

PIOTR JAŚKOWSKI, MICHAŁ TOMCZAK\*

## ASSIGNMENT PROBLEM AND ITS EXTENSIONS FOR CONSTRUCTION PROJECT SCHEDULING

### ZAGADNIENIE PRZYDZIAŁU I JEGO MODYFIKACJE W HARMONOGRAMOWANIU PROCESÓW BUDOWLANYCH

#### Abstract

The assignment problem consists of allocating renewable resources (construction equipment, crews, or contractors) of limited availability to a set of activities. The classical model for this problem minimizes the total time or cost of completing all activities with the assumption that each activity is assigned to one particular resource. This paper systematizes and describes extensions of these assumptions, considering the effects of task sequence: parallel, serial and hybrid (modeled by means of network methods). This study proposes algorithms for the solution of presented models, which can be used in construction project scheduling.

*Keywords: assignment problem, project scheduling, mathematical modeling, renewable resources, bottleneck assignment problem*

#### Streszczenie

W zagadnieniu przydziału (znanym również w literaturze pod nazwą zagadnienia rozmieszczenia) rozważa się problem alokacji ograniczonej liczby zasobów odnawialnych (maszyn, brygad, wykonawców) do realizacji zbioru zadań. W klasycznym modelu tego problemu jest minimalizowany łączny czas lub koszt wykonania wszystkich zadań przy założeniu, że każda jednostka organizacyjna jest przydzielona do realizacji innego procesu. W artykule przedstawiono usystematyzowanie i modyfikacje tych założeń z uwzględnieniem różnej kolejności realizacji procesów: równoczesnej, kolejnej i mieszanej (modelowanej za pomocą metod sieciowych). Zaproponowano algorytmy rozwiązania opracowanych modeli, które mogą być stosowane w harmonogramowaniu procesów budowlanych.

*Słowa kluczowe: zagadnienie przydziału, harmonogramowanie, modelowanie matematyczne, zasoby odnawialne, metoda minimaksowa*

\* Ph.D. Piotr Jaśkowski, M.Sc. Michał Tomczak, Department of Construction Project Engineering, Faculty of Civil Engineering, Lublin University of Technology.

### 1. Introduction

Assignment problems (selection and allocation of resources) are one of the primary task in construction process harmonization [9]. In its classical formulation  $n$  teams (crews, sets of construction plant, contractors) are assigned to  $n$  activities, in order to minimize the total time or cost of completing all of them. Arrangement of these activities depends on technological and organizational requirements. These requirements concurrently affect the allocation constraints, such as assigning only one job to each crew. With respect to technological constraints, options for conducting construction processes in parallel, in series, or in a hybrid way may exist (Fig. 1).

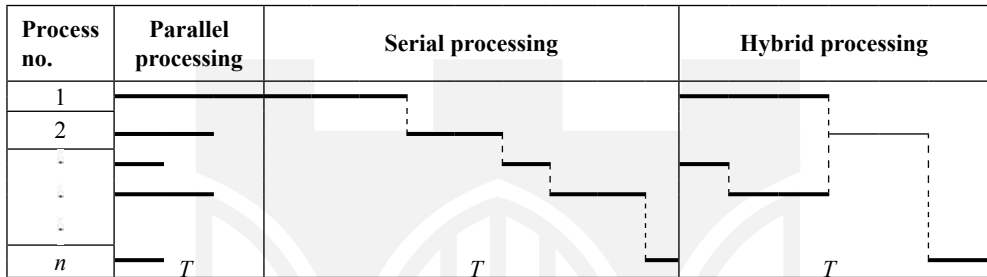


Fig. 1. Comparisons of different processing options in processes scheduling

Parallel processing consists of the simultaneous execution of activities by different resources in separate building units (work zones). The greatest advantage of this technique is the shortest project time span  $T$  – when compared with other approaches. The drawbacks of the parallel technique are: the lack of teams’ work continuity and unlevelled daily demand for building materials or plant.

Serial processing consists of performing a sequence of processes in one work zone. The advantage of this method is the lowest maximum level of daily employment of renewable resources and daily usage for building materials. Each activity may be realized by a different crew, but the total duration is incommensurately long. Another disadvantage, similar to the parallel technique is discontinuity of the team’s work and unlevelled daily consumption of resources.

Hybrid processing is a combination the two previous approaches. The precedence constraints, modeled by means of network techniques, enable serial processing of some activities (on the same network path) and, simultaneously, concurrent processing of other jobs on parallel paths. The two optimization approaches are used for resource management in project networks: allocation of limited resources (in order to minimize the project makespan) and leveling resource requirements profile (in order to improve economic efficiency).

This study considers different processing options for assignment problem formulation. It has been assumed that an activity requires one resource type for its execution (crew, contractor), and the resource can perform one activity at a time. In the case of parallel processing, each activity has to be carried out by a different crew. With serial processing, each activity can, but does not have to be conducted by a different crew. Thus, in the case of hybrid processing, parallel activities have to be entrusted to different crews, and activities scheduled in sequence can but do not have to be carried out by different crews. A discrete time/cost/resource function implies the representation of an activity in different modes of operation. For each activity  $i$  (construction process), a set of

modes of execution is defined. Each mode, is described by the following parameters:  $t_{ij} \in N$  – duration of the activity  $i$  realized by crew  $j$  and  $k_{ij} \in R^+$  – cost of the activity  $i$  realized by crew  $j$ . The problem consists of choosing the optimal process modes in order to minimize the project duration or cost. Binary linear programs are developed to model the assignment problem for different processing options and recommended approach solutions are presented.

## 2. Assignment problem formulation for parallel processing

The classical assignment problem was formulated in 1952 by D.F. Votaw and A. Orden [1] as a type of transportation problem. The mathematical model of the assignment problem for parallel processing can be described by an objective function minimizing the total cost of realizing processes:

$$\min z : z = \sum_{i=1}^n \sum_{j=1}^n k_{ij} \cdot x_{ij}, \quad (1)$$

and the following constraints:

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n, \quad (2)$$

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n, \quad (3)$$

$$x_{ij} \in \{0, 1\}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, n, \quad (4)$$

where:

$x_{ij}$  – a binary variable for modeling a decision of selecting processes' modes; the variable assumes the value of 1 if the activity  $i$  is to be executed by the resource (crew)  $j$ , and equals 0 in the other case.

According to equations (2) and (3), considering equal numbers of crews and activities, each crew may be assigned only to one process and each process may be realized by only one crew.

H.W. Kuhn [2] in 1955 created the Hungarian method – exact algorithm for solving the model. He combined the ideas of two Hungarian mathematicians: D. König [3] and J. Egerváry [4]. The method finds an optimal assignment for a given square cost matrix, and consist of five steps [2, 5–7]:

Step 1 – Subtract the smallest entry in each row from all the entries of its row.

Step 2 – Subtract the smallest entry in each column from all the entries of its column.

Step 3 – Construct a minimal number of lines, which covers all the zero entries of the cost matrix with  $k_{ij}$ .

Step 4 – If number of covering lines is  $n$ , then it is complete. Otherwise, proceed to Step 5.

Step 5 – Determine the smallest, uncovered entry and subtract it from uncovered rows, and then add this entry to each covered column. Repeat Step 3.

The bottleneck assignment problem is one of the extensions of the classical formulation. It consist of the minimization of the project makespan – maximal duration of processes realized in parallel (each demanding a different crew):

$$\min z : z = \max \sum_{i=1}^n \sum_{j=1}^n t_{ij} \cdot x_{ij} \quad (5)$$

subject to the same constraints and definitions as in the classic assignment problem.

In 1959 O. Gross [8] created an algorithm which is used for solving this kind problem. It may be described as follows:

Step 1 – Begin with any of feasible solution, e.g.  $x_{ii} = 1, I = 1, 2, \dots, n$ , set of chosen entries –  $B = \{t_{ij} : x_{ij} = 1\}$ .

Step 2 – Compute  $V = \max \{t_{ij} : x_{ij} = 1\}$ .

Step 3 – Locate a cycle, which begins and ends at  $V$  as follows:

Step 3a – from  $V$  go to entry in its column with cost less than  $V$ ,

Step 3b – from not chosen entry go to the entry from  $B$  in this row; if such cycle no exist, then is done.

Step 4 – Reverse the assignments along entries, which take part in previous steps. Let  $x'_{ij} = 1 - x_{ij}$  for participating entries in Step 2. Proceed to Step 2.

### 3. Assignment problem formulation for serial processing

After completing each serial process, the renewable resources become available again and may be assigned to succeeding activities. Therefore, the constraint (2) in the classical formulation doesn't hold in this case. For each process, the whole set of crews is considered as the feasible solution space – in the problem of minimizing project duration as well as cost, appropriate modes should be selected with the shortest duration for each process or with minimal cost, respectively.

Assuming that the processes sequence will be repeated continuously in  $d$  identical work zones (units), the objective function of minimizing project duration should be modified as follows:

$$\min z : z = (d-1) \cdot \max \{t_{ij} \cdot x_{ij}\} + \sum_{i=1}^n \sum_{j=1}^n t_{ij} \cdot x_{ij} \quad (6)$$

The mathematical model (converted to linear form) with binary variables for many practical instances may be solved using any commercial optimization software.

### 4. Assignment problem formulation for hybrid processing

A construction project can be modeled as an activity–on–node network. Precedence relations between activities are modeled by a graph  $G = \langle V, E \rangle$ , directed and acyclic, with

a single initial node and a single final node, where  $V = \{1, 2, \dots, n\}$  is a set of activities, the edges (or arcs)  $E \subset V \times V$  represent precedence relations between activities.  $R$  is the set of resources – crews or contractors – available to the project. Variables  $s_i, \forall i \in V$ , stand for activities' start times.

Resources can be assigned to a number of processes, but not at the same time. Therefore, a set of processes' pairs  $J \subset V \times V$  can be defined, which can potentially be executed in parallel ( $(u, v) \in J \Leftrightarrow u < v$  and activities  $u$  and  $v$  do not lie on the same path of the project network). In the case that the resource  $j$  ( $x_{ij} = 1 \wedge x_{vj} = 1$ ) is assigned to a pair of processes  $(u, v) \in J$ , these processes cannot run at the same time, but have to be completed in sequence. The sequence is modeled by means of binary variables:  $y_{uv} \in \{0, 1\}$ , defined for  $\forall (u, v) \in J$ . The variable  $y_{uv}$  equals 1 if the activity  $u$  is to be completed before activity  $v$ , and it equals 0 in the other case.

The decision making process is aimed at selecting options of resource assignment and scheduling them in such a way that project duration is minimal. To solve the problem, a mixed integer (binary) linear program is developed to model the construction project scheduling problem. The mathematical model used for this problem is described as follows:

$$\min T : T = s_n + D_n \quad (7)$$

$$D_i = \sum_{j \in R} t_{ij} \cdot x_{ij}, \quad \forall i \in V \quad (8)$$

$$\sum_{j \in R} x_{ij} = 1, \quad \forall i \in V \quad (9)$$

$$s_1 = 0 \quad (10)$$

$$s_i + D_i \leq s_l, \quad \forall (i, l) \in E \quad (11)$$

$$s_u + D_u \leq s_v + M \cdot (1 - y_{uv}) + M \cdot (2 - x_{uj} - x_{vj}), \quad \forall (u, v) \in J, \forall j \in R \quad (12)$$

$$s_v + D_v \leq s_u + M \cdot y_{uv} + M \cdot (2 - x_{uj} - x_{vj}), \quad \forall (u, v) \in J, \forall j \in R. \quad (13)$$

$$s_i \geq 0, \quad \forall i \in V \quad (14)$$

$$x_{ij} \in \{0, 1\}, \quad \forall i \in V, \forall j \in R \quad (15)$$

$$y_{uv} \in \{0, 1\}, \quad \forall (u, v) \in J. \quad (16)$$

The objective function (7) minimizes total project duration. Equation (8) determines duration  $D_i$  of a activity  $i$  – it has been introduced as an auxiliary formula to simplify the formulas (7) and (11)–(13). According to condition (9), each activity can be executed in only one way – as selected from available options. Execution of the first activity of the project (i.e. a activity that has no predecessors) starts at the moment of 0 (10). Condition (11) defines the successors’ start dates as “not earlier than their predecessors have been completed”.

Formulas (12) and (13) are introduced to define process start times  $(u, v) \in J$ . If these processes are not to be executed by the same resource  $j$ ,  $(x_{ij} = 0 \vee x_{vj} = 0)$ , both of these conditions are automatically met ( $M$  is an arbitrarily assumed, sufficiently large constant), and the processes may run concurrently. If the same resource  $j$  is assigned to them  $(x_{uj} = 1 \wedge x_{vj} = 1)$ , and if the variable  $y_{uv}$  assumes the value of 1, then (in accordance with condition (12)), process  $v$  can only start after process  $u$  has been completed; in this case, condition (13) is automatically fulfilled. If the variable  $y_{uv}$  equals 0, then process  $v$  must be completed before  $u$  has been started – according to condition (13) and condition (12) is met automatically.

The problem of project cost minimization is trivial and in the optimal solution the crews with the lowest cost for each process are assigned to realize it. Because the project duration for this assignment may be unacceptably long, the objective could be modified as follows:

$$\min K : K = \sum_{j \in V} \sum_{j \in R} k_{ij} \cdot x_{ij} \tag{17}$$

and the following constraint needs to be added to assure not exceeding the deadline  $T_d$ .

$$s_n + D_n \leq T_d . \tag{18}$$

### 5. Example

Durations (in days) and costs (in monetary units) for completing a particular processes by particular crews in an example project is presented in matrices **T** and **K** respectively.

$$\mathbf{T} = \begin{bmatrix} 9 & 7 & 6 & 5 & 4 \\ 6 & 5 & 8 & 6 & 4 \\ 3 & 5 & 2 & 5 & 5 \\ 4 & 4 & 4 & 3 & 4 \\ 6 & 5 & 8 & 7 & 6 \end{bmatrix}, \quad \mathbf{X}_1^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The hybrid precedence relations of the processes is settled according to the technological constraints and shown in Fig. 2.

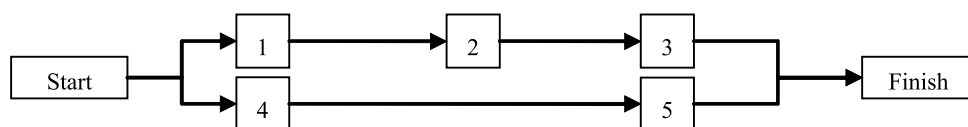


Fig. 2. Project network (example)

There are two optimal solutions of the assignment problem for parallel processes (with minimal project cost of 19 monetary units) obtained using the Hungarian algorithm:

$$\mathbf{X}_I^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{X}_{II}^1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The Hungarian algorithm prompts that there are three different optimal solutions obtained for the original assignment problem with minimal sum of processes' durations (20 days):

$$\mathbf{X}_{II}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{X}_{II}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{X}_{II}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The solution  $\mathbf{X}_{II}^1$  is also obtained by means of the algorithm by Gross. The minimal duration for the bottleneck assignment problem is 5 days. Because the solutions for both the bottleneck and the original assignment problems are the same, so this is also the optimal crew assignment for the sequence of processes repeated in different units (regardless of the units number).

The optimal solution for crew assignment processes with hybrid precedence constraints are as follows:

$$\mathbf{X}_{III} = [x_{ij}^{opt}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The minimal project duration for a given network is 10 days. Solution of the mathematical model (7)–(16) based on the above input values was found by means of LINGO 12.0 Optimization Modelling Software by Lingo Systems Inc.

## 6. Conclusions

Scheduling with the allocation of constrained resources, particularly for skilled labor, is a major challenge for almost all construction projects. Most existing techniques for project scheduling consider a single-skilled resource strategy where each worker or crew is assumed to be of one particular trade. This strategy may lead to inefficiencies in labor utilization, which can also be reflected in increased project durations and unnecessary costs. In practice each construction contractor or crew can possess several skills at different proficiency levels, i.e. they are able to perform more than one type of work (construction process), each at specified times and costs. The assumption that each worker may possess multiple skills which could allow them to participate in any activity that fits one of their skills, can improve project efficiency in terms of project cost and duration.

The resources assignment models presented in the paper can help managers in determining a strategy for crew or bids selection.

For the case of parallel processing there are exact algorithms available for solving assignment problems with low computational effort. For hybrid processing options, the problems analyzed in this paper can be considered as an extension of the Resource-Constrained Project Scheduling Problem (RCPSp) which is NP-hard. Exact algorithms may be not efficient to solve complex practical problems, therefore developing heuristic solving procedures is recommended for further research.

*This work was financially supported by Ministry of Science and Higher Education within the statutory research number S/63/2014*

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