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DEGENERATE SINGULARITIES AND THEIR MILNOR NUMBERS

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Abstract. We give an example of a curious behaviour of the Milnor number with respect to evolving degeneracy of an isolated singularity in $\mathbb{C}^2$.

1. Introduction. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *isolated singularity*, i.e. let f be a holomorphic function germ with an isolated critical point at 0. The *Milnor number of f* is defined as

$$\mu(f) := \dim_{\mathbb{C}} \left(\frac{\mathcal{O}^n}{(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)\mathcal{O}^n} \right).$$

The number turns out to be a topological invariant of a singularity (see [**7**]). We also put $\mu(f) = 0$ for a smooth f.

To the singularity f a combinatorial object is also associated: – its Newton diagram $\mathcal{N}(f)$. Under some non-degeneracy conditions on f (see Section 2) the Milnor number $\mu(f)$ can be computed from its Newton diagram. It is the celebrated Kouchnirenko Theorem (see [**5**] or [**8**] for the case $n = 2$).

Theorem 1. *There exists a number $\nu(f)$, called the Newton number of f, depending on the Newton diagram of f only and such that*

1. $\nu(f) \leqslant \mu(f)$,
2. *if f is non-degenerate,* $\nu(f) = \mu(f)$.

Although Theorem 1 is valid in any dimension, the inverse implication in (2), as observed by Płoski [**8, 9**] and [**1**], is true for $n = 2$ only.

Theorem 2. *If $n = 2$ and $\nu(f) = \mu(f)$, then f is non-degenerate.*

The fact that Theorem 2 is not true in general, was already noticed by Kouchnirenko [**5**, Remarque 1.21], see also Example 1.

In light of Theorem 2, the two-variable case seems to be very special. Indeed, it turns out that in this case there exists a complete characterisation of non-degeneracy of a singularity f in a coordinate system, in terms of intrinsic topological invariants of f (see [**1**]).

Let us explicitly list some other properties of Milnor numbers.

i. $\mu(\cdot)$ is upper semi-continuous w.r.t. holomorphic unfoldings (see [**2**, Theorem 2.6]).

ii. $\mu(\cdot)$ is an increasing function on the set of *non-degenerate* singularities partially ordered by the relation

$$f \preccurlyeq g \Leftrightarrow \mathcal{N}(f) \supset \mathcal{N}(g),$$

where $f, g \in \mathcal{O}^n$; a simple proof of this fact can be found in [**3**].

iii. Let $n = 2$ and f be non-degenerate. A simple consequence of Theorem 1 and Theorem 2. is that if f 'gets degenerated' on *any* face $\mathcal{S}$ of $\mathcal{N}(f)$, then its Milnor number increases. Precisely, if $g = f + r$ is another isolated singularity such that $\mathcal{N}(f) = \mathcal{N}(g)$ and g is degenerate on $\mathcal{S} \in \mathcal{N}_0(f)$ (see Section 2 for definitions) then $\mu(g) > \mu(f)$.

In the paper we examine the possibility of extending property (iii) onto the case of degenerate singularities (Section 3). Our first result is that it cannot be done in a verbatim way. Namely, we give an example (Example 2) of a singularity f such that, f having been degenerated on one segment of its Newton boundary, its Milnor number decreases. The second result gives such a method of degenerating f under which the Milnor number increases (Proposition 2).

2. Definitions and auxiliary properties. In this section we briefly recall the necessary basics. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with an expansion of the form

$$f = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha x^\alpha,$$

where the usual multi-index notation is applied. We define the *support of f* as $\operatorname{Supp} f := \{\alpha \in \mathbb{N}_0^n : f_\alpha \neq 0\}$ and the *Newton diagram of f* as $\mathcal{N}(f) := \operatorname{conv}(\operatorname{Supp} f + \mathbb{N}_0^n)$. The set of the compact faces of $\mathcal{N}(f)$ of positive dimension is called the *Newton boundary of f* and is denoted by $\mathcal{N}_0(f)$. f is said to be *convenient* if $\mathcal{N}_0(f)$ meets each of the coordinate axes. For a convenient f we denote by $\mathcal{N}_-(f)$ the compact polytope defined as $\overline{\mathbb{R}_+^n \setminus \mathcal{N}(f)}$. Then the *Newton number $\nu(f)$ of f* is defined by

$$\nu(f) := n! V_n - (n-1)! V_{n-1} + \ldots + (-1)^{n-1} V_1 + (-1)^n,$$

where V_n is the n-dimensional volume of $\mathcal{N}_-(f)$ and for $1 \leqslant k \leqslant n-1$, V_k is the sum of the k-dimensional volumes of the intersections of $\mathcal{N}_-(f)$ with the coordinate planes of dimension k. If f is an isolated singularity and f is not convenient, it can be made convenient by adding to it high enough powers of the missing variables and then the formula above makes sense for the changed f. It can be shown that such operations on f lead to the same Newton numbers and so – to the definition of $\nu(f)$ in the general case of isolated singularities (cf. [**5, 6, 11**]).

The non-degeneracy condition, which is the key to the Kouchnirenko Theorem, can be formulated as follows.

DEFINITION 1. For $\mathcal{S} \in \mathcal{N}_0(f)$ let

$$\mathrm{in}_{\mathcal{S}}(f) := \sum_{\alpha \in \mathcal{S} \cap \mathrm{Supp}\, f} f_\alpha x^\alpha.$$

We say that f is non-degenerate on $\mathcal{S}$ if the system

$$\nabla\, \mathrm{in}_{\mathcal{S}}(f) = 0$$

has no solutions in $(\mathbb{C}^*)^n$, where ∇ denotes the gradient of a function. If f is non-degenerate on every $\mathcal{S}$, we say that f is (Kouchnirenko) non-degenerate. In the opposite case, we say f is degenerate.

Let us recall the fol lowing simple properties.

PROPOSITION 1. *Let* $\mathcal{S} \in \mathcal{N}_0(f)$. *Then:*

i. $f_{\mathcal{S}}$ *is quasi-homogeneous,*
ii. if $f_{\mathcal{S}}$ *has two terms only, then* f *is non-degenerate on* $\mathcal{S}$,
iii. if $f_{\mathcal{S}}$ *has a multiple factor that is not a monomial, then* f *is degenerate on* $\mathcal{S}$,
iv. for $n = 2$ *the converse of (iii) also holds.*

PROOF. Items (i)–(iii) are straightforward. The item (iv) follows from Euler's formula for quasi-homogeneous polynomials. □

We cite the Kouchnirenko example, which shows that, when $n \geqslant 3$, the above-defined nondegeneracy condition is actually too strong for the conclusion of (2) in Theorem 1 to be true.

EXAMPLE 1. Let $f = (x_1 + x_2)^2 + x_1 x_3 + x_3^2$. Note that f is convenient and quasi-homogeneous. By Proposition 1 we conclude that f is degenerate on the face $\mathcal{S} = \overline{(1,0,0)(0,1,0)}$, because $f_S = (x_1 + x_2)^2$. A simple calculation shows that $\nabla f = (2(x_1 + x_2) + x_3, 2(x_1 + x_2), x_1 + 2x_3)$ generates the ideal (x_1, x_2, x_3) in $\mathcal{O}^3$ and so there is $\mu(f) = 1$, by the definition of the Milnor number. On the other hand, it is easy to see that $\nu(f) = 3! \cdot \frac{4}{3} - 2! \cdot 6 + 1! \cdot 6 - 1 = 1$, so $\nu(f) = \mu(f)$, although f is degenerate.

In the case of $n = 2$, we recall a notion of non-degeneracy for pairs of holomorphic germs. We say that a pair $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is *non-degenerate* (see [**8**]) if for every $\mathcal{S} \in \mathcal{N}_0(f), \mathcal{T} \in \mathcal{N}_0(g)$ one of the following conditions holds

a) $\mathcal{S} \nparallel \mathcal{T}$,
b) $\mathcal{S} \| \mathcal{T}$ and the system $f_{\mathcal{S}} = g_{\mathcal{T}} = 0$ has no solutions in $(\mathbb{C}^*)^2$.

Then the following is true (see [**8**, Theorem 1.2]).

Theorem 3. *For any non-degenerate pair $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of convenient germs the intersection multiplicity $(f, g)_0$ of f and g depends on the pair $(\mathcal{N}_0(f), \mathcal{N}_0(g))$ only.*

Finally, the following formula holds (it is an immediate consequence of [**4**, Proposition 4] and [**7**, Theorem 10.5]; see also [**2**, Lemma 3.32] and [**2**, Proposition 3.35], respectively).

Theorem 4. *If $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and fg is an isolated singularity then*

$$\mu(fg) = \mu(f) + 2(f, g)_0 + \mu(g) - 1.$$

3. Degenerate singularities and their Milnor numbers. First we give an example showing that item (iii) of Introduction cannot be generalised in the form analogical to the degenerate case, even for $n = 2$.

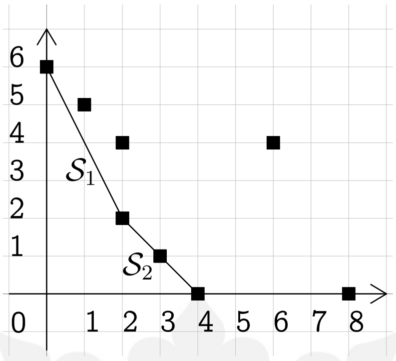
Example 2. Consider three germs $f_1, f_2, f_3 \in \mathcal{O}^2$:

$$\begin{aligned} f_1 &:= y - x - x^3, \\ f_2 &:= y - x + x^3, \\ f_3 &:= x^2 + y^4. \end{aligned}$$

We define

$$\begin{aligned} f &:= f_1 f_2 f_3 \\ &= (y^6 + x^2y^2 - 2x^3y + x^4) - 2xy^5 + (x^2 - x^6)y^4 - x^8. \end{aligned}$$

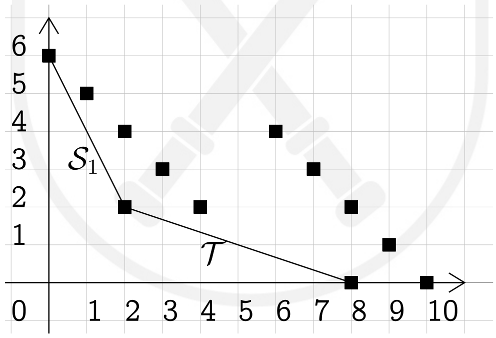
The Newton diagram of f is of the following form.



From the definition of f it follows that $f_{\mathcal{S}_1} = y^6 + x^2y^2 = y^2(y^4 + x^2)$ and $f_{\mathcal{S}_2} = x^2y^2 - 2x^3y + x^4 = x^2(y - x)^2$, and so by Proposition 1, f is non-degenerate on $\mathcal{S}_1$ and degenerate on $\mathcal{S}_2$. It is easily seen that $\nu(f) = 11$. In order to compute $\mu(f)$ we change the coordinates: $x \mapsto x, y \mapsto x + y$. Then f takes the form

$$\begin{aligned}\hat{f} := f(x, x+y) = (y^6 + x^2y^2 - x^8) + (4xy^5 + 6x^2y^4 + 4x^3y^3 + x^4y^2)\\ + (-x^6y^4 - 4x^7y^3 - 6x^8y^2 - 4x^9y - x^{10})\end{aligned}$$

and the Newton diagram of $\hat{f}$ is



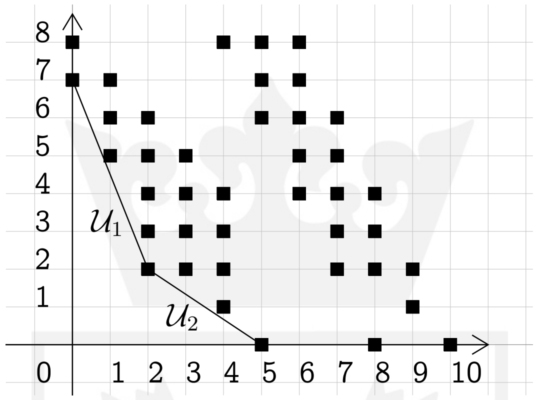
Now it is $\hat{f}_{\mathcal{S}_1} = y^6 + x^2y^2$ and $\hat{f}_{\mathcal{T}} = x^2y^2 - x^8$, so again by Proposition 1 we conclude that $\hat{f}$ is non-degenerate. Since the Milnor number is an analytic invariant of a singularity then by Kouchnirenko Theorem, $\mu(f) = \mu(\hat{f}) = \nu(\hat{f}) = 15$.

The next step is to 'degenerate' f on $\mathcal{S}_1$. Namely, we consider $g := f + 2xy^4$. It is obvious that from the point of view of the Newton boundary, the only difference between f and g is on the segment $\mathcal{S}_1$: $g_{\mathcal{S}_1} = f_{\mathcal{S}_1} + 2xy^4 = y^2(y^2+x)^2$ and $g_{\mathcal{S}_2} = f_{\mathcal{S}_2}$. We conclude that g is degenerate on the both of its segments – on $\mathcal{S}_1$ and $\mathcal{S}_2$. Since $\mathcal{N}_0(g) = \mathcal{N}_0(f)$ then $\nu(g) = \nu(f) = 11$. To compute $\mu(g)$

we change the coordinates once again: $x \mapsto x - y^2, y \mapsto x + y$. We obtain

$$\hat{g} := g(x - y^2, x + y) = (4y^7 + x^2y^2 + 2x^5) - 8xy^5 - 5x^2y^4 - 4x^3y^3 + 10x^2y^3 - x^4y^2 + 12x^3y^2 + 8x^4y + \text{other terms of degree} \geqslant 7.$$

The (truncated above 8) Newton diagram of $\hat{g}$ is of the form



and we can see that $\hat{g}_{\mathcal{U}_1} = 4y^7 + x^2y^2$ and $\hat{g}_{\mathcal{U}_2} = x^2y^2 + 2x^5$. It means that $\hat{g}$ is nondegenerate and we can apply Theorem 1 to compute its Milnor number. We obtain $\mu(\hat{g}) = \nu(\hat{g}) = 13$. By the invariancy of the Milnor number, we conclude that

$$\mu(g) = \mu(\hat{g}) = 13 < 15 = \mu(f).$$

Summing up, we have found f and g such that:

1. $\operatorname{Supp} g = \operatorname{Supp} f \cup \{\text{single point}\}$,
2. $\mathcal{N}_0(f) = \mathcal{N}_0(g) = \{\mathcal{S}_1, \mathcal{S}_2\}$,
3. f is non-degenerate on $\mathcal{S}_1$ and degenerate on $\mathcal{S}_2$,
4. g is degenerate on $\mathcal{S}_1$ and $\mathcal{S}_2$,
5. $\mu(f) > \mu(g)$.

The above shows that the Milnor number is not 'monotonic with respect to degeneracy,' in general.

A positive result concerning the problem can also be given. Let f be an isolated and convenient singularity with $\#\mathcal{N}_0(f) \geqslant 2$, and let $\mathcal{S}_0 \in \mathcal{N}_0(f)$. Let g be the factor of f associated to this segment. It means that there exists a decomposition of f of the form $f = gh$ such that:

i. $\mathcal{N}_0(g) = \{\mathcal{S}_1\}$ and $\mathcal{S}_1 \| \mathcal{S}_0$,

ii. $\bigwedge_{\mathcal{T} \in \mathcal{N}_0(h)} \mathcal{T} \nparallel \mathcal{S}_0$

(see e.g. [**12**, Lemma 2.44]). The following holds.

PROPOSITION 2. *Assume that f is non-degenerate on $\mathcal{S}_0$ and that there exists an isolated singularity $\tilde{g}$ such that $\mathcal{N}_0(\tilde{g}) = \{\mathcal{S}_1\}$ and $\tilde{g}$ is degenerate. Define $\tilde{f} := \tilde{g}h$. Then:*
1. $\mathcal{N}_0(f) = \mathcal{N}_0(\tilde{f})$,
2. $\tilde{f}$ is degenerate on $\mathcal{S}_0$,
3. $\mu(f) < \mu(\tilde{f})$.

PROOF. By assumption $\mathcal{N}_0(g) = \mathcal{N}_0(\tilde{g})$. Since f is convenient, g and $\tilde{g}$ are convenient, too. This implies that $\mathcal{N}(g) = \mathcal{N}(\tilde{g})$. Now the first assertion follows from the known properties of Newton diagrams (see [**10**, Section 3.6] or [**1**])

$$\mathcal{N}(f) = \mathcal{N}(gh) = \mathcal{N}(g) + \mathcal{N}(h) = \mathcal{N}(\tilde{g}) + \mathcal{N}(h) = \mathcal{N}(\tilde{g}h) = \mathcal{N}(\tilde{f}).$$

We claim that conditions (i) and (ii) of the assumption imply that the following is true:

$$f \text{ is non-degenerate on } \mathcal{S}_0 \Leftrightarrow g \text{ is non-degenerate on } \mathcal{S}_1 \tag{1}$$

and similarly

$$\tilde{f} \text{ is degenerate on } \mathcal{S}_0 \Leftrightarrow \tilde{g} \text{ is degenerate on } \mathcal{S}_1. \tag{2}$$

Indeed, if $v \perp \mathcal{S}_0$ is a vector with positive integer coefficients and we consider the v-gradation on $\mathcal{O}^2$, then denoting by in_v the initial form operator with respect to this gradation, we see that

$$f_{\mathcal{S}_0} = \mathrm{in}_v f = \mathrm{in}_v g \cdot \mathrm{in}_v h = g_{\mathcal{S}_1} \cdot \text{(a monomial)}.$$

Using Proposition 1 (iii)–(iv) we arrive at (1). The argument for $\tilde{f}$ runs in the same way so also (2) holds, and in particular the second assertion is proved.

Now note that by assumption (ii) the pair (g, h) is non-degenerate, and since $\mathcal{N}_0(\tilde{g}) = \{\mathcal{S}_1\}$ the same is true for the pair $(\tilde{g}, h)$. By Theorem 3, it means that $(\tilde{g}, h)_0 = (g, h)_0$. On the other hand, since $\tilde{g}$ is degenerate (by assumption) and g is not (by (1)), Theorem 2 asserts that $\mu(\tilde{g}) > \mu(g)$. Summing up

$$\begin{aligned}\mu(\tilde{f}) = \mu(\tilde{g}h) &= \mu(\tilde{g}) + 2(\tilde{g}, h)_0 + \mu(h) - 1 \\ &> \mu(g) + 2(g, h)_0 + \mu(h) - 1 = \mu(gh) = \mu(f).\end{aligned}$$

□

We illustrate the proposition with the following example.

EXAMPLE 3. Consider f of Example 2. We can take $g = f_3 = x^2 + y^4$ and $h = f_1 f_2$. Then $\mathcal{N}_0(g) = \{\mathcal{S}_0\} = \{\overline{(0,4),(2,0)}\}$ and $\mathcal{S}_0 \| \overline{(0,6),(2,2)} = \mathcal{S}_1 \in \mathcal{N}_0(g)$. We consider a degenerated g, e.g. $\tilde{g} := (x + y^2)^2 + x^3$ and define $\tilde{f} := \tilde{g}h = f + (2xy^2 + x^3) f_1 f_2$. It is easy to see that $\mu(\tilde{f}) = 17 > 15 = \mu(f)$ (one can use Theorem 4 to perform the calculation).

Remark 1. It is possible to give a parametric version of Example 2. Namely, one can consider $f_1 := y - x - sx^3$, $f_2 := y - x + sx^3$, $f_3 := y^4 + s^2x^2$ and ${}^s f := f_1 f_2 f_3 + x^7$, where $|s| \ll 1$. Additionally, let ${}^{st} f := {}^s f + 2txy^4$, for $|s|, |t| \ll 1$. Then ${}^{st} f$ is a holomorphic unfolding of $f_0 := (y - x)^2 y^4 + x^7$. It is easy to see that $\mathcal{N}_0({}^{st} f) = \mathcal{N}_0({}^s f) = \{\mathcal{S}_1, \mathcal{S}_2\}$, where $\mathcal{S}_1, \mathcal{S}_2$ are the segments as in Example 2. One can check that ${}^{st} f_{\mathcal{S}_1} = y^2(y^4 + 2txy^2 + s^2x^2)$ and ${}^{st} f_{\mathcal{S}_2} = s^2x^2(y - x)^2$. It means that ${}^{st} f$ is non-degenerate on $\mathcal{S}_1$ for $s \neq t$ and degenerate on $\mathcal{S}_2$. Here $\mu({}^{st} f) = 12$ for $s \neq t$ different from 0. In accordance with property (i) of Introduction, $\mu({}^{st} f\,|_{t=0}) = \mu({}^s f) = 14 > 12$ and $\mu({}^{st} f\,|_{t=s}) = 13 > 12$. However, ${}^{st} f\,\big|_{t=s} = {}^s f + 2sxy^4$ is degenerate on $\mathcal{S}_1$, while ${}^s f$ is not, and yet $\mu({}^{st} f\,|_{t=s}) < \mu({}^s f)$. The skipped calculations are similar to those of Example 2.

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