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SOME REMARKS ON HAUSDORFF GAPS AND AUTOMORPHISMS OF $P(\omega)/FIN$

O NIEROZDZIELALNYCH LUKACH W ALGEBRZE $P(\omega)/FIN$

A b s t r a c t

We present, under the Continuum Hypothesis (CH), a construction of an automorphism of $P(\omega)/fin$ which maps a Hausdorff gap onto increasingly ordered gap of type (ω_1, ω_1) which is not a Hausdorff gap.

Keywords: compactification, automorphism, Boolean algebra

S t r e s z c z e n i e

Artykuł przedstawia, przy założeniu Hipotezy Continuum, konstrukcję automorfizmu algebry $P(\omega)/fin$, który przeprowadza lukę Hausdorffa na lukę niemającą własności Hausdorffa.

Słowa kluczowe: uzwarcenie, automorfizm, algebra Boole'a

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It is known that cardinality of the group of automorphisms of $P(\omega)/fin$ depends on some additional axioms of ZFC. Under CH (the Continuum Hypothesis) the cardinality of the group is the largest posssible – it is equal to2^c, where c denotes the continuum (cf. [2]). On the other hand, there are models of ZFC in which the cardinality is equal to the continuum, for example, in the model constructed by Shelah in [3]. In [4] it was shown by Shelah and Steprans that the assertion PFA + $c = \omega_2$ implies that all automorphisms of $P(\omega)/fin$ are trivial (i.e. induced by a bijection between co-finite subsets of ω). Velickovic [7] proved that the same thesis follows from OCA+MA. One of the methods of elimination an automorphim of $P(\omega)/fin$ is adding a new real which fills a non-separable gap in such a way that the image of the gap under that automorphism remains unfilled. It is known that each forcing which adds an element separating a Hausdorff gap collapses the to the ω. One can ask if the image of a Hausdorff gap under a automorphism must be a Hausdorff gap .

Basic facts and definitions. Byω we denote the set of all natural numbers (and the first infinite ordinal) and by fin the ideal of all its finite subsets.$P(\omega)/fin$ is the factor Boolean algebra and for $A, B \in P(\omega)$ we shall use the following notation: $A =_* B$ if $A \div B \in fin$, $A \subseteq_* B$ if $A \setminus B \in fin$.

Let λ, κ be ordinals. ***A gap of the type*** (λ, κ) in a Boolean algebra $(A,+,\bullet,0,1)$ is a pair $(\{a_\gamma: \gamma < \lambda\},\{b_\beta: \beta < \kappa\})$ of subsets of A such that $a_\gamma \bullet b_\beta = 0$. If for every $\gamma_1 < \gamma_2 < \lambda$, $\beta_1 < \beta_2 < \kappa$, $a_{\gamma 1} \bullet a_{\gamma 2} = a_{\gamma 1}$ and $b_{\beta 1} \bullet b_{\beta 2} = b_{\beta 1}$ the gap is said to be ***incresingly ordered***. Element$c \in A$ ***fills (separates)*** the gap if $a_\gamma \bullet c = a_\gamma$ and $b_\beta \bullet c = 0$ for every $\gamma < \lambda$, $\beta < \kappa$. If there is no such an element, the gap is called ***non-separable***. A (strictly) decreasing sequence $(a_\beta: \beta < \gamma)$ of elements of the A of the lengthγ is called ***g-limit*** if there is no non-zero element $a \in A$ such that for every $\beta < \gamma$, $a \bullet a_\beta = a$.

Assume that $\mathcal{L}=(\{X_\alpha: \alpha<\omega_1\},\{Y_\beta: \beta < \omega_1)$ is an increasingly ordered gap in $P(\omega)/fin$. $\mathcal{L}$is a ***Hausdorff gap*** if $\{\gamma < \beta: \max X_\gamma \cap Y_\beta < k\} = {}^*\varnothing$ for every $\beta < \omega_1$ and $k \in \omega$.

It is known that

1. In the algebra $P(\omega)/fin$ every countable gap (i.e. $\mathrm{card}(\lambda) = \mathrm{card}(\kappa) = \omega$ is filled.
2. There is no ω-limits (i.e.γ-limits with card $(\gamma) = \omega$ in the $P(\omega)/fin$.
3. A Hausdorff gap is non-separable (thus there exist non-separable gaps of the type (ω_1, ω_1) in the $P(\omega)/fin$.

In the following construction we shall apply the Sikorski's theorem (to define a required automorphism).

Theorem 1 (Sikorski [5, 6]**)** Let A, B be Boolean algebras, A_0 a subalgebra of A and $a_0 \in A \setminus A_0$. Assume that $T: A_0 \to B$ is a homomorphism. If there exists an element $b \in B$ which fills a gap:

$$\mathcal{L} = (\{T(x): x \in A_0, x \le a_0\}, \{T(x): x \in A_0, x \bullet a_0 = 0\}),$$

then T can be extended to a homomorphism $T^*: A_1 \to B$ (where A_1 is a subalgebra generated by $A_0 \cup \{a_0\}$ with $T^*(a_0) = b$.

Moreover if T is monomorphism then T^* is monomorphism if and only if the following condition holds:

$$(*) \text{ for all } x, y \in A_0 [\, (x \le a_0 \Leftrightarrow T(x) \le b) \quad \text{and} \quad (y \ge a_0 \Leftrightarrow T(y) \ge b)].$$

Thus, in order to extend a monomorphism, we have to ensure that an image of a (separeted) gap under the monomorphism satisfies the condition (*). Let us remind a (well known) method how to find the required (in the (*)) element in the range of the monomorphism. Although the method can be applied in the case of Boolean algebras in which there are no countable gaps nor countable limits, we present it in the particular case of $P(\omega)/fin$.

Claim 1 (cf. [6]**)** Let T: $A \to B$ be a monomorphism of countable subalgebras of $P(\omega)/\text{fin}$ and let $G \in P(\omega)/\text{fin}\backslash A$. Then there is a gap in B such that any element which fills the gap satisfies the condition (*).

Proof: Let $J = \{Z_n: n \in \omega\}$ be an enumeration of all elements $Z \in A$ with $Z \cap G \neq_* \emptyset$ and $Z \backslash G \neq_* \emptyset$. Fix $n \in \omega$. For a $Z_n \in J$ and $X, Y \in A$ we have:

If $Y \cap G =_* \emptyset$ then $Z_n \cap Y \cap G =_* \emptyset$. Since $Z \cap G \neq_* \emptyset$, it follows that $Z_n \backslash Y \neq_* \emptyset$. T is a monomorphism thus we have $T(Z_n) \backslash T(Y) \neq_* \emptyset$. In a similar way we show that if $X \subseteq_* G$ then $T(Z_n) \backslash T(X) \neq_* \emptyset$.

Since A is countable, there exists an enumeration $\{Y_m \in A: m \in \omega\}$ of all elements which are almost disjoint with G. Thus:

$$\{T(Z_n) \backslash (T(Y_1) \cup \ldots \cup T(Y_m)): m \in \omega\}$$

is a countable decreasing chain in $P(\omega)/fin$. Since there are no countable limits in $P(\omega)/fin$, we can choose an infinite subset $S(Z_n) = S_n$ which is almost contained in each

$$T(Z_n) \backslash (T(Y_1) \cup \ldots \cup T(Y_m)).$$

In a similar way we can choose $I(Z_n) = I_n \subseteq_* T(Z_n) \backslash (T(X_1) \cup \ldots \cup T(X_m))$ for $X_m \in A$, $X_m \subseteq_* G$ Consider the gap $\mathcal{P} = (\mathcal{M}, \mathcal{O})$ where:

$$\mathcal{M} = \{T(X) \in A: X \subseteq_* \text{G}\} \cup \{T(I_n): n \in \omega\},$$

$$\mathcal{O} = \{T(Y) \in A: Y \cap G =_* \emptyset\} \cup \{T(S_n): \quad n \in \omega\}.$$

Since P is countable, there exists element H which fills the gap. It is easy to see that such an element H satisfies the condition.

Main theorem. We prove that:

Theorem 2 If CH holds then there exists an automorphism T of $P(\omega)/\text{fin}$ and two increasingly ordered gaps of the type (ω_1, ω_1):

$$\mathcal{L}_H = (\{X_\alpha: \alpha < \omega_1\}, \{Y_\beta: \beta < \omega_1\}), \quad \mathcal{L} = (\{A_\alpha: \alpha < \omega_1\}, \{B_\beta: \beta < \omega_1\})$$

such that

1. for all $\beta < \omega_1$ and every $k \in \omega$, a set $\{\alpha < \beta: \max(X_\alpha \cap Y_\beta) < k\}$ is finite,
2. if $\beta = \lambda + \omega$ for some limit ordinal $\lambda < \omega_1$ then there exists $k \in \omega$ such that a set $\{\alpha < \beta: \max(A_\alpha \cap B_\beta) < k\}$ is infinite,
3. for every $\alpha, \beta < \omega_1$, $T(A_\alpha) = X_\alpha$, $T(B_\beta) = Y_\beta$.

Proof: We construct the required automorphism and gaps by using transfinite induction. Fix a set $\{G_\alpha: \alpha < \omega_1\}$ of generators of $P(\omega)/\text{fin}$. At the step $\alpha = 0$ fix two pairs of disjoint infinite subsets of ω: A_0, B_0 and X_0, Y_0 such that both sets $\omega \backslash (A_0 \cup B_0)$ and $\omega \backslash (X_0 \cup Y_0)$ are infinite. Let:

$$T_0(A_0) = X_0, \quad \text{and} \quad T_0(B_0) = Y_0.$$

Denote by $\underline{\mathbf{D}}_0$ and $\underline{\mathbf{P}}_0$ the Boolean algebras generated by $\{A_0, B_0\}$ and $\{X_0, Y_0\}$ (respectively) and extend T_0 to the isomorphism from $\underline{\mathbf{D}}_0$ onto $\underline{\mathbf{P}}_0$. Consider the first generator G_0 and the algebra $\underline{\mathbf{D}}_0$. In the way described in the Claim find element H_r in the $\underline{\mathbf{P}}_0$ and put $T_0(G_0) = H_r$. Then apply the claim to the G_0, the algebra $\underline{\mathbf{P}}_0$ and the T_0^{-1}. Choose any element H_d which fills the obtained gap in $\underline{\mathbf{D}}_0$ and define $T_0^{-1}(G_0) = H_d$. Then, using Sikorski's theorem, extend the isomorphism T_0 to an isomorphism from $\mathbf{D}_0$ (the subalgebra generated by $\{A_0, B_0, G_0, T^{-1}(G_0)\}$) onto $\mathbf{P}_0$ (the subalgebra generated by $\{X_0, Y_0, G_0, T_0(G_0)\}$).

Assume inductively that for every $\beta < \alpha$ we have defined increasing sequences of subalgebras $\mathbf{D}_\beta$, $\mathbf{P}_\beta$ of $P(\omega)/fin$, isomorphisms $T_\beta\colon \mathbf{D}_\beta \to \mathbf{P}_\beta$ and gaps

$$\mathcal{L} = (\{A_\gamma\colon \gamma < \alpha\}, \{B_\gamma\colon \gamma < \alpha\}) \text{ and } \mathcal{L}_H = (\{X_\gamma\colon \gamma < \alpha\}, \{Y_\gamma\colon \gamma < \alpha\})$$

such that

1. For all $\beta < \alpha$, $A_\beta, B_\beta, G_\beta \in \mathbf{D}_\beta$ and $X_\beta, Y_\beta, G_\beta \in \mathbf{P}_\beta$,
2. $\mathcal{L}$, $\mathcal{L}_H$ are increasingly ordered gaps,
3. For all $\beta < \alpha$, $A_\beta \cap B_\beta = \varnothing$ and $X_\beta \cap Y_\beta = \varnothing$; both sets $\omega\setminus(A_\beta \cup B_\beta)$ and $\omega\setminus(X_\beta \cup Y_\beta)$ are infinite,
4. For all $\beta < \alpha$ and every $k \in \omega$, the set $\{\gamma < \beta\colon \max(Y_\beta \cap X_\gamma) < k\}$ is finite,
5. If $\beta < \alpha$ is equal to $\lambda + \omega$, for some limit ordinal λ, then there exists $k \in \omega$ such that $\{\gamma < \beta\colon \max(B_\beta \cap A_\gamma) < k\}$ is infinite,
6. If $\beta = \lambda + n$, for some limit ordinal λ and a natural number $n > 0$, then
$$\left(\bigcup_{k \leq n} A_{\lambda+k}\right) \cap \left(\bigcup_{k \leq n} B_{\lambda+k}\right) = \varnothing.$$
7. For $\gamma < \beta < \alpha$, $T_\beta|\mathbf{D}_\gamma = T_\gamma$ and $T(A_\beta) = X_\beta$, $T(B_\beta) = Y_\beta$.

Assume that α is a successor ordinal, $\alpha = \beta + 1$. Then there exist a limit ordinal λ and a natural number $n > 0$ such that $\alpha = \lambda + n$. Choose infinite and disjoint subsets A, B of ω such that:

$$(A_\beta \cup B_\beta) \cap (A \cup B) = \varnothing \text{ and } \omega\setminus(A_\beta \cup B_\beta \cup A \cup B) \text{ is infinite}$$

and both sets $B \cap \bigcup_{k \leq n} A_{\lambda+k}$ and $A \cap (\bigcup_{k \leq n} B_{\lambda+k})$ are empty. Put:

$$A_\alpha = A_\beta \cup A, \quad B_\alpha = B_\beta \cup B.$$

Let $\underline{\mathbf{D}}_\alpha$ be subalgebra generated by $\mathbf{D}_\beta$ and the elements A_α, B_α. Apply The Claim to choose candidates for images of A_α, B_α (and then Sikorski's theorem to extend T_β). Denote this extension by T^*_β. Note, that each of sets $T^*_\beta(A_\alpha)$ and $T^*_\beta(B_a)$ separates $\mathcal{L}_H$ and we may assume that they are disjoint. Define

$$X_\alpha = T^*_\beta(A_\alpha), \quad Y_\alpha = T^*_\beta(B_\alpha).$$

Since $Y_\beta \subseteq_* Y_\alpha$ and for every natural number k, the set $\{\gamma < \beta\colon \max(Y_\beta \cap X_\gamma) < k\}$ is finite, it follows that the set $\{\gamma < \alpha\colon \max(Y_\alpha \cap X_\gamma) < k\}$ is finite as well. If $G_\alpha \in \underline{\mathbf{D}}_\alpha$ then we add, in the same way, an image $T^*_\beta(G_\alpha)$. Let $\underline{\mathbf{P}}_\alpha$ be the subalgebra generated by $\mathbf{P}_\beta$ and the elements X_α, Y_α and $T^*_\beta(G_\alpha)$. If $G_\alpha \in \underline{\mathbf{P}}_\alpha$, then a preimage $T^{*}_\beta{}^{-1}(G_\alpha)$ of a generator G_α has to be added. We fix the preimage in the way described above. We conclude the successor step with definitions of $\mathbf{D}_\alpha$ and $\mathbf{P}_\alpha$. $\mathbf{D}_\alpha$ is a subalgebra generated by $\mathbf{D}_\beta$ and A_α, B_α, G_α, $T^{*}_\beta{}^{-1}(G_\alpha)$

while $\mathbf{P}_\alpha$ is generated by $\mathbf{P}_\beta$ and X_α, Y_α, G_α, $T^*_\beta(G_\alpha)$. Moreover $T_\alpha = T^*_\beta : \mathbf{D}_\alpha \to \mathbf{P}_\alpha$. It is obvious that all inductive assumptions are satisfied.

Assume that α is a limit ordinal. Put:

$$\underline{\mathbf{D}}_\alpha = \bigcup_{\beta<\alpha} \mathbf{D}_\beta,\ \ \underline{\mathbf{P}}_\alpha = \bigcup_{\beta<\alpha} \mathbf{P}_\beta,\quad T = \bigcup_{\beta<\alpha} T_\beta.$$

In order to construct elements X_α and Y_α we modify slightly Hausdorff argument (presented in [1]). The sequence $(\omega\backslash(X_\beta \cup Y_\beta)\colon \beta < \alpha)$ is countable and decreasing; it follows that there exists an infinite set $D \subseteq \omega$ with $D \subseteq_* \omega\backslash(X_\beta \cup Y_\beta)$ for all $\beta < \alpha$. Thus $X_\beta \cup Y_\beta \subseteq_* D^c = \omega\backslash D$. Since:

$$\mathcal{L}_H = (\{X_\gamma : \gamma < \alpha\},\ \{Y_\gamma : \gamma < \alpha\})$$

is countable, one can choose a set F which separates the gap i.e. for all $\beta < \alpha$, $Y_\beta \subseteq_* F$ and $X_\beta \cap F =_* \varnothing$. Moreover, we may assume that $F \subseteq D^c$ (replacing F with $F \cap D$, if necessary). Applying the claim and Sikorski's theorem we fix a \$$T^{-1}(F)$, which fills the gap $\mathcal{L}$. Note that for every $\beta < \alpha$ and $k \in$ the set $\{\gamma < \beta\colon \max(F \cap X_\gamma) < k\}$ is finite however it does not follow that for each $k \in \omega$ the set $\{\gamma < \alpha\colon \max(F \cap X_\gamma) < k\}$ is finite. In order to ensure that the assertion holds we have to enlarge the set F. For $k \cap \omega$ let

$$\mathcal{J}_k = \{\gamma < \alpha\colon \max(F \cap X_\gamma) < k\}.$$

We define (inductively) a (countable) increasing sequence $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that for every $n, k \in \omega$ the set $\{\gamma \in \mathcal{J}_n\colon \max(F_{n+1} \cap X_\gamma) < k\}$ is finite and $F_n \cap X_\gamma =_* \varnothing$.

Assume that sets $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ and their preimages under T have been defined. Denote by $\mathbf{P}_{\alpha,n+1}$ the subalgebra generated by $\mathbf{P}_{\alpha,n}$ and F_{n-1} and by $\mathbf{D}_{\alpha,n+1}$ the subalgebra generated by $\mathbf{D}_{\alpha,n}$ and $T^{-1}(F_{n-1})$, for $n > 0$.

If $\mathcal{J}_{n+1}$ is finite, then $F_{n+1} = F_n$. So suppose that $\mathcal{J}_{n+1}$ is infinite. Then $\mathcal{J}_{n+1}$ can be increasigly enumerated with natural numbers and $\sup \mathcal{J}_{n+1} = \alpha$. Indeed, this is implied by the fact, that for each $\beta < \alpha$ the set $\beta \cap \mathcal{J}_{n+1} = \{\gamma < \beta\colon \max(F \cap X_\gamma) < k\}$ is finite.

The subalgebra $\mathbf{P}_{\alpha,n}$ is countable thus we can fix an enumeration $\{K_i\colon i \in \omega\}$ of all elemets $K \in \mathbf{P}_{\alpha,n}$ such that $K \cap X_\gamma =_* \varnothing$ for $\gamma < \alpha$.

Thus let $\mathcal{J}_{n+1} = \{l\colon l \in \omega\}, \gamma_l < \gamma_{l+1}$. The sequence $\{X_\gamma\colon \gamma < \alpha\}$ is increasing, which implies that $X_{\gamma l}\backslash(X_{\gamma 0} \cup X_{\gamma 1} \cup \ldots \cup X_{\gamma l-1}) \neq {*}\varnothing$. Moreover, $X_{\gamma l} \subseteq_* D^c$, thus

$$D^c \cap [X_{\gamma l}\backslash(X_{\gamma 0} \cup X_{\gamma 1} \cup \ldots \cup X_{\gamma l-1})] \neq^* \varnothing.$$

It follows that for every $l \in \omega$ there exists a natural number

$$j_l \in D^c \cap [X_{\gamma l}\backslash(X_{\gamma 0} \cup X_{\gamma 1} \cup \ldots \cup X_{\gamma l-1})] \cap (\omega\backslash \textstyle\bigcup_{i\le l} K\_i)$$

with $j_l > l$. Put

$$F_{n+1} = F_n \cup \{j_l\colon l \in \omega\}.$$

It is easy to note that $\{j_l\colon l \in \omega\} \cap X_{\gamma i}$ is finite for every $i \in \omega$. Since the sequence $\{X_\gamma\colon \gamma < \alpha\}$ is increasing it follows that $\{j_l\colon l \in \omega\} \cap X_\gamma =\_{*}\varnothing$ and $F_{n+1} \cap X_\gamma =_* \varnothing$ for each $\gamma < \alpha$. Moreover for each $k \in \omega$ the set $\{\gamma \in \mathcal{J}_n\colon \max(F_{n+1} \cap X_\gamma) < k\}$\$ is finite. Extend the range of the isomorphism T in the following way: $\mathcal{D} = (\mathcal{M}, \mathcal{O})$

$$\mathcal{M} = \{T^{-1}(X) \in \mathsf{D}_{\alpha,n}\colon X \subseteq_* F_n\} \cup \{T^{-1}(I(Z))\colon Z \in J\},$$

$$O = \{T^{-1}(Y) \in \mathsf{D}_{\alpha,n}\colon Y \cap F_n =_* \varnothing\} \cup \{T^{-1}(S(Z))\colon Z \in J\},$$

where $J = \{Z \in \mathsf{P}_{\alpha, n+1}\colon Z \cap F_n \neq_* \varnothing, Z\setminus F_n \neq_* \varnothing\}$ and elements $S(Z)$, $I(Z)$ are defined for each $Z \in J$ in the way described in proof of the Claim.

Since $\mathcal{D}$ is countable, then there exist infinite sets C_n, H_n such that C_n separates the gap and H_n is almost disjoint with every element of the sets forming the gap. Note that since $F_n \cap X_\beta =_* \varnothing$ and $Y_\beta \subseteq_* F_n$ then, by the Claim, $C_n \cap A_\beta =_* \varnothing$ and $C_n \subseteq_* B_\beta$, for $\beta < \alpha$. If, for some limit ordinal λ, $\alpha = \lambda + \omega$ then, by inductive assumption, $(\bigcup_{k\in\omega} A_{\lambda+k}) \cap (\bigcup_{k\in\omega} B_{\lambda+k}) = \varnothing$. Moreover, since for every $L \in \mathsf{D}_{\alpha, n}$ with $L \cap A_\gamma =_* \varnothing$ there exists $i \in \omega$ such that $T(L) = K_{i,}$ it follows that $C_n \cap L =_* \varnothing$. Thus we may assume that

$$C_n \cap \bigcup_{k\in\omega} A_{\lambda+k} = \varnothing.$$

Put $T^{-1}(F_n) = C_n$.
Since

$$\bigcup_{n\in\omega} C_n \cap \bigcup_{k\in\omega} A_{\lambda+k} = \varnothing$$

then in the case $\alpha = \lambda + \omega$, for some limit ordinal λ, we can choose B_α which fills the gap:

$$(\{B_\gamma\colon \gamma < \alpha\} \cup \{C_n\colon n \in \omega\}, \{A_\gamma\colon \gamma < \alpha\})$$

and $B_\alpha \cap A_{\lambda+k} = \varnothing$ for each $k \in \omega$. Apply the the Claim theorem to determine a Y_α and Sikorski's theorem to extend the T. Note, that the element separates the gap

$$\mathcal{P} = (\{F_n\colon n \in \omega\}, \{X_\gamma\colon \gamma < \alpha\}).$$

We may assume that $F \subseteq Y_\alpha \subseteq D^c$. We have to show that for each $k \in \omega$ the set $\{\gamma < \alpha\colon \max (Y_\alpha \cap X_\gamma) < k\}$ is finite. Assume to the contrary that for some $k \in \omega$ the set is infinite. Since $F \subseteq Y_\alpha$ then $\{\gamma < \alpha\colon \max (Y_\alpha \cap X_\gamma) < k\} \subseteq \mathcal{J}_k$. The latter assumtion implies that the set $I = \{\gamma \in \mathcal{J}_k\colon \max (Y_\alpha \cap X_\gamma) < k\}$ is infinite as well. But since $F \subseteq Y_\alpha$, $I \subseteq \{\gamma \in \mathcal{J}_n\colon \max F_{n+1} \cap X_\gamma < k\} =_* \varnothing$, a contradiction.

Put $X_\alpha = D^c \setminus Y_\alpha$. Now, apply the Claim to define A_α and using Sikorski's theorem extend the T.

Define D_α (P_α) as the subalgebra generated by $\bigcup_{n\in\omega} \mathsf{D}_{\alpha, n}$, A_α and B_α ($\bigcup_{n\in\omega} \mathsf{P}_{\alpha,n}$ and X_α and Y_α). The automorphism T_α is equal to the (extended) $T\colon \mathsf{D}_\alpha \to \mathsf{P}_\alpha$.

This finishes the the limit step of the construction.

After ω_1 steps each of algebras $\mathsf{P} = \bigcup_{\alpha<\omega1} \mathsf{P}_\alpha$ and $\mathsf{D} = \bigcup_{\alpha<\omega1} \mathsf{D}_\alpha$ contains all of the generators G_α, thus $\mathsf{P} = \mathsf{D} = P(\omega)/\mathsf{fin}$. $T = \bigcup_{\alpha<\omega} 1\ T_\alpha$ is an isomorphism of $P(\omega)/\mathsf{fin}$. The gap

$$\mathcal{L}_H = (\{X_\gamma\colon \gamma < \omega_1\}, \{Y_\gamma\colon \gamma < \omega_1\})$$

is a Hausdorff gap, while the gap

$$\mathcal{L} = (\{A_\gamma\colon \gamma < \omega_1\}, \{B_\gamma\colon \gamma < \omega_1\})$$

does not satisfy the condition.

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