

KATARZYNA PAŁASIŃSKA*

EXPANSION BY A NEW CONSTANT MAY CHANGE THE FINITE AXIOMATIZATION PROPERTY OF A MATRIX

ROZSZERZENIE SYGNATURY MATRYCY LOGICZNEJ O STAŁE WPŁYWA NA WŁASNOŚĆ SKOŃCZONEJ AKSJOMATYZACJI

Abstract

We give an example of a finite matrix with the property that expanding its language with a constant changes its finite axiomatization property: in the language with one binary operation the tautologies of the matrix are finitely axiomatizable while in the expanded language they are not. The constant we add is not definable in the original language. The deductive system generated by this matrix is not algebraizable.

Keywords: logical matrix, finite axiomatization

Streszczenie

Podajemy przykład skończonej matrycy logicznej, która jest skończenie aksjomatyzowalna, ale po dodaniu stałej do sygnatury tej matrycy, własność ta się psuje. Dodawana stała nie jest definiowalna w języku matrycy, a operator konsekwencji wyznaczony przez tę matrycę nie jest algebraizowalny

Słowa kluczowe: matryca logiczna, skończona aksjomatyzowalność

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* Katarzyna Pałasińska (kpalasin@pk.edu.pl), Institute of Mathematics, Faculty of Physics, Mathematics and Computer Science, Cracow University of Technology.

1. Introduction

By a (logical) *matrix* we mean an algebra with a designated subset. *Tautologies* of the matrix are the terms that under every valuation take a designated value. By a *valid rule*, or simply a *rule*, we mean a pair $\langle X, \alpha \rangle$, where $X \cup \{\alpha\}$ is a finite set of terms such that for every valuation assigning designated values to all members of X , the term α also takes a designated value. For $X = \emptyset$ the rule $\langle X, \alpha \rangle$ is called *axiomatic* and is identified with α . The set of all tautologies of a matrix \mathfrak{M} is denoted by $E(\mathfrak{M})$; the set of all valid rules of \mathfrak{M} is denoted by $R(\mathfrak{M})$. We say that a matrix is *finitely axiomatizable* if there exists a finite set of rules valid in this matrix from which all its tautologies can be derived. This differs from the *finite basis property*, which is the property that there exists a finite set of rules from which all valid rules can be derived.

Consider the 5-element matrix

$$\mathfrak{M} = \langle \{0, 1, 2, 3, 4\}, \{\cdot\}, \{2, 3\} \rangle$$

with \cdot given by the following table.

\cdot	0	1	2	3	4
0	1	2	2	2	2
1	1	2	2	2	2
2	1	2	2	2	2
3	4	3	3	3	3
4	3	3	3	3	3

Although this matrix is finitely axiomatizable (Proposition 1), we will show that the matrix

$$\mathfrak{M}_1 = \langle \{0, 1, 2, 3, 4\}, \{\cdot, 3\}, \{2, 3\} \rangle$$

does not have a finite axiomatization for the set of its tautologies (Theorem 2). The constant 3 is not a definable constant of \mathfrak{M} , so \mathfrak{M} and \mathfrak{M}_1 are not term equivalent. Let us observe that the deductive systems determined by these matrices are not algebraizable.

Proposition 1. *The consequence operation of neither \mathfrak{M} nor \mathfrak{M}_1 is algebraizable.*

Proof. Let \mathfrak{N} be either \mathfrak{M} or \mathfrak{M}_1 . We will show that \mathfrak{N} is not even protoalgebraic, a weaker condition than algebraizable. Suppose that there is a finite set of binary terms $\Delta(x, y)$ such that all terms in $\Delta(x, x)$ are tautologies and such that y is a consequence of $\Delta(x, y)$ and x . Such a set $\Delta(x, y)$ must exist for a protoalgebraic deductive system, [1]. As no variable is a tautology of \mathfrak{N} , it follows that no term in $\Delta(x, x)$ is a variable, so neither is any term in $\Delta(x, y)$. Evaluating x as 3 and y as 4, we get that x and $\Delta(x, y)$ evaluate to 3, while y is 4. This contradicts the condition that y is a consequence of $\Delta(x, y)$ and x . ■

In [4] Katarzyna Idziak has shown a finite equational algebra with a similar property: the quasi-equational theory of this algebra is finitely based but adding a nondefinable constant to the language of the algebra results in a nonfinitely based quasi-equational theory. Her example and ours differ in two aspects. First, the deductive system generated by the

matrix \mathfrak{M} is not algebraizable, while the deductive system equivalent to any equivalential algebra is obviously algebraizable. The role of rules in the deductive system associated with this algebra is played by the quasi-identities, while the role of tautologies – by identities valid in it. Therefore the second difference lies in the difference between finite axiomatization and finite basis: the example of [4] is an example that the finite basis property is fragile under adding new constants, while ours shows the same for the finite axiomatization property. Every finitely based deductive system is finitely axiomatizable, but a system that is not finitely based may still be finitely axiomatizable. The example given in [4] has 6 elements, so ours is smaller by one element.

2. Main result

Let $V = \{x_1, x_2, \dots\}$ be a countable set of pairwise distinct variables. Let Te denote the set of all terms written by means of these variables in the language $\{\cdot, 3\}$. When writing terms we omit the symbol of the binary operation \cdot and assume the association to the left. The length of a term t is denoted by $|t|$. By θ we mean the valuation in the algebra $\langle \{0, 1, 2, 3, 4\}, \{\cdot, 3\} \rangle$ assigning 0 to every variable.

Observe that every term $t \in \text{Te}$ is of the form

$$t = t_0 t_1 \cdots t_n, \quad (1)$$

where n is a nonnegative integer, all t_i 's are terms and t_0 is either a variable or the constant 3. Immediately from the table we see that for a term of the form (1):

$$\text{if } t_0 \text{ is variable, then } \theta(t) \in \{0, 1, 2\}. \quad (2)$$

By our next proposition the set $E(\mathfrak{M})$ is a consequence of one single axiom, so \mathfrak{M} is finitely axiomatizable.

Proposition 2. *The tautologies of the matrix \mathfrak{M} all follow from the axiom $x(yz)$.*

Proof. Clearly, $x(yz)$ is a tautology of \mathfrak{M} and no variable is. If a term of the form $t = rs$ is in $E(\mathfrak{M})$ then s cannot be a variable; for otherwise, by (2), $\theta(r) \in \{0, 1, 2\}$ and $\theta(rs) = \theta(r)\theta(s) = \theta(r)0 = 0$. Therefore $E(\mathfrak{M})$ contains only terms of the form $r(su)$. ■

Theorem 3. *The matrix \mathfrak{M}_1 is not finitely axiomatizable.*

Proof. Let E be the set of all tautologies of \mathfrak{M}_1 in Te . We will call a term t *left associated* if t is a variable or is of the form $t_1 x$, where t_1 is left associated and $x \in V$. For the proof by contradiction let R be a finite subset of $R(\mathfrak{M}_1)$ and assume that all tautologies of \mathfrak{M}_1 are derivable from R . Then there is a number n such that the length of the conclusion of any rule in R is no longer than $2n$. Let

$$\alpha_0 := 3x_1 x_2 \dots x_{2n}$$

and consider the set F consisting of all left associated tautologies of \mathfrak{M}_1 having α_0 as a sub-term. Notice that the term

$$\alpha_0 x_1 x_1 x_3 x_3 \cdots x_{2i-1} x_{2i-1} \cdots x_{2n-1} x_{2n-1}$$

belongs to F . So

$$F \neq \emptyset. \quad (3)$$

Lemma 4

Assume that $\alpha = \alpha_0 y_l y_{l-1} \cdots y_2 y_1 \in F$, where l is a nonnegative integer and $y_1, y_2, \dots, y_l \in V$.

Then l is even (4)

$$\forall_{i \leq l} [i \text{ is even} \Rightarrow \exists_{j < i} y_i = y_j] \quad (5)$$

$$\forall_{i \leq 2k} [i \text{ is odd} \Rightarrow \exists_{j < i} x_i = y_j] \quad (6)$$

$$l \leq 2n. \quad (7)$$

Proof of the Lemma. Use the valuation θ to see (4). For (5) assume that for some even i there is no $j < i$ such that $y_i = y_j$. Let i be the smallest such. Assign 3 to y_i and 0 to every variable other than y_i . Then the value of α is $3 \cdot 0 \cdots 0$, with an odd number of 0's in this expression. Hence α takes 4 under this valuation, a contradiction. Condition (6) is proved similarly and (7) follows from (6) and (5). ■

By (3) there exists a proof using the rules from the set R that proves some term $\alpha \in F$. Consider a shortest such proof π and let $\alpha \in F$ be the term proved by this proof. Consider the last rule $\langle X, \beta \rangle \in R$ used in π . So

$$|\beta| \leq 2n \quad (8)$$

and there is a substitution σ such that:

$$\sigma(\beta) = \alpha \quad (9)$$

and all terms $\sigma(\gamma)$ for $\gamma \in X$ occur in the proof π earlier than α . Since π is a shortest proof proving a formula in F , it follows that for every $\gamma \in X$

$$\sigma(\gamma) \in E \setminus F. \quad (10)$$

Since α satisfies the assumptions of Lemma 4, by (7), (8) and (9) we get that

$$\beta = uv_m \cdots v_1,$$

where $u, v_1, \dots, v_m \in V$, $m < l$, $\sigma(u) = \alpha_0 y_1 \cdots y_{m+1}$ and $\sigma(v_i) = y_i$ for each $i = 1, \dots, m$.

Obviously, $u \neq v_i$, for any $i \in \{1, \dots, m\}$.

Let us define the valuation φ such that $\varphi(u) = 3$ if m is odd, $\varphi(u) = 4$ if m is even and $\varphi(x) = \theta(\sigma(x))$ for every $x \in V \setminus \{u\}$. Notice that then for $i \in \{1, \dots, m\}$, $\varphi(v_i) = 0$, so $\varphi(\beta) = 4$. Since the rule $\langle X, \beta \rangle$ is valid in \mathfrak{M}_1 , there must be a term $\gamma \in X$ such that

$$\varphi(\gamma) \in \{0, 1, 4\}. \quad (11)$$

By (10), $\theta(\sigma(\gamma)) \in \{2, 3\}$. So

$$\varphi(\gamma) \neq \theta(\sigma(\gamma)). \quad (12)$$

By the definition of φ and by (12), the term γ contains u . Moreover, by (1), the term γ takes one of the following three forms: $\gamma = ut_1 \cdots t_k$, $\gamma = xt_1 \cdots t_k$ with $x \in V$ and $x \neq u$, or $\gamma = 3t_1 \cdots t_k$, for some k and some sequence t_1, \dots, t_k of terms. In the last two cases, $\varphi(\gamma) = \theta(\sigma(\gamma))$, because on positions other than the initial one, the value 3 behaves the same as the value 4. Similarly, if any of the terms t_i would be composed, then we would have $\varphi(\gamma) = \theta(\sigma(\gamma))$. So it follows by (12) that the only form γ may take is

$$\gamma = uz_1 \dots z_k,$$

where z_1, \dots, z_k are variables. But then $\sigma(\gamma)$ is a left associated tautology of \mathfrak{M}_1 with a subterm α_0 which contradicts (10). ■

The technique of the proof is similar to the one used in [2, 6, 7]. The idea of the example is similar to that of [5].

3. Questions

One may ask if there is a matrix of a smaller size or a matrix with a smaller number of designated values that has the same property as presented here.

Question 5. *Is there a non-algebraizable matrix with less than 5 elements with the property that its tautologies are finitely axiomatizable while the tautologies of the same matrix in the language expanded by a constant are not finitely axiomatizable?*

Question 6. *Find such a matrix with only one designated value.*

The finite basis property mentioned in the Introduction is related but different from the finite axiomatization property. Our example does not answer the following

Question 7. *Find a non-algebraizable finitely based matrix that expanded by a constant becomes non-finitely based.*

An open problem, due to W. Rautenberg is whether the finite basis property of finite matrices is independent of the language. More precisely, given a finite finitely based matrix, is every matrix term-equivalent to it also finitely based? See [3]. The constant 3 added to the language of our matrix \mathfrak{M} is clearly not definable.

Question 8. *Is there a finitely based (resp. finitely axiomatizable) matrix $\mathfrak{M} = \langle M, F, D \rangle$ and a constant c definable in its language such that the consequence operation of the matrix $\mathfrak{M}_1 = \langle M, F \cup \{c\}, D \rangle$ is not finitely based (not finitely axiomatizable, resp.)?*

If such a matrix \mathfrak{M} exists its consequence operation is necessarily non-algebraizable.

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