subseteq R_T - L^0_+$ then it is enough to show that $\overline{R_T - L^0_+} \subseteq \overline{A}_T$. We will show that $R_T - L^0_+ \subseteq \overline{A}_T$. Let $x = (HS_T - r) \in R_T - L^0_+$, where $HS_T \in R_T$ and $r \in L^0_+$. Since r is a non-negative random variable we can write

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$r = \lim r_n$, where r_n are certain simple functions. Hence $x = \lim(HS_T - r_n)$, where $(HS_T - r_n) \in A_T$ due to the assumption that A_T is a convex cone and finally $x \in \overline{A}_T$. As $R_T - L^0_+ \subseteq \overline{A}_T$ then $\overline{R_T - L^0_+} \subseteq \overline{A}_T$.

Now we show that the above implications easily follow from the Dalang–Morton–Willinger Theorem. Condition (d) is equivalent to

$$\overline{R_T - L^0_+} \cap \mathcal{L}_+ = \{0\}.$$

It implies the condition

(a2) $\quad (R_T - L^0_+) \cap \mathcal{L}_+ = \{0\}$.

We prove that (a2) is equivalent to (a). Since $\mathcal{L}_+ \subseteq L^0_+$ then it is enough to show the implication (a2) $\Rightarrow$ (a). Assume that (a2) holds and suppose that there exists $x \in (R_T - L^0_+) \cap L^0_+$ such that $x \neq 0$. Then $x = HS_T - h$ where $HS_T \in R_T$ and $h \in L^0_+$. Moreover, there exists $\varepsilon > 0$ such that $P(x \geq \varepsilon) > 0$. Define $A := \{x \geq \varepsilon\}$. Since $x - \varepsilon I_A \geq 0$ we get $HS_T - h - (x - \varepsilon I_A) \in R_T - L^0_+$ and the equality $\varepsilon I_A = HS_T - h - (x - \varepsilon I_A)$ contradicts (a2). $\square$

Now we present some lemmas which will be used in the proof of Theorem 1.

Lemma 1. *Let X_n be a sequence of random vectors taking values in $\mathbb{R}^d$ such that for almost all $\omega \in \Omega$ we have $\liminf \|X_n(\omega)\|_d < \infty$. Then there is a sequence of random vectors Y_n taking values in $\mathbb{R}^d$ satisfying the following conditions:*

(1) Y_n converges pointwise to Y almost surely where Y is a random vector taking values in $\mathbb{R}^d$,

(2) $Y_n(\omega)$ is a convergent subsequence of $X_n(\omega)$ for almost all $\omega \in \Omega$.

Remark 4. The above claim can be formulated as follows: there exists an increasing sequence of integer-valued random variables σ_k such that X_{σ_k} converges a.s. The proof of this lemma can be found e.g. in [**2**] and [**4**]. For the reader's convenience we enclose it here.

Proof. Define random variable $X_* := \liminf \|X_n\|_d$.
Let $\sigma(0) := 0$ and $\sigma(k) := \inf\{n > \sigma(k-1) \colon |\|X_n\|_d - X_*| \leq \frac{1}{k}\}$. For the sequence $\tilde{X}_n : X_{\sigma(n)}$ (notice that $\tilde{X}_n$ is a well-defined random variable) we will have $\sup_n \|\tilde{X}_n\|_d < \infty$ a. s. on Ω. In particular, $X^1_* := \liminf X^1_n < \infty$. Let

$$\tau_1(0) := 0, \qquad \tau_1(k) := \inf\{n > \tau_1(k-1) \colon |\tilde{X}^1_n - X^1_*| \leq \frac{1}{k}\}, \quad k \geq 1.$$

In a similar way, working with the second component of the sequence $\tilde{X}_{\tau_1(n)}$ whose first component converges, we construct an increasing sequence $\tau_2(k)$ and so on. Finally, the sequence $\tau_n := \tau_d \circ \ldots \circ \tau_1 \circ \sigma(n)$ has the claimed property. $\square$

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LEMMA 2. *Let $r_n = \lambda_n I_{A_n} \in \mathcal{L}_+$ and suppose that $r_n \to \xi$ pointwise almost surely. Then $\xi \in \mathcal{L}_+$.*

PROOF. Notice that ξ is a random variable as the limit of a convergent a.s. sequence of measurable functions. If $\lambda_n I_{A_n}$ converges pointwise to zero a.s. then ξ is equal to zero a.s. on Ω and the claim is proved. Suppose that there exists a set of positive measure such that $I_{A_n} \not\to 0$ a.s. on it. Accordingly, define the measurable set $A := \{\omega\colon \xi(\omega) \neq 0\}$ and suppose that $P(A) > 0$. It is enough to consider the situation on A, because ξ is equal to zero on $\Omega \setminus A$. By the definition of a convergence, for almost all $\omega \in \Omega$

$$\forall \varepsilon > 0\ \exists N(\omega)\ \forall n \geq N(\omega)\colon \quad |\lambda_n I_{A_n}(\omega) - \xi(\omega)| < \varepsilon.$$

Moreover, for almost all $\omega \in A$ there exists $N(\omega)$ such that for all $n \geq N(\omega)$ there holds $I_{A_n}(\omega) = 1$. Summing up, for almost all $\omega \in A$ the following condition holds:

$$\forall \varepsilon > 0\ \exists N(\omega)\ \forall n \geq N(\omega)\colon \quad |\lambda_n - \xi(\omega)| < \varepsilon.$$

In particular, the sequence λ_n converges, so there exists a number $\lambda \geq 0$ such that $\lambda = \lim \lambda_n$. Furthermore, for almost all $\omega \in A$ the equality $\xi(\omega) = \lambda$ holds true. Thus, $\xi = \lambda I_A$ a.s. on Ω. □

For $t = 1, \dots, T$ we define the following sets

$$\mathcal{R}_t := \{\sum_{n=t}^{T} H_n \Delta S_n | H_n \text{ is } \mathcal{F}_{n-1} - \text{measurable}\},$$

$$\mathcal{A}_t := \mathcal{R}_t - \mathcal{L}_+.$$

LEMMA 3. *If $A_T \cap \mathcal{L}_+ = \{0\}$ then $\mathcal{A}_t \cap \mathcal{L}_+ = \{0\}$ for all $t = 1, \dots, T$. Similarly, if $(R_T - L^0_+) \cap L^0_+ = \{0\}$ then $(\mathcal{R}_t - L^0_+) \cap L^0_+ = \{0\}$ for all $t = 1, \dots, T$.*

PROOF. Suppose that $T > 1$ and $A_T \cap \mathcal{L}_+ = \{0\}$. The case $t = 1$ is trivial: fix any $t = 2, \dots, T$. To show that $\mathcal{A}_t \cap \mathcal{L}_+ = \{0\}$ it is enough to prove that $\mathcal{A}_t \subset A_T$. Suppose that $\xi \in \mathcal{A}_t$. We can assume that $\xi \sum_{n=t}^{T} H_n \Delta S_n - l$ where $(H_n)_{n=t}^{T}$ is $\mathbb{R}^d$-valued process which is predictable with respect to the filtration $\mathbb{F}$ and $l \in \mathcal{L}_+$. Let $H_n : 0$ for $n < t$ and consider the process $(H_n)_{n=1}^{T} \in \mathcal{P}$. There is $\xi = \sum_{n=t}^{T} H_n \Delta S_n - l \sum_{n=1}^{T} H_n \Delta S_n - l \in A_T$. Using analogous reasoning we prove the second implication. □

LEMMA 4. *Let $(H^n_t)_{t=1}^{T} \in \mathcal{P}$ and $r_n \in \mathcal{L}_+$. We can assume that $r_n = \lambda_n I_{A_n}$, where $A_n \in \mathcal{F}$ and $\lambda_n > 0$. Suppose that $A_T \cap \mathcal{L}_+ = \{0\}$ and $\sum_{t=1}^{T} H^n_t \Delta S_t - r_n \to \zeta$ a.s. on Ω where ζ is a random variable. Then $\liminf r_n(\omega) < \infty$ for almost all $\omega \in \Omega$. Moreover, if we assume that $(R_T - L^0_+) \cap L^0_+ = \{0\}$ and $\zeta \notin R_T$ then $\liminf \lambda_n < \infty$.*

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Proof. Define $A := \{\omega\colon \ \liminf r_n(\omega) = \infty\}$ and suppose that $P(A) > 0$. There immediately follows that $\lambda_n \to \infty$. Dividing by λ_n we get the following convergence $\sum_{t=1}^{T} \frac{H_t^n}{\lambda_n} \Delta S_t - I_{A_n} \to 0$, where the process $(\frac{H_t^n}{\lambda_n})_{t=1}^{T} \in \mathcal{P}$. Using the same procedure as in the proof of implication (a) $\Rightarrow$ (b) of Theorem 1 below (see also [**3, 4**]) we can construct sequences $(\tilde{H}_t^n)_{t=1}^{T} \in \mathcal{P}$ and $\tilde{u}_n$ which converge a.s. on Ω and $\sum_{t=1}^{T} \tilde{H}_n \Delta S_t - \tilde{u}_n \to 0$. Moreover, $\tilde{u}_n(\omega)$ is a convergent subsequence of the sequence $I_{A_n}(\omega)$ for almost all $\omega \in \Omega$. Hence $\tilde{u}_n = I_{\tilde{A}_n}$ where $\tilde{A}_n \in \mathcal{F}$. Denote the limits $\tilde{H} := \lim \tilde{H}_n$ and $\tilde{u} := \lim \tilde{u}_n$ in the sense of a.s. Notice that $\tilde{H} \in \mathcal{P}$ and $\tilde{u} \in \mathcal{L}_+$. Hence, in particular, $\tilde{u} = I_{\tilde{A}}$ for a certain set $\tilde{A} \in \mathcal{F}$ and $P(\tilde{A}) > 0$ due to the fact that $I_{\tilde{A}_n} \to 1$ a.s. on A. Therefore, $\tilde{H} S_T = I_{\tilde{A}}$ a.s. on Ω, which contradicts the condition $A_T \cap \mathcal{L}_+ = \{0\}$.

Now we show that if $(R_T - L_+^0) \cap L_+^0 = \{0\}$ and $\zeta \notin R_T$ then $\liminf \lambda_n < \infty$. As we assumed $\sum_{t=1}^{T} H_t^n \Delta S_t - r_n \to \zeta$. We know that $\liminf r_n < \infty$ a.s. on Ω. Suppose that $\lambda_n \to \infty$. Hence $\liminf r_n 0$ a.s. on Ω. Using the procedure from the proof of implication (a) $\Rightarrow$ (b) in the Dalang–Morton–Willinger Theorem (see [**3, 4**]) we can construct sequences $(\hat{H}_t^n)_{t=1}^{T} \in \mathcal{P}$ and $\hat{u}_n \in L_+^0$ which converge a.s. on Ω and $\sum_{t=1}^{T} \hat{H}_n \Delta S_t - \hat{u}_n \to \zeta$. (Notice that we use Lemma 3 in such a construction.) Moreover, $\hat{u}_n(\omega)$ is a convergent subsequence of the sequence $r_n(\omega)$ for almost all $\omega \in \Omega$. Hence $\hat{u}_n \to 0$ a.s. on Ω and by the closedness of R_T we get $\zeta \in R_T$ which contradicts the assumption. □

Remark 5. Notice that the set R_T is always closed irrespective of the absence of arbitrage (see e.g. [**4**]). If we are able to prove Lemma 4 without using the assumption $(R_T - L_+^0) \cap L_+^0 = \{0\}$ then only if A_T is convex, conditions (a)–(e) in Theorem 1 are equivalent to $A_T \cap \mathcal{L}_+ = \{0\}$. It is because we use the fact that $\liminf \lambda_n < \infty$ in the proof of the closedness of A_T.

Lemma 5. *Let $r_n = \lambda_n I_{A_n} \in \mathcal{L}_+$ be a sequence such that $\lambda_n \to \lambda \geq 0$. Moreover, let X_n be a sequence of random variables convergent almost surely on Ω such that for almost all $\omega \in \Omega$ the sequence $X_n(\omega)$ is a convergent subsequence of the sequence $r_n(\omega)$. Define $X := \lim X_n$ where the convergence is pointwise a.s. on Ω. Then $X \in \mathcal{L}_+$.*

Proof. The case of $\lambda = 0$ is trivial. For simplification assume that $\lambda > 0$. Define $A := \{\omega \in \Omega\colon X(\omega) \neq 0\}$. Since X is a random variable (as the limit of a sequence of random variables convergent pointwise a.s.), we obtain $A \in \mathcal{F}$. Therefore, it suffices to consider the situation on A. We know that $X_n(\omega)$ is the almost surely convergent subsequence of $r_n(\omega)$. Thus, for almost all $\omega \in A$ the following equalities hold:

$$X(\omega) = \lim_{n\to\infty} X_n(\omega) = \lim_{k\to\infty} r_{n_k}(\omega) = \lim_{k\to\infty} \lambda_{n_k} I_{A_{n_k}}(\omega).$$

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Since $X \neq 0$ a.s. on A, hence for almost all $\omega \in A$ there exists $N(\omega)$ such that for $k \geq N(\omega)$ we have that $I_{A_{n_k}}(\omega) = 1$. Summing up and taking into consideration the convergence of λ_n, the following equalities $X(\omega) = \lim_{k\to\infty} \lambda_{n_k} = \lim_{n\to\infty} \lambda_n = \lambda$ hold true for almost all $\omega \in A$. It means that $X \in \mathcal{L}_+$. □

Now we formulate a lemma which is similar to the well-known result due to Kreps and Yan (see e.g. [**3, 4**]).

Lemma 6. *Let $K \supseteq -\mathcal{L}_+$ be a closed convex cone in L^1 such that $K \cap \mathcal{L}_+ = \{0\}$. Then there is a probability $\widetilde{P} \sim P$ with $d\widetilde{P}/dP \in L^\infty$ such that $\widetilde{E}\xi \leq 0$ for all $\xi \in K$.*

Proof. By the Hahn–Banach Separation Theorem for any $x \in \mathcal{L}_+$, $x \neq 0$, there is $z_x \in L^\infty$ such that $Ez_x\xi < Ez_x x$ for all $\xi \in K$. Since K is a cone, there follows that $Ez_x\xi \leq 0$ for all $\xi \in K$. Moreover, $Ez_x x > 0$ for $\xi = 0$. To show that $z_x \geq 0$, define $A : \{z_x < 0\}$ and suppose that $P(A) > 0$. Considering the sequence $\xi_n := -\lambda_n I_A \in K$, where $\lambda_n \to \infty$, we get $Ez_x\xi_n \to \infty$, which contradicts the inequality $Ez_x\xi_n \leq 0$. Normalizing, we assume that $z_x \leq 1$. The Halmos–Savage Theorem asserts that the family of measures $\{z_x P\}$ contains a countable equivalent subfamily $\{z_{x_i}P, i \in \mathbb{N}\}$. It means (see [**1**] for more details) that for every $E \in \mathcal{F}$ we have the following equivalence

$$(\forall z_x P\colon\ z_x P(E) = 0) \quad \Leftrightarrow \quad (\forall z_{x_i} P\colon\ z_{x_i} P(E) = 0).$$

Put $\rho := \sum 2^{-i} z_{x_i}$ and $\tilde{x} := I_{\{\rho=0\}}$. Then $Ez_{x_i}\tilde{x} = 0$ for all i and, hence, $Ez_x\tilde{x} = 0$ for all $x \in \mathcal{L}_+$. Thus, $\tilde{x} = 0$ (otherwise we would have $Ez_{\tilde{x}}\tilde{x} > 0$) and the measure $\tilde{P} := c\rho P$ with $c = 1/E\rho$ meets the requirements. Indeed, for any $\xi \in K$ there is $\tilde{E}\xi = E(c\rho\xi) = c\sum 2^{-i} Ez_{x_i}\xi \leq 0$. □

Remark 6. The proof of the Halmos–Savage Theorem can be found e.g. in [**1**]. One can see that Lemma 6 easily follows from Kreps–Yan Theorem. Indeed, we can show that if the assumptions of the above lemma are satisfied, then the assumptions of Kreps–Yan Theorem are valid too. The inclusion $K \supseteq -L^0_+$ follows from the fact that K is a closed convex cone and that every non-negative random variable is the limit of a sequence of non-negative simple functions. Now we prove that $K \cap L^0_+ = \{0\}$. Suppose that there exists $x \in K \cap L^0_+$ such that $x \neq 0$. Hence there exists $\varepsilon > 0$ such that $P(x \geq \varepsilon) > 0$. Define $A := \{x \geq \varepsilon\}$. Since $x - \varepsilon I_A \geq 0$ and $K \supseteq -L^0_+$ we have $-(x - \varepsilon I_A) \in K$. Because K is a convex cone, then $-(x - \varepsilon I_A) + x\varepsilon I_A \in K$, which contradicts $K \cap \mathcal{L}_+\{0\}$.

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Proof of the Theorem 1.

(a) $\Rightarrow$ (b) Consider the case $T = 1$ first. To simplify the notation define $K_n := H_1^n$ and $\Delta S : \Delta S_1$. We will respectively use the notation K^i for the i-th coordinate of the random vector K. Suppose that $K_n \Delta S - r_n \to \zeta$ in probability, where $K_n \in L^0(\mathbb{R}^d, \mathcal{F}_0)$, $r_n \in \mathcal{L}_+$ (we can assume that $r_n = \lambda_n I_{A_n}$ for some $A_n \in \mathcal{F}$ and $\lambda_n > 0$) and ζ is a random variable. By the Riesz Theorem the sequence $K_n \Delta S - r_n$ contains a subsequence convergent to ζ a.s. Thus, perhaps confining ourselves to this subsequence, we can assume that $K_n \Delta S - r_n \to \zeta$ a.s. The closedness of A_1 means that $\zeta = K \Delta S - r$ for some $K \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ and $r \in \mathcal{L}_+$. Therefore, it is enough to show that ζ has the above form. Assume first that $\liminf \lambda_n < \infty$ (we consider opposite case at the end of the proof). Hence, from the sequence λ_n, we can choose a convergent subsequence (let λ_{n_k} be such a subsequence, i.e. $\lambda_{n_k} \to \lambda \geq 0$). In particular, $K_{n_k} \Delta S - r_{n_k} \to \zeta$ pointwise a.s. on Ω. For simplification we will use again the same notation $K_n \Delta S - r_n \to \zeta$ for this subsequence. We will consider separately certain sets belonging to the σ-algebra.

Consider the measurable set $\Omega_1 := \{\omega\colon \liminf \|K_n(\omega)\|_d < \infty\}$. By Lemma 1, we can find a sequence $L_k \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ which converges a.s. on Ω_1 and $L_k(\omega)$ is a convergent subsequence of the sequence $K_n(\omega)$ for almost all $\omega \in \Omega_1$. More precisely, by Lemma 1 there exists an increasing sequence of integer-valued random variables τ_k such that $L_k(\omega) = K_{\tau_k(\omega)}(\omega)$ and the sequence has the above properties. We define the sequence p_k corresponding to the sequence L_k and having the form $p_k(\omega) = r_{\tau_k(\omega)}(\omega)$. In particular, p_k is convergent a.s. on Ω_1 (it follows from the convergence of L_k and $K_n \Delta S - r_n$). Furthermore, since $L_k(\omega)\Delta S(\omega) - p_k(\omega) = K_{\tau_k(\omega)}(\omega)\Delta S(\omega) - r_{\tau_k(\omega)}(\omega)$ and $K_n \Delta S - r_n \to \zeta$, $L_k \Delta S - p_k \to \zeta$ a.s. on Ω_1. Because the limit of a convergent sequence of measurable functions is a measurable function, hence $L := \lim L_k$ is $\mathcal{F}_0$-measurable random vector taking values in $\mathbb{R}^d$. It is enough to show that the random variable on Ω_1 of the form $p := \lim p_k$ belongs to $\mathcal{L}_+$. Notice that the assumptions of Lemma 5 hold. Namely, $\lambda_n \to \lambda$, p is a non-negative random variable on Ω_1 (as the limit of non-negative random variables convergent a.s. on Ω_1) and for almost all $\omega \in \Omega_1$, $p_k(\omega)$ is a convergent subsequence of $r_n(\omega)$. By Lemma 5, we receive that $p \in \mathcal{L}_+$. Moreover, if we define the (measurable) set $A := \{\omega \in \Omega_1 \colon p(\omega) \neq 0\}$, then $p = \lambda I_A$ a.s. on Ω_1.

Thus, if Ω_1 is of full measure, we end our proof for $T = 1$. If not, we continue the proof on the (measurable) set $\Omega_2 := \{\omega\colon \liminf \|K_n(\omega)\|_d = \infty\}$ working further with the sequence $K_n \Delta S - r_n \to \zeta$ a.s. on Ω (let us recall we have previously selected it as a subsequence of the original sequence such that $\lambda_n \to \lambda$). Notice that if Ω_1 is of measure zero, then our operations on Ω_1 should be omitted and we should only consider the case of Ω_2. Put $G_n := \frac{K_n}{\|K_n\|_d}$ and $h_n := \frac{r_n}{\|K_n\|_d}$ and observe that $G_n \Delta S - h_n \to 0$ a.s. on Ω_2.

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Moreover, by the choice of the sequence r_n (such that $\lambda_n \to \lambda$) and the equality $\liminf \|K_n\|_d = \infty$ a.s. on Ω_2, there is $h_n \to 0$ a. s. on Ω_2. Now we will apply the similar reasoning as on the set Ω_1. By Lemma 1 there exists a sequence $J_k \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ such that $J_k(\omega)$ is a convergent subsequence of the sequence $G_n(\omega)$ for almost all $\omega \in \Omega_2$. More precisely, by Lemma 1 there exists an increasing sequence of integer-valued random variables τ_k such that $J_k(\omega) = G_{\tau_k(\omega)}(\omega)$ and the above properties are satisfied. In particular, $J_k \Delta S \to 0$ a.s. on Ω_2. Because the limit of a convergent sequence of measurable functions is a measurable function, hence $J := \lim J_k$ (the limit a.s. on Ω_2) is $\mathcal{F}_0$-measurable random vector taking values in $\mathbb{R}^d$.

Therefore, $J\Delta S = 0$ a.s. on Ω_2. Since $J(\omega) \neq 0$ a.s. on Ω_2 (because $J_k(\omega)$ is a convergent subsequence of the sequence $G_n(\omega)$ for almost all $\omega \in \Omega_2$, then $\|G_n(\omega)\|_d = 1$ for almost all $\omega \in \Omega_2$), then there exists a partition of Ω_2 into at most d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $J^i(\omega) \neq 0$ a.s. on Ω_2^i. (Such a partition can be receive putting $\Omega_2^1 := \{\omega \in \Omega_2 \colon J^1(\omega) \neq 0\}$ and then continuing the partition on the set $\Omega_2 \setminus \Omega_2^1$ only, putting $\Omega_2^2 := \{\omega \in \Omega_2 \setminus \Omega_2^1 \colon J^2(\omega) \neq 0\}$ and so on.) Define $\overline{K}_n := K_n - \beta_n J$, where $\beta_n := \frac{K_n^i}{J^i}$ on the set Ω_2^i. Then on each of the sets Ω_2^i there is $\overline{K}_n \Delta S = K_n \Delta S$. It is so, because $\overline{K}_n \Delta S = K_n \Delta S - \beta_n J \Delta S$ and β_n is a certain random variable on Ω_2^i. Summing up, since $J\Delta S = 0$ a.s. on Ω_2, then $\overline{K}_n \Delta S = K_n \Delta S$ on Ω_2. We repeat the entire procedure on each Ω_2^i with the sequence $\overline{K}_n$ knowing that $\overline{K}_n^i = 0$ a.s. on Ω_2^i. Remember that on each subset of Ω_2^i such that $\liminf \|\overline{K}_n\|_d < \infty$ a.s. we work with the sequence $\overline{K}_n \Delta S - r_n$ selected earlier so that $\lambda_n \to \lambda$. n these subsets of Ω_2^i on which $\liminf \|\overline{K}_n\|_d = \infty$ a.s. we apply the same arguments based on the elimination of non-zero components. (Notice that if on a certain subset of Ω_2^i we eliminate all coordinates, i.e. we receive the sequence $\overline{K}_n$ such that $\overline{K}_n^i = 0$ a.s. on this set for all $i = 1, \ldots, d$ (such a situation might not appear), then on this set, in particular, $r_n \to \zeta$ pointwise a.s. and by Lemma 2 the limit belongs to $\mathcal{L}_+$. Since $\lambda_n \to \lambda$, then it is of the form λI_B, where $B \in \mathcal{F}$).

After a finite number of steps we construct the desired sequences.We can denote them by $\tilde{K}_n$ (it is the sequence of $\mathcal{F}_0$-measurable random vectors taking values in $\mathbb{R}^d$ and convergent pointwise a.s. on Ω) and $\tilde{r}_n$ (it is the sequence of non-negative random variables converges pointwise a.s. on Ω). Furthermore, these sequences were constructed so that $\tilde{K}_n \Delta S - \tilde{r}_n \to \zeta$ a.s. on Ω. Therefore, our limit is of the form $\zeta = \tilde{K}\Delta S - \tilde{r}$ a.s. on Ω, where $\tilde{K} := \lim \tilde{K}_n$ and $\tilde{r} := \lim \tilde{r}_n$ (limits are pointwise a.s. on Ω). Moreover, $\tilde{K} \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ as the limit of a convergent pointwise a.s. sequence of random vectors taking values in $\mathbb{R}^d$. The second limit is of the form $\tilde{r} = \lambda I_{A \cup (\cup_i A_i) \cup (\cup_j B_j)}$, where $A_i, B_j \in \mathcal{F}$. The first union of the sets A_i is finite and appears during continuing the procedure

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on the sets Ω_2^i and its subsets such that at a certain stage $\liminf \|\overline{K}_n\|_d < \infty$ on these sets. The second union of the sets B_j is finite and appears during continuing the procedure on the sets Ω_2^i and its subsets such that at every stage $\liminf \|\overline{K}_n\|_d = \infty$ and finally $\overline{K}_n^i = 0$ a.s. on these sets for $i = 1, \dots, d$. Notice that random vectors $\overline{K}_n$ are the new sequences constructed using the procedure performed in detail on Ω_2. All in all (using Lemma 5), by the choice of the sequence r_n such that $\lambda_n \to \lambda$ and taking into account that the countable union of sets from σ-algebra belongs to σ-algebra, we conclude that $\tilde{r} \in \mathcal{L}_+$.

Now we show that the claim is true for any $T > 1$. To this end suppose that $\sum_{t=1}^T H_t^n \Delta S_t - r_n \to \zeta$ pointwise a.s. on Ω where $(H_t^n)_{t=1}^T \in \mathcal{P}$ and $r_n \in \mathcal{L}_+$. Assume first that $\liminf \lambda_n < \infty$ (we consider the opposite case at the end of the proof). Therefore, we can select a subsequence r_{n_k} from the sequence r_n such that $\sum_{t=1}^T H_t^{n_k} \Delta S_t - r_{n_k} \to \zeta$ a.s. on Ω and $\lambda_{n_k} \to \lambda \geq 0$. For simplicity, we will again use the same notations $\sum_{t=1}^T H_t^n \Delta S_t - r_n \to \zeta$ and $\lambda_n \to \lambda$ for these subsequences. First we will work with H_1^n using the similar reasoning as in the case $T = 1$ and sometimes the same notation.

Consider the (measurable) set $\Omega_1 := \{\omega\colon \liminf \|H_1^n(\omega)\|_d < \infty\}$. By Lemma 1 there exists an increasing sequence of integer-valued random variables τ_k such that $H_1^{\tau_k} \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ and $H_1^{\tau_k(\omega)}(\omega)$ is a convergent subsequence of the sequence $H_1^n(\omega)$ for almost all $\omega \in \Omega_1$. Moreover, $(H_t^{\tau_k})_{t=1}^T \in \mathcal{P}$. By the construction $\sum_{t=1}^T H_t^{\tau_k} \Delta S_t - r_{\tau_k} \to \zeta$ and by the convergence (a.s. on Ω_1) of the first term of the sum we have $\sum_{t=2}^T H_t^{\tau_k} \Delta S_t - r_{\tau_k} \to \zeta_1$ a.s. on Ω_1, where ζ_1 is a random variable.

Consider now the situation on the set $\Omega_2 := \{\omega\colon \liminf \|H_1^n(\omega)\|_d = \infty\}$. Put $G_t^n := \frac{H_t^n}{\|H_1^n\|_d}$, $h_n := \frac{r_n}{\|H_1^n\|_d}$ and notice that $\sum_{t=1}^T G_t^n \Delta S_t - h_n \to 0$ a.s. on Ω_2. Furthermore, by the choice of the sequence r_n (i.e. such that $\lambda_n \to \lambda$) and taking into consideration that $\liminf \|H_1^n\|_d = \infty$ a.s. on Ω_2 we coclude that $h_n \to 0$ a.s. on Ω_2. By Lemma 1 there exists an increasing sequence of integer-valued random variables σ_k such that $G_1^{\sigma_k} \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ and $G_1^{\sigma_k(\omega)}(\omega)$ is a convergent subsequence of the sequence $G_1^n(\omega)$ for almost all $\omega \in \Omega_2$. Moreover, $(G_t^{\sigma_k})_{t=1}^T \in \mathcal{P}$. By the construction $\sum_{t=1}^T G_t^{\sigma_k} \Delta S_t \to 0$ and by the convergence ($G_1^{\sigma_k} \to G_1$ a.s. on Ω_2, where $G_1 \in L^0(\mathbb{R}^d, \mathcal{F}_0)$) of the first term of the sum we conclude that $\sum_{t=2}^T G_t^{\sigma_k} \Delta S_t \to \zeta_2$ a.s. on Ω_2. By the closedness of R_T there is $\zeta_2 = \sum_{t=2}^T G_t \Delta S_t$, where $G_t \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. Summing up, $\sum_{t=1}^T G_t \Delta S_t = 0$ a. s. on Ω_2, where $(G_t)_{t=1}^T \in \mathcal{P}$.

Since $G_1(\omega) \neq 0$ a.s. on Ω_2 (it is because $G_1^{\sigma_k}(\omega)$ is a convergent subsequence of the sequence $G_1^n(\omega)$ for almost all $\omega \in \Omega_2$ and for almost all $\omega \in \Omega_2$ there is $\|G_1^n(\omega)\|_d = 1$), then there exists a partition of Ω_2 into at most d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $G_1^i(\omega) \neq 0$ a.s. on Ω_2^i. (We may obtain

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such a partition putting $\Omega_2^1 := \{\omega \in \Omega_2 \colon G_1^1(\omega) \neq 0\}$ and then continuing the partition on the set $\Omega_2 \setminus \Omega_2^1$ only, putting $\Omega_2^2 := \{\omega \in \Omega_2 \setminus \Omega_2^1 \colon G_1^2(\omega) \neq 0\}$ and so on.) Define $\overline{H}_t^n := H_t^n - \beta_n G_t$, where $\beta_n := \frac{H_1^{ni}}{G_1^i}$ on the set Ω_2^i (H_1^{ni} denotes the i-th coordinate of H_1^n). Then on each Ω_2^i the following equality holds $\sum_{t=1}^T \overline{H}_t^n \Delta S_t = \sum_{t=1}^T H_t^n \Delta S_t$. It is so, because $\sum_{t=1}^T \overline{H}_t^n \Delta S_t = \sum_{t=1}^T H_t^n \Delta S_t - \beta_n \sum_{t=1}^T G_t \Delta S_t$ and β_n is a random variable on Ω_2^i. Summing up, since $\sum_{t=1}^T G_t \Delta S_t = 0$ a.s. on Ω_2, $\overline{H}_n \Delta S = H_n \Delta S$ a.s. on Ω_2. We repeat the entire procedure on each Ω_2^i with sequences $\overline{H}_t^n$ and original r_n (i.e. such as $\lambda_n \to \lambda \geq 0$), knowing that $\overline{H}_1^{ni} = 0$ a.s. on Ω_2^i. It means that on each subset of Ω_2^i such that $\liminf \|\overline{H}_1^n\|_d < \infty$ a.s. we apply Lemma 1. On these subsets of Ω_2^i on which $\liminf \|\overline{H}_1^n\|_d = \infty$ a.s. we apply the same arguments based on the elimination of non-zero components.

After a finite number of steps we construct the sequence $(\tilde{H}_t^n)_{t=1}^T \in \mathcal{P}$. We also get a sequence $\tilde{r}_n$ of non-negative random variables such that $\sum\limits_{t=1}^{T} \tilde{H}_t^n \Delta S_t - \tilde{r}_n \to \zeta$ a.s. on Ω. Moreover, $\tilde{H}_1^n$ converges a.s. on Ω and the limit is an $\mathcal{F}_0$-measurable random vector taking values in $\mathbb{R}^d$. By the convergence of the first term of the sum $\sum_{t=2}^T \tilde{H}_t^n \Delta S_t - \tilde{r}_n \to \zeta'$ a.s. on Ω, where ζ' is a certain random variable. Working further with this sequence, we continue our reasoning with sequences $\tilde{H}_2^k, \ldots, \tilde{H}_T^k$ applying analogous arguments. Summing up, we receive the sequences $\overline{H}_1^k, \ldots, \overline{H}_T^k$ of random vectors taking values in $\mathbb{R}^d$ which converge a.s. on Ω and every $\overline{H}_t^k$ is $\mathcal{F}_{t-1}$-measurable (in particular, $\overline{H}_1^k = \tilde{H}_1^k$). Furthermore, we get the sequence $\overline{r}_n$ of non-negative random variables such that by the construction $\sum_{t=1}^T \overline{H}_t^k \Delta S_t - \overline{r}_k \to \zeta$ pointwise a.s. on Ω. It follows from the convergence of previous sequences that the sequence $\overline{r}_n$ converges pointwise a.s. on Ω and for almost all $\omega \in \Omega$ there is $\overline{r}_n(\omega)$ is a convergent subsequence of the sequence $r_n(\omega)$ where $\lambda_n \to \lambda$. By Lemma 5, $\overline{r} := \lim \overline{r}_k \in \mathcal{L}_+$. Summing up, $\zeta = \sum_{t=1}^T \overline{H}_t \Delta S_t - \overline{r} \in A_T$, where $\overline{H}_t := \lim \overline{H}_t^k$. This ends the proof of the closedness for the case $T \geq 1$ and $\liminf \lambda_n < \infty$.

We now consider the opposite case. Assume that $\sum_{t=1}^T H_t^n \Delta S_t - r_n \to \zeta$ pointwise a.s. on Ω and $\lambda_n \to \infty$. Then from condition (a) and Lemma 4 there follows $\zeta \in R_T \subseteq A_T$.

(b) $\Rightarrow$ (c) Notice that $A_T \subseteq R_T - L_+^0$ and $\mathcal{L}_+ \subseteq L_+^0$.

(c) $\Rightarrow$ (d) Trivial.

(d) $\Rightarrow$ (e) Notice that for any random variable η there is an equivalent probability P' with bounded density such that $\eta \in L^1(P')$ (e.g., $P' = Ce^{-|\eta|}P$). Property (d) is invariant under an equivalent change of probability. This consideration allows as to assume that all S_t are integrable. The convex set

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$A^1_T := \overline{A}_T \cap L^1$ is closed in L^1. Moreover, $A^1_T \supseteq -\mathcal{L}_+$. Since $A^1_T \cap \mathcal{L}_+ = \{0\}$, Lemma 6 ensures the existence of $\tilde{P} \sim P$ with a bounded density and such that $\tilde{E}\xi \le 0$ for all $\xi \in A^1_T$, in particular, for $\xi = \pm H_t \Delta S_t$ where H_t is bounded and $\mathcal{F}_{t-1}$-measurable. Thus, $\tilde{E}(\Delta S_t|\mathcal{F}_{t-1}) = 0$.

(e) $\Rightarrow$ (a) It follows from the Dalang–Morton–Willinger Theorem. □

Remark 7. Notice that we use Lemma 4 in the proof of the implication (a) $\Rightarrow$ (b) only in the case of $\lambda_n \to \infty$. Hence, if $\sum_{t=1}^T H^n_t \Delta S_t - r_n \to \zeta$ a.s. on Ω where $(H^n_t)_{t=1}^T \in \mathcal{P}$, $r_n = \lambda_n I_{A_n} \in \mathcal{L}_+$ and $\liminf \lambda_n < \infty$ then $\zeta \in A_T$ irrespective of the absence of arbitrage.

Remark 8. We here enclose the proof of the implication (e) $\Rightarrow$ (a) from the classical Dalang–Morton–Willinger Theorem (see e.g. [**3, 4**]). By Remark 2 we can without lost of generality assume that all strategies H are bounded. Let $\xi \in (R_T - L^0_+) \cap L^0_+$, i.e. $0 \le \xi \le H \cdot S_T$. Since $H_t\tilde{E}(\Delta S_t|\mathcal{F}_{t-1}) = \tilde{E}(H_t\Delta S_t|\mathcal{F}_{t-1}) = 0$, we obtain by conditioning that $\tilde{E}H \cdot S_T = 0$. Thus, $\xi = 0$.

3. Characterisation of the convexity of A_T and examples.

Remark 9. Assume that $A_T \cap \mathcal{L}_+ = \{0\}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. If the market is complete, i.e. for every contingent claim $X \in L^0(\mathcal{F}_T)$ there exists replicable strategy (i.e. $x \in \mathbb{R}$ and $H = (H_t)_{t=1}^T \in \mathcal{P}$ such that $X = x + HS_T$) then A_T is convex.

Proof. Let $x - \mu_1 I_{A_1}, y - \mu_2 I_{A_2} \in A_T$ where $x, y \in R_T$ and $\mu_1 I_{A_1}, \mu_2 I_{A_2} \in \mathcal{L}_+$. It is enough to show that $\lambda_1(x - \mu_1 I_{A_1}) + \lambda_2(x - \mu_2 I_{A_2}) \in A_T$ for every $\lambda_1, \lambda_2 \ge 0$. Since the market is complete, then $\mu_1 I_{A_1} = a + \tilde{x}$ and $\mu_2 I_{A_2} = b + \tilde{y}$, where $a, b \in \mathbb{R}$ and $\tilde{x}, \tilde{y} \in R_T$. Notice that $a, b \ge 0$ due to the condition $A_T \cap \mathcal{L}_+ = \{0\}$. Hence, using the fact that R_T is a convex cone, we have

$$\begin{aligned}\lambda_1(x - \mu_1 I_{A_1}) + \lambda_2(y - \mu_2 I_{A_2}) &= \lambda_1 x + \lambda_2 y - \lambda_1(a + \tilde{x}) - \lambda_2(b + \tilde{y}) \\ &= \lambda_1(x - \tilde{x}) + \lambda_2(y - \tilde{y}) - (\lambda_1 a + \lambda_2 b) I_\Omega \in (R_T - \mathcal{L}_+) = A_T.\end{aligned}$$

□

We now formulate a proposition stating that certain conditions prevent A_T from being convex.

Proposition 1. *Suppose that $\overline{A}_T \subseteq R_T - L^0_+$ and $\overline{A}_T \ne R_T - L^0_+$. Then A_T is not convex.*

Proof. By the assumptions there exists $x \in R_T - L^0_+$ such that $x \notin \overline{A}_T$. Let $x = HS_T - l$, where $HS_T \in R_T$ and $r \in L^0_+$. Since l is a non-negative random variable we can write $l = \lim l_n$ where $l_n = \sum_{i=1}^{k_n} \lambda^n_i I_{A^n_i}$ are certain simple functions and $k_n \ge 2$ for n large enough. Hence $x = \lim(HS_T - l_n)$,

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where $(HS_T - l_n) \in R_T - L^0_+$. Since $x \notin \overline{A}_T$ then there is no sequence $y_n \in A_T$ such that $x = \lim y_n$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $HS_T - l_n \notin A_T$. In particular, $HS_T - l_{n_0} = HS_T - \sum_{i=1}^{k_{n_0}} \lambda_i^{n_0} I_{A_i^{n_0}} \notin A_T$. Notice that $x_i := HS_T - k_{n_0}\lambda_i^{n_0} I_{A_i^{n_0}} \in A_T$ but the convex combination $\sum_{i=1}^{k_{n_0}} \frac{1}{k_{n_0}} x_i = (HS_T - l_{n_0}) \notin A_T$. Hence A_T is not convex. □

Remark 10. If we assume that $A_T \neq R_T - L^0_+$ and $(R_T - L^0_+) \cap L^0_+ = \{0\}$ then by Theorem 1 the assumptions of Proposition 1 are satisfied. Notice that in the proof of the closedness of A_T we did not use the assumption of the convexity of A_T. Similarly, when we assume that $A_T \neq R_T - L^0_+$ and $A_T \overline{A}_T$ then the assumptions of Proposition 1 are also valid. Hence the corollary below follows easily.

Corollary 1. *If A_T is convex then $A_T = R_T - L^0_+$ or $(R_T - L^0_+) \cap L^0_+ \neq \{0\}$. In particular, when A_T is convex and $A_T \neq R_T - L^0_+$ then we have an arbitrage in the model, i.e. $(R_T - L^0_+) \cap L^0_+ \neq \{0\}$.*

Example 1. Now we show that the assumption of A_T being convex can not be omitted. Consider the probability space $([0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and λ is the Lebesgue measure. Fix $T = 1$, $d = 1$ and assume that the probability space is equipped with the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{B}$. Define $\Delta S_1(\omega)\omega$ where $\omega \in [0,1]$. Then $(R_T - \mathcal{L}_+) \cap \mathcal{L}_+ = \{0\}$, because $h\Delta S_1 - \alpha I_A \neq \beta I_B$ for $\alpha, \beta > 0$, $h \in \mathbb{R}$ and $A, B \in \mathcal{F}$ (of course, except for case 0). Moreover, the set $R_T - \mathcal{L}_+$ is closed in probability in this particular case. It can be checked easily. For example, one can take a sequence $h_n \Delta S_1 - \lambda_n I_{A_n} \in R_T - \mathcal{L}_+$ such that $h_n \Delta S_1 - \lambda_n I_{A_n} \to \zeta$ a.s. on Ω, where ζ is a random variable and show that $\zeta \in R_T - \mathcal{L}_+$. Because $h_n \in \mathbb{R}$ and $\Delta S_1(\omega) = \omega$, then by the convergence to ζ the sequence h_n is bounded and we can take a convergent subsequence $h_{n_k} \to h \in \mathbb{R}$. Then $h_{n_k} \Delta S_1 - \lambda_{n_k} I_{A_{n_k}} \to \zeta$ and also $\lambda_{n_k} I_{A_{n_k}} \to \lambda I_A \in \mathcal{L}_+$. Hence $\zeta \in R_T - \mathcal{L}_+$. Therefore, $\overline{R_T - \mathcal{L}_+} \cap \mathcal{L}_+ = \{0\}$. While there is no arbitrage in the model, the set $R_T - \mathcal{L}_+$ is not convex.

Example 2. Assume that $T = 1$, $d = 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, A, \Omega \setminus A, \Omega\}$ and $A \in \mathcal{F}$ is such that $0 < P(A) < 1$. Define $\Delta S_1 = \alpha I_A - \beta I_{\Omega \setminus A}$ where $\alpha, \beta > 0$. Then $R_T = \{h(\alpha I_A - \beta I_{\Omega\setminus A})\colon\ h \in \mathbb{R}\}$ and $(R_T - L^0_+) \cap L^0_+ = \{0\}$. In particular, A_T is convex and $A_T \cap \mathcal{L}_+ = \{0\}$. Therefore, there is no arbitrage in the model, but $A_T = R_T - L^0_+$, even though $\mathcal{L}_+ \neq L^0_+$.

4. Concluding remarks. We can also adapt these new no-arbitrage criteria to the case of a model where the investor's decisions are based on a partial

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information. Such a model was proposed in [**5**]. It corresponds to the situation when a filtration can be smaller than a filtration generated by the price process.

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