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## LUDWIK SILBERSTEIN AND THE OPERATOR CALCULUS

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### LUDWIK SILBERSTEIN I RACHUNEK OPERATOROWY

#### Abstract

In the article we outline the life of Ludwik Silberstein (1872–1948). We present his approach to the matrix calculus and its application to the operator form of relativity. We also give the list of books on the subject written by him as well as translated by him.

*Keywords:* Ludwik Silberstein, matrix calculus, theory of relativity

#### Streszczenie

W artykule przedstawiamy zarys życiorysu Ludwika Silbersteina (1872–1948). Omawiamy podejście do rachunku macierzowego i jego zastosowanie do podania operatorowej postaci teorii względności. Prezentujemy również listę książek i tłumaczeń z różnych języków dotyczących tej tematyki.

*Słowa kluczowe:* Ludwik Silberstein, rachunek macierzowy, teoria względności

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## 1. Historical background

Ludwik Silberstein is regarded by historians of science as a Polish-American mathematical physicist.

Silberstein was born on May 15<sup>th</sup>, 1872 in Warsaw, Poland, and died in Rochester, USA on January 17<sup>th</sup>, 1948. His parents were Samuel Silberstein and Emilia, nee Steinkalk. The father took great care of the education of the three children. Henryk (1858–1890) obtained his PhD with Lothar Meyer 1882 in Tubingen in chemistry [4], while Ludwik's sister Adela (1874–?) obtained a doctorate in philosophy in Zurich [3].

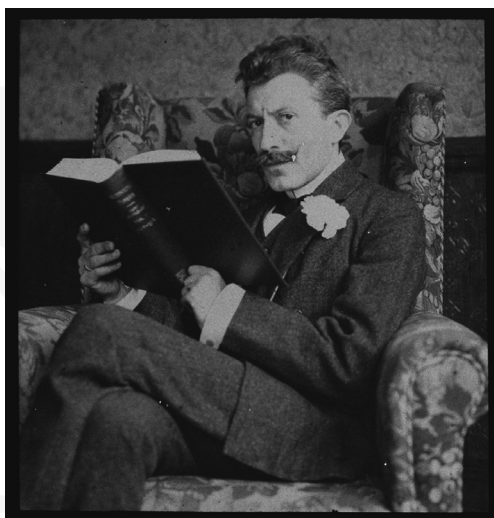


Fig. 1. Ludwik Silberstein in his home about 1915 (source: Courtesy of George Eastman House, International Museum of Photography and Film, Rochester, USA)

Ludwik went to a high school in Warsaw, and later passed the examination to attend the sixth class of the St. Hyacinth Imperial-Royal Gymnasium (c.k. Gimnazjum św. Jacka) in Cracow. His given name was Lazar, which he used until the end of high school. In 1890, he studied at Cracow's Jagiellonian University, even before completing his secondary education. He was attending calculus and laboratory classes conducted by professors August Witkowski (1854–1913) and Franciszek Karliński (1830–1906) respectively, and later always listed them among his most important teachers. There, at the Jagiellonian University, the name 'Ludwik' appeared for the first time. The name 'Lazar' is seen in the matriculation document, because all school documents contained this name; however, afterwards the name 'Lazar' never appeared. From the document it follows that he regarded himself as a Pole (Fig. 2).

After a short period of study in Cracow, Silberstein spent one semester in Heidelberg and several years in Berlin, where he obtained his doctorate at the faculty of philosophy (where Max Planck was the dean) at Berlin's Friedrich-Wilhelms-University. Von Helmholtz wrote an opinion of Silberstein's dissertation *Ueber die mechanische Auffassung*

Rok, szk. 1897/98 RODOWÓD. Półrocze: I.

Imię i nazwisko Ucznia <i>Ludwik Silberstein</i>		Wystąpienie czy jest uczniem rezerwowym lub zastępczym <i>rezerwowym</i>		Oznaczenie Wydziału <i>fizykochemii</i>	
Miejsce urodzenia <i>Warszawa</i>	Wiek <i>18</i>	Religia <i>Żydowski</i>	Narodowość <i>Polak</i>	Poddaństwo Ucznia <i>Russki</i>	Mieszkanie <i>Kraków ul. Krakowska 11</i>
Imię, stan i miejsce zamieszkania ojca jego <i>Samuel Silberstein, kupiec - Warszawa</i>					
Imię, nazwisko, stan i miejsce za- mieszkania opiekuna jego					
Wskazanie zakładu naukowego, na któ- rem Uczeń strawił ostatnie półrocze <i>Genowegum w Jassi.</i>					
Pobiera stypendium udzielone mu przez państwo		w ilości — str. — kr. 188 — Nr.		Wzrost w centymetrach w dniu 21/12 1897, północy 664 DDD	
Przytoczenie zasady na jakiej Uczeń zdał Instrukcyjności (wzrostu) lub Instrukcyjności (wzrostu)					
<i>1000/1000 = 100% prawa</i>					
Wykaz odczytów, na które Uczeń uczęszczać zamierza:					
Przedmiot wykładu	Liczba godzin tygodniowa	Nazwisko Profesora lub Starożytności	Wniosek podpisany Asystentem		
<i>Rachunek różniczkowy</i>	<i>3</i>	<i>Prof. Kertész</i>	<i>Silberstein</i>		
		<i>Dr. Dr. 5/11 1897</i>			
<i>Pracownia fizyczna</i>	<i>4</i>	<i>Prof. Böttcher</i>	<i>Silberstein</i>		

Fig. 2. Matriculation document of Ludwik Silberstein Wpis Ludwika Silbersteina na studia w Uniwersytecie Jagiellońskim (source: Jagiellonian University Archive, Cracow, Poland)

*elektromagnetischer Erscheinungen in Isolatoren und Halbleitern* [5], which circulated in the faculty from December 18, 1893.

Von Helmholtz wrote: “Mr Silberstein’s work shows clear understanding of difficult and abstract problems and aptness for mathematical treatment. I propose its assessment as: “sollertiae et ingenii specimen laudabile”. Below this document is a short note by the dean: “I agree in all points” and the signature of Max Planck. On July 5<sup>th</sup>, the PhD examination took place – the examiners were Max Planck in physics, Karl Hermann Schwarz in mathematics, Hans Landolt in chemistry, and Carl Stumpf in philosophy.

From 1895–1897 Silberstein was an assistant at the Politechnika Lwowska (Lviv Technical School) where Prof. Olearski was the head of the physics department. The scientific work and collaboration with Prof. Olearski were fruitful, but it was impossible for Silberstein to continue his work in Lviv. At that time Poland did not exist, having been partitioned by three empires: Russian, Prussian and Austro-Hungarian. As he was born in Warsaw, Silberstein was a Russian subject, while both Cracow and Lvov (Lviv, Lemberg) were situated in the Austro-Hungarian Empire. The authorities did not want to employ foreigners. His best-known paper from that period was probably *O tworzeniu się wirów w płynie doskonałym* [7], which allowed Bjerknes to formulate his theorem on turbulent motion, which formed the foundation of meteorological investigations and weather forecasting based on numerical calculations (1904) (application to eddy currents in the atmosphere).

In 1899 Silberstein took the position of lecturer (*libero docente*) in mathematical physics in Bologna, in 1904 moving to La Sapienza University in Rome. He was affiliated with Rome University until 1920, even though in the period 1912–1913 he was a lecturer in general relativity at London’s University College, where the textbook *General Relativity* was written and published (1914). In 1920 he left Europe and went to the USA, where he remained until the end of his life, living in Rochester and collaborating with the Eastman Kodak Laboratory. He published several books, with the phrase “former lecturer at Rome University” appearing below his name, which shows he was proud of this work.

On 29 June 1905 he married Rose Eisenman from Warsaw, and they had three children (Georg P., Hedwig and Hannah).

He maintained close connections with Poland practically until the end of his stay in Europe. He corresponded with several physicists, was the organizer of the summer school of theoretical physics in Zakopane in 1904, and was very helpful in the organization of Polish science. In a letter to Banachiewicz, Sierpiński (1917) advised him to ask Silberstein in order to obtain books published in Europe.

## 2. Silberstein and matrix calculus

Silberstein translated several books dealing with methodology and methods of scientific investigations, written in several languages: Polish, German, French and English (see appendix A). Also, throughout his life he wrote textbooks on vectors and operators (see appendix B).

Matrix calculus was his most prominent subject of study, and he was one of the first to notice the advantages of using this calculus in physics. There are several papers written as early as 1901 and 1902. Interestingly, he proposed the quaternion form of general relativity [10]. As we know, almost twenty years later this calculus found its main application in quantum mechanics. The novelty and importance of these papers was noted by as great a specialist as Jammer [1].

In his book *The conceptual development of quantum mechanics* [1] Max Jammer wrote:

“In concluding our brief survey on the early development of operators in mathematical physics, we should like to draw the reader’s attention to a rather unknown paper by Ludwik Silberstein [9], which anticipated to some extent the formal aspects of the operational approach in modern quantum mechanics. The study of symbolic integrals of the equations of the electromagnetic field suggested to Silberstein [8], who is known mostly for his writings on the theory of relativity, a theory of “physical operators”, in terms of which he attempted to give a unified representation of such disparate phenomena as mechanical oscillations, heat conduction, and electrodynamic process. Defining the “state” of a physical system by a time-dependent function  $\psi(t)$ ,

Silberstein introduced what he called “chrono-operators”  $\left\{ \begin{matrix} H \\ t \end{matrix} \right\}$  by means of

which the state at time  $t = t$  can be determined from the knowledge of the state at time  $t = 0$ , in accordance with the equation  $\psi(t) = \left\{ \begin{matrix} H \\ t \end{matrix} \right\} \psi(0)$ . He defined

the inverse operator  $\left\{ \begin{matrix} H \\ t \end{matrix} \right\}^{-1}$  by  $\left\{ \begin{matrix} H \\ -t \end{matrix} \right\}$ , showed that  $\left\{ \begin{matrix} H \\ nt \end{matrix} \right\} = \left\{ \begin{matrix} H \\ t \end{matrix} \right\}^n$  and that

$\left\{ \begin{matrix} H \\ t \end{matrix} \right\} = \{e^{Ft}\}$ , where  $F$  is another operator, connected with  $H$  in a definite

manner, but independent of  $t$ . Silberstein even spoke of the superposition of states and mentioned numerous other details which, including the notation, made their appearance in the operational formalism of quantum mechanics some twenty-five years later. "Every class of physical phenomena", declared Silberstein in 1901, "or at least those amenable to quantitative treatment, will be characterized by corresponding physical operators; and the scientific study of natural phenomena will proceed by detailed examinations of the properties of these operators, based on observation and nurtured by experiment".

### 3. Quaternion form of general relativity

Quaternions are quadrupels of real numbers. Quaternions can be written in different ways:

1. The matrix form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

where  $z = a + bi$ ,  $w = c + di$ .

2. The algebraic form

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

with a unit matrix  $2 \times 2$  for real  $r$ .  $r = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ . Each quaternion can be explicitly write as  $q = a + bi + cj + dk$ , where  $a, b, c, d$  are real.

Quaternions satisfy the following relations:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

$$1q = q1 = q \text{ for each } q$$

$rq = qr$  if  $r \in \mathbb{R}$ ,  $q$  a quaternion

$$q\bar{q} = |q|^2$$

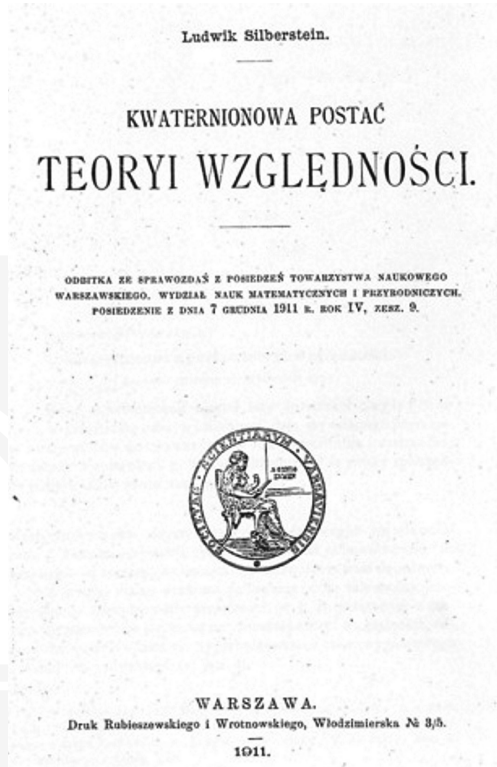


Fig. 3. The cover of the book *Kwaternionowa Postać Teorii Względności*

We thank several archives for giving us access to the documents. This work started with the collaboration of dr Hilmar Duerbeck (1948–2014), whose absence is very sad and unfortunate.

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