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AN $O(h_k^5)$ ACCURATE FINITE DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF FOURTH ORDER TWO POINT BOUNDARY VALUE PROBLEMS ON GEOMETRIC MESHES

METODA RÓŻNICOWA O DOKŁADNOŚCI $O(h_k^5)$, DO ROZWIĄZYWANIA DWUPUNKTOWYCH ZAGADNIEŃ BRZEGOWYCH CZWARTEGO RZĘDU NA SIATKACH GEOMETRYCZNYCH

Abstract

Two point boundary value problems for fourth order, nonlinear, singular and non-singular ordinary differential equations occur in various areas of science and technology. A compact, three point finite difference scheme for solving such problems on nonuniform geometric meshes is presented. The scheme achieves a fifth or sixth order of accuracy on geometric and uniform meshes, respectively. The proposed scheme describes the generalization of Numerov-type method of Chawla (IMA J Appl Math 24:35-42, 1979) developed for second order differential equations. The convergence of the scheme is proven using the mean value theorem, irreducibility, and monotone property of the block tridiagonal matrix arising for the scheme. Numerical tests confirm the accuracy, and demonstrate the reliability and efficiency of the scheme. Geometric meshes prove superior to uniform meshes, in the presence of boundary and interior layers.

Keywords: Geometric mesh, finite difference method, compact scheme, singularity, stiff equations, Korteweg-de Vries equation, maximum absolute errors

Streszczenie

Dwupunktowe zagadnienia z warunkami brzegowymi, dla nieliniowych, osobliwych i nieosobliwych równań różniczkowych zwyczajnych czwartego rzędu, występują w różnych obszarach nauki i techniki. Zaprezentowano kompaktowy, trzypunktowy schemat różnicowy do rozwiązywania takich problemów na niejednorodnych siatkach geometrycznych. Schemat ten osiąga dokładność piątego lub szóstego rzędu, odpowiednio na siatkach geometrycznych lub jednorodnych. Proponowany schemat przedstawia uogólnienie metody typu Numerowa, autorstwa Chawli (IMA J Appl Math 24:35-42, 1979), opracowanej dla równań różniczkowych drugiego rzędu. Udowodniono zbieżność schematu, korzystając z twierdzenia o własności średniej, nieredukowalności oraz monotoniczności macierzy blokowo-trójdiagonalnej wynikającej ze schematu. Testy numeryczne potwierdzają dokładność, oraz demonstrują niezawodność i wydajność schematu. Siatki geometryczne wykazują przewagę nad siatkami jednorodnymi, w obecności warstw brzegowych i wewnętrznych.

Słowa kluczowe: Siatka geometryczna, metoda różnic skończonych, schemat kompaktowy, osobliwość, równania sztywne, równanie Kortewega-de-Vriesa, maksymalne błędy bezwzględne

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1. Introduction

In this paper we consider a numerical solution of the fourth order ordinary differential equation (ODE):

$$-U^{(4)}(r)+g(r,U(r),U^{(1)}(r),U^{(2)}(r),U^{(3)}(r))=0,-\infty<a<r<b<\infty \tag{1.1}$$

subject to the boundary conditions $U(a)=m_1, U(b)=m_2, U^{(2)}(a)=m_3, U^{(2)}(b)=m_4$, where m_1, m_2, m_3, m_4 are finite real constants. We assume that $g \in C^{(6)}(a,b)$, with the possibility that $g(.)$ can be singular inside and on the boundaries of the domain $[a, b]$.

Boundary value problems of this kind play an important role in various areas of science and technology. The mathematical formulation of noise removal and edge preservation (Yu--Li and Kaveh [1]), Kirchhoff plates (Zhong [2]), theory of plates and shell (Timoshenko and Krieger [3]), waves on a suspension bridge (Chen and McKenna [4]), geological folding of crock layers (Budd [5]) and hydrodynamics equation (Wasow [6]) are some examples of such problems.

The solvability, existence and uniqueness of the solutions of fourth order boundary value problems have been discussed by O'Regan [7], Agarwal [8] and Atabizadeh [9]. For solving Eq. (1.1) a number of approaches have been proposed, such as differential transform (Momani et. al. [10]), Adomian decomposition (Wazwaz [11]), homotopy perturbation (Din et. al. [12]), variational iteration (Noor et. al. [13]), exponential spline (Zahra [14]) and finite difference approximations (Usmani [15], Schroder [16] and Shanthi [17]).

Possible approaches to solving Eq. (1.1) can be roughly divided into two categories. The first category includes methods which solve Eq. (1.1) as is, either analytically as in [10–13] or numerically as in [14–17]. The second category includes methods in which Eq. (1.1) is first converted to a system of second order ODEs:

$$-U^{(2)}(r)+V(r)=0, \tag{1.2}$$

$$-V^{(2)}(r)+g(r,U(r),U^{(1)}(r),V(r),V^{(1)}(r))=0,-\infty<a<r<b<\infty. \tag{1.3}$$

Subsequently, one solves system (1.2) and (1.3) by a technique appropriate to second order ODEs (see, for example Twizell and Boutayeb [18]).

In the present paper we describe a new method that belongs to the second category. The method uses a fifth order accurate, compact three point finite difference scheme that approximates system (1.2) and (1.3) on a specific nonuniform mesh called a geometric mesh (Jain et. al. [19], Kadalbajoo [20] and Mohanty [21]); in some application areas, like electrochemistry the name "exponentially expanding grid" is also used (Britz [22]). The geometric mesh is defined by the formulae: $a=r_0<\ldots<r_{n+1}=b$, $h_k=r_k-r_{k-1}$, $k=1(1)n+1$, $h_{k+1}=\tau h_k$, where $\tau > 0$ is a constant mesh ratio parameter and $n + 2$ is the total number of nodes. Such a mesh is particularly suitable when ODEs such as Eq. (1.1) or (1.2) and (1.3) are singularly perturbed, so that their solutions possess boundary or interior layers (Roos [23], Farrell et. al. [24]). The compact, three point character of the scheme makes it particularly convenient. This is because in the process of the numerical solution of the resulting nonlinear algebraic equation systems (for example, by the Newton method) one obtains linear algebraic systems with block tridiagonal matrices. Such systems are easy to solve, using standard algorithms, for example the generalized Thomas algorithm (Thomas [25], Bieniasz [26]). In contrast, higher order discretizations associated with non-compact stencils lead to the increase of the bandwidth of the resultant coefficient matrix, which implies a larger number of arithmetic operations.

There exists an ample literature devoted to the development of compact schemes for solving two point boundary value problems for single second order ODEs. In particular, we mention here the various improvements of the classical Numerov scheme (Numerov [27], Agarwal [28]) and the arithmetic average schemes, obtained by (Chawla [29, 30], Wang [31], Bieniasz [32], Mohanty [33], Zhang [34] and Jha [35, 36]). The new scheme proposed in the present work, can be regarded as an extension, and adaptation to the nonuniform mesh, of the sixth order compact scheme of Chawla [30]. Minor modifications of the scheme are required for the singular problems.

The paper is organized as follows: In section 2, we develop the higher order finite difference scheme on the geometric mesh. The convergence analysis is contained in section 3. In section 4, some computational experiments are described that show the reliability of the algorithm. In the last section, the findings are summarized.

2. Formulation of the $O(h_k^5)$ finite difference scheme on the geometric mesh

Let U_k, V_k be the exact solution values and u_k, v_k be the approximate values of $U(r)$ and $V(r)$ at the mesh node r_k respectively. With the help of finite Taylor's expansions, we first obtain the following relation that approximates the second order derivative at r_k using geometric meshes:

$$\begin{aligned} h_k^2 c_0 U_k^{(2)} &= -U_{k+1} + (1+\tau)U_k - \tau U_{k-1} \\ &\quad - h_k^2\left(c_1 U_{k+1}^{(2)} + c_2 U_{k-1}^{(2)} + c_3 U_{k+1/2}^{(2)} + c_4 U_{k-1/2}^{(2)}\right) + O(h_k^7), \end{aligned} \tag{2.1}$$

where:

$$c_0 = -(1+\tau)(3\tau^2 + 7\tau + 3)/60,$$

$$c_1 = -(2\tau^3 + \tau^2 - \tau + 1)/[60(1+2\tau)],\ c_3 = -2(1+\tau)(2\tau^2 + 2\tau - 1)/[15(2+\tau)],$$

$$c_2 = -\tau(\tau^3 - \tau^2 + \tau + 2)/[60(2+\tau)],\ c_4 = 2\tau(1+\tau)(\tau^2 - 2\tau - 2)/[15(1+2\tau)]$$

As Eq. (1.3) involves first solution derivatives, we need certain approximations to these derivatives. Consider the following geometric mesh approximations to $U^{(1)}$:

$$\tilde{U}_k^{(1)} = [U_{k+1} - (1-\tau^2)U_k - \tau^2 U_{k-1}]/[h_k\tau(1+\tau)], \tag{2.2}$$

$$\tilde{U}_{k+1}^{(1)} = [(1+2\tau)U_{k+1} - (1+\tau)^2 U_k + \tau^2 U_{k-1}]/[h_k\tau(1+\tau)], \tag{2.3}$$

$$\tilde{U}_{k-1}^{(1)} = [-U_{k+1} + (1+\tau)^2 U_k - \tau(2+\tau)U_{k-1}]/[h_k\tau(1+\tau)], \tag{2.4}$$

In a similar manner, we can obtain approximations $\tilde{V}_k^{(1)}$ and $\tilde{V}_{k\pm1}^{(1)}$ to $V^{(1)}$. We denote

$$\tilde{G}_{k+\theta} = g(r_{k+\theta}, U_{k+\theta}, \tilde{U}_{k+\theta}^{(1)}, V_{k+\theta}, \tilde{V}_{k+\theta}^{(1)}), \theta = 0, \pm 1. \tag{2.5}$$

With the help of Eqs. (2.2)–(2.5), we obtain

$$\begin{aligned} \tilde{G}_k &= g_k + h_k^2 \tau (A_k U_k^{(3)} + D_k V_k^{(3)})/6 + h_k^3 \tau(\tau-1)(A_k U_k^{(4)} + D_k V_k^{(4)})/24 \\ &\quad + h_k^4 \tau^2 [B_k (U_k^{(3)})^2 + 2C_k U_k^{(3)} V_k^{(3)} + E_k (V_k^{(3)})^2]/72 \\ &\quad + h_k^4 \tau(\tau^2 - \tau + 1)(A_k U_k^{(5)} + D_k V_k^{(5)})/120 + O(h_k^5), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \tilde{G}_{k+1} &= g_{k+1} - h_k^2 \tau(1+\tau)[A_k U_k^{(3)} + D_k V_k^{(3)} + h_k \tau (A_k^{(1)} U_k^{(3)} + D_k^{(1)} V_k^{(3)})]/6 \\ &\quad - h_k^3 \tau(\tau+1)(2\tau-1)[A_k U_k^{(4)} + D_k V_k^{(4)} + \tau (A_k^{(1)} U_k^{(4)} + D_k^{(1)} V_k^{(4)})]/24 \\ &\quad - h_k^4 \tau(\tau+1)[(3\tau^2 - 2\tau + 1)(A_k U_k^{(5)} + D_k V_k^{(5)}) + 10\tau^2 (A_k^{(2)} U_k^{(3)} + D_k^{(2)} V_k^{(3)})]/120 \\ &\quad + h_k^4 \tau^2 (\tau+1)^2 [\mathrm{B}_k (U_k^{(3)})^2 + 2C_k U_k^{(3)} V_k^{(3)} + E_k (V_k^{(3)})^2]/72 + O(h_k^5), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \tilde{G}_{k-1} &= g_{k-1} - h_k^2 (1+\tau)[A_k U_k^{(3)} + D_k V_k^{(3)} - h_k (A_k^{(1)} U_k^{(3)} + D_k^{(1)} V_k^{(3)})]/6 \\ &\quad - h_k^3 (\tau^2 - \tau - 2)[A_k U_k^{(4)} + D_k V_k^{(4)} - h_k (A_k^{(1)} U_k^{(4)} + D_k^{(1)} V_k^{(4)})]/24 \\ &\quad - h_k^4 (\tau+1)[(\tau^2 - 2\tau + 3)(A_k U_k^{(5)} + D_k V_k^{(5)}) + 10(A_k^{(2)} U_k^{(3)} + D_k^{(2)} V_k^{(3)})]/120 \\ &\quad + h_k^4 (\tau+1)^2 [B_k (U_k^{(3)})^2 + 2C_k U_k^{(3)} V_k^{(3)} + E_k (V_k^{(3)})^2]/72 + O(h_k^5), \end{aligned} \tag{2.8}$$

where:

$$A_k = (\partial g / \partial U^{(1)})_{r_k},\ B_k = (\partial^2 g / \partial U^{(1)^2})_{r_k},\ C_k = (\partial^2 g / \partial U^{(1)} \partial V^{(1)})_{r_k},$$

$$D_k = (\partial g / \partial V^{(1)})_{r_k} \text{ and } E_k = (\partial^2 g / \partial V^{(1)^2})_{r_k}.$$

By using $\tilde{G}_k$ and $\tilde{G}_{k\pm1}$, one can look for the approximations to the solution values and derivatives;

$$\begin{aligned} &[\hat{U}_{k+1/2}, \hat{U}_{k-1/2}, \hat{U}_{k+1}^{(1)}, \hat{U}_{k-1}^{(1)}, \hat{U}_{k+1/2}^{(1)}, \hat{U}_{k-1/2}^{(1)}]^T = \\ &\qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \begin{bmatrix} U_{k-1} \\ U_k \\ U_{k+1} \end{bmatrix} + h_k^2 \begin{bmatrix} a_{14} & a_{15} & a_{16} \\ \vdots & \vdots & \vdots \\ a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} V_{k-1} \\ V_k \\ V_{k+1} \end{bmatrix}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &[\hat{V}_{k+1/2}, \hat{V}_{k-1/2}, \hat{V}_{k+1}^{(1)}, \hat{V}_{k-1}^{(1)}, \hat{V}_{k+1/2}^{(1)}, \hat{V}_{k-1/2}^{(1)}]^T = \\ &\qquad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{61} & b_{62} & b_{63} \end{bmatrix} \begin{bmatrix} V_{k-1} \\ V_k \\ V_{k+1} \end{bmatrix} + h_k^2 \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ \vdots & \vdots & \vdots \\ b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} \tilde{G}_{k-1} \\ \tilde{G}_k \\ \tilde{G}_{k+1} \end{bmatrix}, \end{aligned} \tag{2.10}$$

where $a_{lm}, b_{lm}, l, m = 1(1)6$ are free parameters to be determined in such a way that we can achieve the following high order approximations

$$\widehat{U}_{k\pm 1/2} - U_{k\pm 1/2} = O(h_k^5), \quad \widehat{V}_{k\pm 1/2} - V_{k\pm 1/2} = O(h_k^5), \tag{2.11}$$

$$\widehat{U}^{(1)}_{k+\theta} - U^{(1)}_{k+\theta} = O(h_k^4), \quad \widehat{V}^{(1)}_{k+\theta} - V^{(1)}_{k+\theta} = O(h_k^4), \quad \theta = \pm 1, \pm 1/2. \tag{2.12}$$

With the help of algebraic calculations using MAPLE (see Ref. [37]), explicit expressions for the free parameters were obtained and they are shown in Table 1, where we have denoted $\sigma = \tau^2 + 3\tau + 1$ and $\rho = \tau^2 + \tau + 1$. Consequently,

$$\widehat{U}^{(1)}_{k+1} = U^{(1)}_{k+1} + h_k^4 \tau^2 (1+\tau)^3 (4+\tau) U_k^{(5)} / (360\sigma) + O(h_k^5), \tag{2.13}$$

$$\widehat{U}^{(1)}_{k-1} = U^{(1)}_{k-1} + h_k^4 (1+\tau)^3 (1+4\tau) U_k^{(5)} / (360\sigma) + O(h_k^5), \tag{2.14}$$

$$\widehat{U}^{(1)}_{k+1/2} = U^{(1)}_{k+1/2} - h_k^4 \tau^2 (4+\tau)(7\tau^3 + 9\tau^2 - 5\tau - 4) U_k^{(5)} / (5760\sigma) + O(h_k^5), \tag{2.15}$$

$$\widehat{U}^{(1)}_{k-1/2} = U^{(1)}_{k-1/2} + h_k^4 (1+4\tau)(4\tau_k^3 + 5\tau_k^2 - 9\tau_k - 7) U_k^{(5)} / (5760\sigma) + O(h_k^5), \tag{2.16}$$

$$\begin{aligned}\widehat{V}^{(1)}_{k+1} = {} & V^{(1)}_{k+1} - h_k^4 \tau^2 (1+\tau)^2 [(2\tau^2 + 2\tau - 1)\{10 A_k^{(1)} U_k^{(3)} + 10 D_k^{(1)} V_k^{(3)} \\ & + 5(A_k U_k^{(4)} + D_k V_k^{(4)})/2\} + (5\tau^2 + 5\tau - 4) V_k^{(5)}] / (360\rho) + O(h_k^5),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\widehat{V}^{(1)}_{k-1} = {} & V^{(1)}_{k-1} + h_k^4 (1+\tau)^2 [(\tau^2 - 2\tau - 2)\{10 A_k^{(1)} U_k^{(3)} + 10 D_k^{(1)} V_k^{(3)} \\ & + 5(A_k U_k^{(4)} + D_k V_k^{(4)})/2\} - (4\tau^2 - 5\tau - 5) V_k^{(5)}] / (360\rho) + O(h_k^5),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\widehat{V}^{(1)}_{k+1/2} = {} & V^{(1)}_{k+1/2} + h_k^4 \tau^2 [(\tau^4 + 3\tau^3 + 2\tau^2 - 2\tau - 1)\{80(A_k^{(1)} U_k^{(3)} + D_k^{(1)} V_k^{(3)}) \\ & + 20(A_k U_k^{(4)} + D_k V_k^{(4)})\} - (23\tau^4 + 63\tau^3 + 31\tau^2 \\ & - 64\tau - 32) V_k^{(5)}] / (5760\rho) + O(h_k^5),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\widehat{V}^{(1)}_{k-1/2} = {} & V^{(1)}_{k-1/2} - h_k^4 [(\tau^4 + 2\tau^3 - 2\tau^2 - 3\tau - 1)\{80(A_k^{(1)} U_k^{(3)} + D_k^{(1)} V_k^{(3)}) \\ & + 20(A_k U_k^{(4)} + D_k V_k^{(4)})\} - (32\tau^4 + 64\tau^3 - 31\tau^2 \\ & - 63\tau - 23) V_k^{(5)}] / (5760\rho) + O(h_k^5).\end{aligned} \tag{2.20}$$

Further, we define

$$\widehat{G}_{k\pm 1} = g(r_{k\pm 1}, U_{k\pm 1}, \widehat{U}^{(1)}_{k\pm 1}, V_{k\pm 1}, \widehat{V}^{(1)}_{k\pm 1}), \tag{2.21}$$

$$\widehat{G}_{k\pm 1/2} = g(r_{k\pm 1/2}, \widehat{U}_{k\pm 1/2}, \widehat{U}^{(1)}_{k\pm 1/2}, \widehat{V}_{k\pm 1/2}, \widehat{V}^{(1)}_{k\pm 1/2}). \tag{2.22}$$

With the help of the above approximations (2.13)–(2.20), we obtain

$$\begin{aligned}\widehat{G}_{k+1} = {} & g_{k+1} - h_k^4 \tau^2 (1+\tau)^2 [(2\tau^2 + 2\tau - 1)(720 D_k^{(1)} (A_k^{(1)} U_k^{(3)} + D_k^{(1)} V_k^{(3)}) \\ & + 180 D_k (A_k U_k^{(4)} + D_k V_k^{(4)})) + 72((\tau^2 + 5\tau + 4)\rho A_k U_k^{(5)} / \sigma \\ & + (5\tau^2 + 5\tau - 4) D_k V_k^{(5)})] / (25920\rho) + O(h_k^5),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\hat{G}_{k-1} = g_{k-1} + h_k^4(1+\tau)^2[(\tau^2 - 2\tau - 2)(720D_k(A_k^{(1)}U_k^{(3)} + D_k^{(1)}V_k^{(3)})\\ +180D_k(A_kU_k^{(4)} + D_kV_k^{(4)})) + 72((4\tau^2 + 5\tau + 1)\rho A_kU_k^{(5)}/\sigma\\ -(4\tau^2 - 5\tau - 5)D_kV_k^{(5)})]/(25920\rho) + O(h_k^5),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\hat{G}_{k+1/2} = g_{k+1/2} + h_k^4\tau^2[20(\tau^4 + 3\tau^3 + 2\tau^2 - 2\tau - 1)D_k(4A_k^{(1)}U_k^{(3)} + 4D_k^{(1)}V_k^{(3)}\\ + A_kU_k^{(4)} + D_kV_k^{(4)}) - (\tau + 4)(7\tau^3 + 9\tau^2 - 5\tau - 4)\rho A_kU_k^{(5)}/\sigma\\ -(23\tau^4 + 63\tau^3 + 31\tau^2 - 64\tau - 32)D_kV_k^{(5)}]/(5760\rho) + O(h_k^5),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\hat{G}_{k-1/2} = g_{k-1/2} - h_k^4[(\tau^4 + 2\tau^3 - 2\tau^2 - 3\tau - 1)20D_k(4A_k^{(1)}U_k^{(3)} + 4D_k^{(1)}V_k^{(3)}\\ + A_kU_k^{(4)} + D_kV_k^{(4)}) - (4\tau + 1)(4\tau^3 + 5\tau^2 - 9\tau - 7)\rho A_kU_k^{(5)}/\sigma\\ -(32\tau^4 + 64\tau^3 - 31\tau^2 - 63\tau - 23)D_kV_k^{(5)}]/(5760\rho) + O(h_k^5).\end{aligned} \tag{2.26}$$

We define additional approximations to the first derivatives:

$$\check{U}_k^{(1)} = \tilde{U}_k^{(1)} + h_k(t_0V_k + t_1V_{k+1} + t_2V_{k-1}) + h_k^3t_3\tilde{G}_{k-1}, \tag{2.27}$$

$$\check{V}_k^{(1)} = \tilde{V}_k^{(1)} + h_k(z_1\tilde{G}_{k+1} + z_2\tilde{G}_{k-1} + z_3\hat{G}_{k+1} + z_4\hat{G}_{k-1} + z_5\tilde{G}_{k+1/2} + z_6\tilde{G}_{k-1/2}), \tag{2.28}$$

where t_k's and z_k's are unknown coefficients to be determined so as to achieve the following final approximations:

$$\begin{aligned}&U_{k+1} - (1+\tau)U_k + \tau U_{k-1}\\ &\qquad + h_k^2(c_0V_k + c_1V_{k+1} + c_2V_{k-1} + c_3\hat{V}_{k+1/2} + c_4\hat{V}_{k-1/2}) = O(h_k^7),\end{aligned} \tag{2.29}$$

$$\begin{aligned}&V_{k+1} - (1+\tau)V_k + \tau V_{k-1}\\ &\qquad + h_k^2(c_0\breve{G}_k + c_1\hat{G}_{k+1} + c_2\hat{G}_{k-1} + c_3\hat{G}_{k+1/2} + c_4\hat{G}_{k-1/2}) = O(h_k^7),\end{aligned} \tag{2.30}$$

where $k = 1(1)n$ and $\breve{G}_k$ is an extra approximation to G_k, to be determined.

The explicit expressions for the unknown coefficients are given in Table 2, where we have denoted $\delta = 3\tau^2 + 7\tau + 3$. From Eqs. (2.7), (2.8) and (2.23)–(2.26), we obtain

$$\begin{aligned}\breve{U}_k^{(1)} = U_k^{(1)} + h_k(t_0 + t_1 + t_2)U_k^{(2)} + h_k^3[(1+12t_1)\tau^2 + 12t_2 + 24t_3 - \tau]U_k^{(4)}/24\\ + h_k^2[(6t_1 + 1)\tau - 6t_2]U_k^{(3)}/6 + h_k^4[(1+20t_1)\tau^3\\ -20t_2 - 120t_3 - \tau^2 + \tau]U_k^{(5)}/120 + O(h_k^5),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\breve{V}_k^{(1)} = V_k^{(1)} + h_k(z_1 + z_2 + z_3 + z_4 + z_5 + z_6)U_k^{(4)} + h_k^2[\tau(1 + 6z_1 + 6z_3 + 3z_5)\\ -3(2z_2 + 2z_4 + z_6)]U_k^{(5)}/6 + h_k^3(1+\tau)(z_1\tau + z_2)(A_kU_k^{(3)} - D_kV_k^{(3)})/6\\ + h_k^3[\tau^2(1 + 3z_5 + 12z_3 + 12z_1) + 3z_6 + 12z_2 + 12z_4 - \tau]U_k^{(6)}/24\\ - h_k^4(1+\tau)(\tau(2\tau - 1)z_1 + (\tau - 2)z_2)(A_kU_k^{(4)} + D_kV_k^{(4)})/24\end{aligned}$$

$$-h_k^4(1+\tau)(z_1\tau^2-z_2)(A_k^{(1)}U_k^{(3)}+D_k^{(1)}V_k^{(3)})/6+h_k^4[2(\tau^3-\tau^2+\tau)$$
$$+40(\tau^3(z_1+z_3)-z_2-z_4)/240+5(z_5\tau^3-z_6)]V_k^{(5)}+O(h_k^5). \tag{2.32}$$

Finally, by using Eqs. (2.27) and (2.28), we define

$$\breve{G}_k=g(r_k,U_k,\breve{U}_k^{(1)},V_k,\breve{V}_k^{(1)}). \tag{2.33}$$

Hence, we have obtained the final geometric mesh finite difference scheme (2.29) and (2.30), which is compact and applicable to the numerical solution of the boundary value problem (1.1) or (1.2) and (1.3). A more detailed analysis reveals that the local truncation error of the scheme is $(\tau-1)O(h_k^7)+O(h_k^8)$ and hence in the case of a uniform mesh ($\tau = 1$), the proposed method is sixth order accurate.

The scheme needs an amendment in the vicinity of a singularity, which arises when, for example, our domain of integration is [0, 1] and we need to evaluate the terms like r_{k-1}^{-1} at $k = 1$. This leads to the division by zero and hence in order to avoid such situations, we need to incorporate the Taylor's approximations $r_{k-1}^{-1}=\sum_{l=0(1)4}h_r^l r_k^{-(1+l)}+O(h_k^5),$ into Eqs. (2.29) and (2.30). The resulting scheme is applicable to singular ODEs such as ODEs involving the Laplacian operator in cylindrical and spherical coordinates. For practical implementations, one replaces the exact values U_k and V_k present in Eqs. (2.29) and (2.30) by approximate values u_k and v_k, and one omits the residual terms $O(h_k^7)$. The resulting system of algebraic equations for u_k and v_k must be extended with boundary conditions.

3. Convergence analysis

In this section, we discuss the convergence property of the proposed finite difference scheme (2.29) and (2.30) for the numerical solution of the two point boundary value problem (1.1). At $r = r_k$, $k = 1(1)n$, Eq. (1.1) can be written as

$$U_k^{(2)}=V_k, V_k^{(2)}=g(r_k,U_k,U_k^{(1)},V_k,V_k^{(1)})\equiv G_k, k=1(1)n. \tag{3.1}$$

Then, the geometric mesh finite difference method (2.29)–(2.30) is given by

$$\begin{cases}\phi_k(U_{k-1},U_k,U_{k+1},V_{k-1},V_k,V_{k+1})+L_k(h_k)=0,\\ \varphi_k(U_{k-1},U_k,U_{k+1},V_{k-1},V_k,V_{k+1})+M_k(h_k)=0, k=1(1)n,\end{cases} \tag{3.2}$$

where

$$\phi_k=-U_{k+1}+(1+\tau)U_k-\tau U_{k-1}$$
$$-h_k^2(c_0V_k+c_1V_{k+1}+c_2V_{k-1}+c_3\widehat{V}_{k+1/2}+c_4\widehat{V}_{k-1/2}),$$

$$\varphi_k=-V_{k+1}+(1+\tau)V_k-\tau V_{k-1}$$
$$-h_k^2(c_0\breve{G}_k+c_1\widehat{G}_{k+1}+c_2\widehat{G}_{k-1}+c_3\widehat{G}_{k+1/2}+c_4\widehat{G}_{k-1/2}),$$

$$L_k(h_k) = O(h_k^7) \quad \text{and} \quad M_k(h_k) = O(h_k^7).$$

The scheme (3.2) in the matrix/vector notation is written as

$$\begin{cases} \boldsymbol{\phi}(\boldsymbol{U},\boldsymbol{V}) + \boldsymbol{L} = \mathbf{0} \\ \boldsymbol{\varphi}(\boldsymbol{U},\boldsymbol{V}) + \boldsymbol{M} = \mathbf{0}, \end{cases} \tag{3.3}$$

where

$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}.$$

We wish to find the approximations $\boldsymbol{u}$ and $\boldsymbol{v}$ for $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, which are determined by solving $2n \times 2n$ systems

$$\begin{cases} \boldsymbol{\phi}(\boldsymbol{u},\boldsymbol{v}) = \mathbf{0} \\ \boldsymbol{\varphi}(\boldsymbol{u},\boldsymbol{v}) = \mathbf{0}. \end{cases} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\begin{cases} \boldsymbol{\phi}(\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{\phi}(\boldsymbol{U},\boldsymbol{V}) = \boldsymbol{L} \\ \boldsymbol{\varphi}(\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{\varphi}(\boldsymbol{U},\boldsymbol{V}) = \boldsymbol{M}. \end{cases} \tag{3.5}$$

Let $\varepsilon_k = u_k - U_k$, $\eta_k = v_k - V_k$, $k = 1(1)n$ be the discretization errors and $\boldsymbol{\varepsilon} = \boldsymbol{u} - \boldsymbol{U}$, $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ be the vectors of these errors. Let us denote

$$\tilde{g}_{k+\theta} = g(r_{k+\theta}, u_{k+\theta}, \tilde{u}_{k+\theta}^{(1)}, v_{k+\theta}, \tilde{v}_{k+\theta}^{(1)}) \simeq \tilde{G}_{k+\theta},\ \theta = 0, \pm 1,$$

$$\widehat{g}_{k\pm1} = g(r_{k\pm1}, u_{k\pm1}, \widehat{u}_{k\pm1}^{(1)}, v_{k\pm1}, \widehat{v}_{k\pm1}^{(1)}) \simeq \widehat{G}_{k\pm1},$$

$$\widehat{g}_{k\pm1/2} = g(r_{k\pm1/2}, \widehat{u}_{k\pm1/2}, \widehat{u}_{k\pm1/2}^{(1)}, \widehat{v}_{k\pm1/2}, \widehat{v}_{k\pm1/2}^{(1)}) \simeq \widehat{G}_{k\pm1/2}$$

$$\breve{g}_k = g(r_k, u_k, \breve{u}_k^{(1)}, v_k, \breve{v}_k^{(1)}) \simeq \breve{G}_k,$$

$$\tilde{E}_{k+\theta} = \tilde{g}_{k+\theta} - \tilde{G}_{k+\theta},\ \theta = 0, \pm 1,$$

$$\widehat{E}_{k\pm\theta} = \widehat{g}_{k\pm\theta} - \widehat{G}_{k\pm\theta},\ \theta = 1, 1/2,$$

$$\breve{E}_k = \breve{g}_k - \breve{G}_k,$$

$$\tilde{\varepsilon}_{k+\theta}^{(1)} = \tilde{u}_{k+\theta}^{(1)} - \tilde{U}_{k+\theta}^{(1)},\ \tilde{\eta}_{k+\theta}^{(1)} = \tilde{v}_{k+\theta}^{(1)} - \tilde{V}_{k+\theta}^{(1)},\ \theta = 0, \pm 1,$$

$$\widehat{\varepsilon}_{k\pm1/2} = \widehat{u}_{k\pm1/2} - \widehat{U}_{k\pm1/2},\ \widehat{\eta}_{k\pm1/2} = \widehat{v}_{k\pm1/2} - \widehat{V}_{k\pm1/2},$$

$$\widehat{\varepsilon}_{k\pm\theta}^{(1)} = \widehat{u}_{k\pm\theta}^{(1)} - \widehat{U}_{k\pm\theta}^{(1)},\ \widehat{\eta}_{k\pm\theta}^{(1)} = \widehat{v}_{k\pm\theta}^{(1)} - \widehat{V}_{k\pm\theta}^{(1)},\ \theta = 1, 1/2,$$

$$\breve{\varepsilon}_k^{(1)} = \breve{u}_k^{(1)} - \breve{U}_k^{(1)},\ \breve{\eta}_k^{(1)} = \breve{v}_k^{(1)} - \breve{V}_k^{(1)},$$

$$\tilde{\xi}_k^{(1)} = [\xi_{k+1} - (1-\tau^2)\xi_k - \tau^2\xi_{k-1}] / [h_k\tau(1+\tau)],\ \xi \in \{\varepsilon, \eta\},$$

$$\tilde{\xi}_{k+1}^{(1)} = [(1+2\tau)\xi_{k+1} - (1+\tau)^2\xi_k + \tau^2\xi_{k-1}] / [h_k\tau(1+\tau)],$$

$$\tilde{\xi}^{(1)}_{k-1} = [-\xi_{k+1} + (1+\tau)^2 \xi_k - \tau(2+\tau)\xi_{k-1}]/[h_k \tau(1+\tau)].$$

By applying the mean value theorem, one obtains:

$$\tilde{E}_{k+\theta} = \alpha_{k+\theta}\tilde{\varepsilon}^{(1)}_{k+\theta} + \beta_{k+\theta}\varepsilon_{k+\theta} + \gamma_{k+\theta}\tilde{\eta}^{(1)}_{k+\theta} + \delta_{k+\theta}\eta_{k+\theta},\ \theta = 0, \pm 1, \tag{3.6}$$

where

$$\alpha_l = \left.\frac{\partial g}{\partial u^{(1)}}\right|_{r=r_1},\quad \beta_l = \left.\frac{\partial g}{\partial u}\right|_{r=r_1},\quad \gamma_l = \left.\frac{\partial g}{\partial v^{(1)}}\right|_{r=r_1},\quad \delta_l = \left.\frac{\partial g}{\partial v}\right|_{r=r_1},\quad l = k, k\pm 1, k\pm 1/2.$$

Let us define:

$$[\hat{\varepsilon}_{k+1/2}, \hat{\varepsilon}_{k-1/2}, \hat{\varepsilon}^{(1)}_{k+1}, \hat{\varepsilon}^{(1)}_{k-1}, \hat{\varepsilon}^{(1)}_{k+1/2}, \hat{\varepsilon}^{(1)}_{k-1/2}]^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \begin{bmatrix} \varepsilon_{k-1} \\ \varepsilon_k \\ \varepsilon_{k+1} \end{bmatrix} + h_k^2 \begin{bmatrix} a_{14} & a_{15} & a_{16} \\ \vdots & \vdots & \vdots \\ a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \eta_{k-1} \\ \eta_k \\ \eta_{k+1} \end{bmatrix}, \tag{3.7}$$

$$[\hat{\eta}_{k+1/2}, \hat{\eta}_{k-1/2}, \hat{\eta}^{(1)}_{k+1}, \hat{\eta}^{(1)}_{k-1}, \hat{\eta}^{(1)}_{k+1/2}, \hat{\eta}^{(1)}_{k-1/2}]^T = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{61} & b_{62} & b_{63} \end{bmatrix} \begin{bmatrix} \eta_{k-1} \\ \eta_k \\ \eta_{k+1} \end{bmatrix} + h_k^2 \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ \vdots & \vdots & \vdots \\ b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} \tilde{E}_{k-1} \\ \tilde{E}_k \\ \tilde{E}_{k+1} \end{bmatrix}, \tag{3.8}$$

where are coefficients given in Table 1 and 2, and

$$\hat{E}_{k\pm1} = \alpha_{k\pm1}\,\hat{\varepsilon}^{(1)}_{k\pm1} + \beta_{k\pm1}\varepsilon_{k\pm1} + \gamma_{k\pm1}\,\hat{\eta}^{(1)}_{k\pm1} + \delta_{k\pm1}\eta_{k\pm1}, \tag{3.9}$$

$$\hat{E}_{k\pm1/2} = \alpha_{k\pm1/2}\,\hat{\varepsilon}^{(1)}_{k\pm1/2} + \beta_{k\pm1/2}\,\varepsilon_{k\pm1/2} + \gamma_{k\pm1/2}\,\hat{\eta}^{(1)}_{k\pm1/2} + \delta_{k\pm1/2}\eta_{k\pm1/2}, \tag{3.10}$$

$$\breve{\varepsilon}^{(1)}_k = \tilde{\varepsilon}^{(1)}_k + h_k(t_0\eta_k + t_1\eta_{k+1} + t_2\eta_{k-1}) + h_k^3 t_3 \tilde{E}_{k-1}, \tag{3.11}$$

$$\breve{\eta}^{(1)}_k = \tilde{\eta}^{(1)}_k + h_k(z_1\tilde{E}_{k+1} + z_2\tilde{E}_{k-1} + z_3\,\hat{E}_{k+1} + z_4\,\hat{E}_{k-1} + z_5\,\hat{E}_{k+1/2} + z_6\,\hat{E}_{k-1/2}), \tag{3.12}$$

$$\breve{E}_k = \alpha_k\breve{\varepsilon}^{(1)}_k + \beta_k\varepsilon_k + \gamma_k\breve{\eta}^{(1)}_k + \delta_k\eta_k. \tag{3.13}$$

In view of the Eq. (3.5), we obtain

$$\begin{aligned} R_k &\equiv \phi_k(u_{k-1}, u_k, u_{k+1}, v_{k-1}, v_k, v_{k+1}) - \phi_k(U_{k-1}, U_k, U_{k+1}, V_{k-1}, V_k, V_{k+1}) \\ &= -\varepsilon_{k+1} + (1+\tau)\varepsilon_k - \tau\varepsilon_{k-1} - h_k^2(c_0\eta_k + c_1\eta_{k+1} + c_2\eta_{k-1} + c_3\,\hat{\eta}_{k+1/2} + c_4\,\hat{\eta}_{k-1/2}), \\ S_k &\equiv \varphi_k(u_{k-1}, u_k, u_{k+1}, v_{k-1}, v_k, v_{k+1}) - \varphi_k(U_{k-1}, U_k, U_{k+1}, V_{k-1}, V_k, V_{k+1}) \\ &= -\eta_{k+1} + (1+\tau)\eta_k - \tau\eta_{k-1} - h_k^2(c_0\breve{E}_k + c_1\,\hat{E}_{k+1} + c_2\,\hat{E}_{k-1} + c_3\,\hat{E}_{k+1/2} + c_4\,\hat{E}_{k-1/2}). \end{aligned}$$

Equivalently, in the matrix notation

$$\begin{bmatrix} \boldsymbol{\phi}(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{\phi}(\boldsymbol{U}, \boldsymbol{V}) \\ \boldsymbol{\varphi}(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{\varphi}(\boldsymbol{U}, \boldsymbol{V}) \end{bmatrix} = \boldsymbol{P}\begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\eta} \end{bmatrix}, \tag{3.14}$$

where

$\boldsymbol{P} =$

$$\text{tridiag}\left(\begin{bmatrix} C(R_k,\varepsilon_{k-1}) & C(R_k,\eta_{k-1}) \\ C(S_k,\varepsilon_{k-1}) & C(S_k,\eta_{k-1}) \end{bmatrix}, \begin{bmatrix} C(R_k,\varepsilon_k) & C(R_k,\eta_k) \\ C(S_k,\varepsilon_k) & C(S_k,\eta_k) \end{bmatrix}, \begin{bmatrix} C(R_k,\varepsilon_{k+1}) & C(R_k,\eta_{k+1}) \\ C(S_k,\varepsilon_{k+1}) & C(S_k,\eta_{k+1}) \end{bmatrix}\right)$$

is a block tridiagonal matrix and $C(R_k, \eta_k)$ = Coefficient of η_k in R_k etc.

From (3.5) and (3.14), one obtains

$$\boldsymbol{P}\xi = \boldsymbol{T}, \quad \boldsymbol{T} = [\boldsymbol{L} \quad \boldsymbol{M}]^T, \quad \xi = [\varepsilon \quad \eta]^T. \tag{3.15}$$

In the limiting case of small h_k, matrix $\boldsymbol{P}$ takes the form

$$\lim_{h_k \to 0} \boldsymbol{P} = \text{tridiag}\left(\begin{bmatrix} -\tau & 0 \\ 0 & -\tau \end{bmatrix}, \begin{bmatrix} 1+\tau & 0 \\ 0 & 1+\tau \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right).$$

Thus, the lower, upper and main diagonal blocks are non-zero, since $\tau > 0$. Hence the graph $G(\boldsymbol{P})$ of the matrix $\boldsymbol{P}$ is strongly connected and consequently, the matrix $\boldsymbol{P}$ is irreducible (Varga [38]).

Let

$$\alpha = \min_k \{\alpha_k, \alpha_{k\pm 1}, \alpha_{k\pm 1/2}\}, \ \beta = \min_k \{\beta_k, \beta_{k\pm 1}, \beta_{k\pm 1/2}\},$$

$$\gamma = \min_k \{\gamma_k, \gamma_{k\pm 1}, \gamma_{k\pm 1/2}\}, \ \delta = \min_k \{\delta_k, \delta_{k\pm 1}, \delta_{k\pm 1/2}\}.$$

Further, let $\sum_l$ be the sum of the l^{th} row elements of the matrix $\boldsymbol{P}$, then

For $l = 1$: $\quad \sum_l \geq \tau + O(h_l^2), \sum_{l+1} \geq \tau + O(h_l)$.

For $l = 3(2)2n-2$: $\quad \sum_l \geq \frac{h_l^2}{2}\tau(1+\tau) + O(h_l^4), \sum_{l+1} \geq \frac{h_1^2}{2}\tau(1+\tau)(\beta+\delta) + O(h_l^3)$.

For $l = 2n-1$: $\quad \sum_l \geq 1 + O(h_l^2), \sum_{l+1} \geq 1 + O(h_l)$.

This implies that for sufficiently small value of h_k, i.e. in the limiting case of $h_k \to 0$,

$$\sum_l \geq \tau > 0, l = 1,2; \quad \sum_l \geq 0, \ l = 3(1)2n-2; \quad \sum_l \geq 1 > 0, l = 2n-1, 2n.$$

Hence, $\boldsymbol{P}$ is monotone (Henrici [39], Young [40]). Consequently $\boldsymbol{P}^{-1}$ exists and is non-negative. Let $P_{i,l}^{-1}$ be the $(i,l)^{th}$ element of $\boldsymbol{P}^{-1}$, and define

$$\left\|\boldsymbol{P}^{-1}\right\|_\infty = \max_{1\leq i\leq 2n} \sum_{l=1}^{2n} \left|P_{i,l}^{-1}\right|, \quad \|T\| = \max_{1\leq l\leq 2n} \sum_{l=1}^{2n} \left|L_l(h_l) + M_l(h_l)\right| = O(h_l^7).$$

From the obvious identity, $\boldsymbol{P}^{-1} = (\boldsymbol{PJ}) = \boldsymbol{J}$, where $\boldsymbol{J} = [1, 1, \ldots, 1]^T$, we obtain

$$\sum_{l=1}^{2n} P_{i,l}^{-1} \sum_l = 1, \ i = 1(1)2n. \tag{3.16}$$

Thus, the following bounds can be estimated by using Taylor series expansions

For $l = 1$:

$$P_{i,l}^{-1} \leq \sum_l^{-1} = \frac{1}{\tau} + O(h_l^2),$$

$$P^{-1}_{i,l+1} \le \Sigma^{-1}_{l+1} \le \frac{1}{\tau} + O(h_l\ ).$$

For $l = 3(2)2n - 2$:

$$P^{-1}_{i,l} \le \min_l \Sigma^{-1}_l \le \frac{2}{\tau(1+\tau)h_l^2} + O(h_l^v),\ v \ge 0,$$

$$P^{-1}_{i,l+1} \le \min_l \Sigma^{-1}_{l+1} \le \frac{2}{\tau(1+\tau)(\beta+\delta)h_l^2} + O(h_l^v),\ v \ge 0.$$

For $l = 2n - 1$:

$$P^{-1}_{i,l} \le \Sigma^{-1}_l = 1 + O(h_l^2),$$

$$P^{-1}_{i,l+1} \le \Sigma^{-1}_{l+1} \le 1 + O(h_l\ ).$$

As a result, from Eqs. (3.15) and (3.16), we obtain the following error estimates:

$$\|\xi\| \le \left\|\boldsymbol{P}^{-1}\right\|_{\infty}.\ \|\boldsymbol{T}\| \le O(h_l^5),\ \text{provided that } \beta + \delta \ne 0. \tag{3.17}$$

This proves the fifth order convergence of the proposed method. Another result is that the coefficients c_k, $k = 0(1)4$ in Eq. (2.1) are negative if $(\sqrt{3}-1)/2 < \tau$ and hence we obtain a lower bound on τ, whereas the upper bound on τ is less than 1.5, otherwise the grid will be too non-uniform to be practical. Thus, we summarise the above result in the following theorem:

Theorem 3.1. The geometric mesh finite difference method (2.29) and (2.30) for the numerical solution of differential equation (1.1) or (1.2) and (1.3) with sufficiently small h_k and $(\sqrt{3}-1)/2 < \tau < 1.5,\ \tau \ne 1,$ gives a fifth order of convergent solution provided that $\frac{\partial g}{\partial U} + \frac{\partial g}{\partial V} \ne 0.$

4. Computational experiment

To verify the theoretical predictions, we have solved several linear and nonlinear problems. We defined the geometric mesh as follows

$$r_0 = a, h_1 = \begin{cases} (b-a)(1-\tau)/(1-\tau^{n+1}), & \tau < 1 \\ (b-a)(\tau-1)/(\tau^{n+1}-1), & \tau > 1 \end{cases}$$

Hence, $h_{k+1} = \tau h_k$, $k = 1(1)n$. If a boundary value problem exhibits a boundary layer at the left boundary, choosing $\tau > 1$ is appropriate. If the layer occurs at the right boundary, we choose $\tau < 1$. If the layer occurs in the interior region, then the mesh can be arranged by choosing $\tau > 1$ in the first half of the interval and $\tau < 1$ in the second half.

The numerical accuracy of the results is expressed using maximum absolute errors $(\varepsilon^{(\infty)}_{u^{(m)}})$ and computational orders of convergence (Θ_m) for m^{th} order derivatives of $u(r)$.

$$\varepsilon^{(\infty)}_{u^{(m)}} = \max\nolimits_{1\le k\le n}\left|u_k^{(m)} - U_k^{(m)}\right|,\ \Theta_m = \log_2\left(\frac{\varepsilon^{(2)}_{u^{(m)}}\Big|_{n\,\text{grids}}}{\varepsilon^{(2)}_{u^{(m)}}\Big|_{2n\,\text{grids}}}\right).$$

Numerical computations were performed using long double arithmetic extended precision variables having 80 bits and 18 digits precision. The code was written in *C* and run under Linux operating system. For solving linear or nonlinear algebric equations resulting from the discretisation, the Newton method and the Thomas algorithm were used, with the error tolerance being $\le 10^{-15}$.

Example 4.1 (Conte [41]) The fourth order two point boundary value problem

$$U^{(4)}(r) - (1+\lambda)U^{(2)}(r) + \lambda U(r) = \frac{\lambda}{2}r^2 + 1, 0 < r < 1,$$

$$U(0) = 1,\ U(1) = \frac{3}{2} + \sinh(1),\ U^{(2)}(0) = 1,\ U^{(2)}(1) = 1 + \sinh(1),$$

possesses analytical solution $U(r) = 1 + \frac{r^2}{2} + \sinh(r).$ We know that ± 1 and $\pm\lambda$ are the eigenvalues of this equation and hence the problem is stiff for large values of λ. We have solved the problem for small as well as for large values of λ. The solution is found accurate for $\lambda < 10^8$ both in the case of uniform and geometric meshes. Table 3 presents errors of the approximate solutions and computational orders of convergence obtained for $\lambda = 10^8$, in the case of uniform meshes $(\tau = 1)$ and geometric meshes $(\tau \ne 1)$. It is evident that the geometric mesh technique is superior to the uniform mesh.

Example 4.2 (Mohanty [33]) The fourth order singular linear problem in polar coordinates

$$\nabla^4 U(r) \equiv \left(\frac{d^2}{dr^2} + \frac{\lambda}{r}\frac{d}{dr}\right)^2 U(r) = \left(1 + \frac{2\lambda}{r} + \frac{\lambda(\lambda-2)}{r^2} - \frac{\lambda(\lambda-2)}{r^3}\right)e^r,\ 0 < r < 1,$$

$$U(0) = U^{(2)}(0) = 1,\ U(1) = U^{(2)}(1) = e,$$

possesses analytical solution $U(r) = e^r$. The choice of $\lambda = 0,1$ and 2, corresponds to Cartesian, cylindrical and spherical coordinates respectively.The errors for the various values of n and λ are reported in Table 4.

Example 4.3 (Elcrat [42]) The nonlinear boundary value problem arising from a model of the axisymmetric flow of an incompressible fluid contained between infinite disks is:

$$U^{(4)}(r) = \lambda U(r)U^{(2)}(r) - \lambda(r^2-1)(1+4r+r^2)e^{2r} - (11+8r+r^2)e^r,\ 0 < r < 1,$$

$$U(0) = 1,\ U(1) = 0,\ U^{(2)}(0) = -1,\ U^{(2)}(1) = -6e.$$

The analytical solution is $U(r) = (1 - r^2)e^r$. The errors obtained are given in Table 5, for various values of n, and for $\lambda = 10^3$.

Example 4.4 (Takaoka [43]) The boundary value problem arising from the steady state form of the Korteweg-de Vries equation of fifth order is:

$$U^{(4)}(r) = \lambda U^{(2)}(r) + \frac{1}{2}U(r)^2 - U(r)$$

$$+\frac{\lambda}{2}\sin(10\pi r)[2 + 200\pi^2(\lambda + 100\pi^2) - \lambda\sin(10\pi r)],$$

$$U(0) = U(1) = U^{(2)}(0) = U^{(2)}(1) = 0,\ 0 < r < 1.$$

The analytical solution is $U(r) = \lambda\sin(10\pi r)$. The maximum absolute errors obtained for $\lambda = 4$ are given in Table 6 for various values of n.

5. Conclusion and remarks

A compact, three point finite difference scheme using geometric mesh has been designed to obtain accurate numerical solutions of fourth order two point regular and singular boundary value problems for nonlinear ordinary differential equations. The theoretical order of accuracy is 5 (or 6 in the limit of uniform meshes).The scheme is shown theoretically to be convergent when the grid ratio τ is $(\sqrt{3}-1)/2 < \tau < 1.5$.

Computational tests confirm that the scheme converges and is applicable both to singular and non singular differential equations. Numerical solutions obtained using geometric meshes prove more accurate than those corresponding to uniform meshes, when local layers are present. The scheme can be effectively combined with the Newton-method and Thomas algorithm for solving block-tridiagonal linear algebraic systems arising in the calculations.

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Table 1

Expressions for the coefficients a_{lm}, b_{lm}, $l, m = 1(1)6$ in Eqs. (2.9) and (2.10)

$a_{11} = -\tau^3(5\tau+12)/[16\sigma(1+\tau)]$	$b_{11} = -3\tau^4/[16\rho(1+\tau)]$
$a_{12} = (\tau+2)(5\tau^2+10\tau+4)/(16\sigma)$	$b_{12} = (\tau+2)(3\tau^2+2\tau+4)/(16\rho)$
$a_{13} = (\tau+2)(3\tau^2+14\tau+4)/[16\sigma(1+\tau)]$	$b_{13} = (\tau+2)(5\tau^2+6\tau+4)/[16\rho(1+\tau)]$
$a_{14} = (\tau+2)(4\tau+3)\tau^3/[96\sigma(1+\tau)]$	$b_{14} = \tau^4(\tau+2)^2/[96\rho(1+\tau)]$
$a_{15} = 0$	$b_{15} = -\tau^2(\tau+2)(\tau^2+2\tau+3)/(96\rho)$
$a_{16} = -\tau^2(\tau+2)(\tau^2+6\tau+6)/[96\sigma(1+\tau)]$	$b_{16} = -\tau^2(\tau+2)(2\tau^2+4\tau+3)/[96\rho(\tau+1)]$
$a_{21} = (2\tau+1)(4\tau^2+14\tau+3)/[16\sigma(\tau+1)]$	$b_{21} = (2\tau+1)(4\tau^2+6\tau+5)/[16\rho(\tau+1)]$
$a_{22} = (2\tau+1)(4\tau^2+10\tau+5)/(16\sigma\tau)$	$b_{22} = (2\tau+1)(4\tau^2+2\tau+3)/(16\rho\tau)$
$a_{23} = -(12\tau+5)/[16\sigma(1+\tau)\tau]$	$b_{23} = -3/[16\rho(\tau+1)\tau]$
$a_{24} = -(2\tau+1)(6\tau^2+6\tau+1)/[96\sigma(1+\tau)]$	$b_{24} = -(2\tau+1)(3\tau^2+4\tau+2)/[96\rho(\tau+1)]$
$a_{25} = 0$	$b_{25} = (2\tau+1)(3\tau^2+2\tau+1)/(96\rho\tau)$
$a_{26} = (3\tau+4)(2\tau+1)/[96\sigma(1+\tau)]$	$b_{26} = (2\tau+1)^2/[96\rho(\tau+1)\tau]$
$a_{31} = (\tau+2)\tau^2/[h_k\sigma(1+\tau)]$	$b_{31} = -\tau^2(\tau+2)/[h_k\rho(\tau+1)]$
$a_{32} = -(\tau+1)^2/(h_k\sigma\tau)$	$b_{32} = (\tau-1)(\tau+1)^2/(h_k\rho\tau)$
$a_{33} = (2\tau^3+6\tau^2+4\tau+1)/[h_k\sigma(1+\tau)\tau]$	$b_{33} = (2\tau+1)/[h_k\rho(\tau+1)\tau]$
$a_{34} = -(\tau+1)\tau^2/(6h_k\sigma)$	$b_{34} = \tau^2(1-\tau^2)/(6h_k\rho)$
$a_{35} = 0$	$b_{35} = \tau(2+\tau)(1+\tau)^2/(6h_k\rho)$
$a_{36} = \tau(\tau+3)(\tau+1)/(6h_k\sigma)$	$b_{36} = \tau(1+\tau)(1+2\tau)/(6h_k\rho)$
$a_{41} = -(\tau^3+4\tau^2+6\tau+2)/[h_k\sigma(1+\tau)]$	$b_{41} = -\tau^2(\tau+2)/[h_k\rho(1+\tau)]$
$a_{42} = (\tau+1)^3/(h_k\sigma\tau)$	$b_{42} = (\tau-1)(\tau+1)^2/(h_k\rho\tau)$
$a_{43} = -(2\tau+1)/[h_k\sigma(1+\tau)\tau]$	$b_{43} = (2\tau+1)/[h_k\rho(\tau+1)\tau]$
$a_{44} = -(\tau+1)(3\tau+1)/(6h_k\sigma)$	$b_{44} = -(\tau+2)(\tau+1)/(6h_k\rho)$
$a_{45} = 0$	$b_{45} = -(2\tau+1)(\tau+1)^2/(6h_k\rho\tau)$
$a_{46} = (\tau+1)/(6h_k\sigma)$	$b_{46} = (1-\tau^2)/(6h_k\rho)$

$a_{51} = \tau^2 / [2h_k\sigma(1+\tau)]$	$b_{51} = \tau^2(\tau+2) / [2h_k\rho(1+\tau)]$
$a_{52} = -(3\tau^2 + 6\tau + 2) / (2h_k\sigma\tau)$	$b_{52} = -(\tau^3 + 4\tau^2 + 2\tau + 2) / (2h_k\rho\tau)$
$a_{53} = (\tau^2 + 4\tau + 2)(2\tau + 1) / [2h_k\sigma(1+\tau)\tau]$	$b_{53} = (3\tau^3 + 6\tau^2 + 4\tau + 2) / [2h_k\rho(\tau+1)\tau]$
$a_{54} = (\tau^2 - \tau - 1)\tau^2 / [24h_k\sigma(1+\tau)]$	$b_{54} = \tau^2(\tau^2 - \tau - 1)(\tau+2) / [24h_k\rho(1+\tau)]$
$a_{55} = 0$	$b_{55} = -\tau(\tau^3 + 4\tau^2 + 6\tau + 1) / (24h_k\rho)$
$a_{56} = -(\tau_k^2 + 5\tau_k + 5)\tau_k^2 / [24h_k\sigma_k(1+\tau_k)]$	$b_{56} = -\tau(2\tau^3 + 5\tau^2 + 3\tau - 1) / [24h_k\rho(1+\tau)]$
$a_{61} = -(2\tau^2 + 4\tau + 1)(\tau+2) / [2h_k\sigma(1+\tau)]$	$b_{61} = -(2\tau^3 + 4\tau^2 + 6\tau + 3) / [2h_k\rho(1+\tau)]$
$a_{62} = (2\tau^2 + 6\tau + 3) / (2h_k\sigma)$	$b_{62} = (2\tau^3 + 2\tau^2 + 4\tau + 1) / (2h_k\rho\tau)$
$a_{63} = -1 / [2h_k\sigma(1+\tau)]$	$b_{63} = -(1 + 2\tau) / [2h_k\rho(1+\tau)\tau]$
$a_{64} = (5\tau^2 + 5\tau + 1) / [24h_k\sigma(1+\tau)]$	$b_{64} = -(\tau^3 - 3\tau^2 - 5\tau - 2) / [24h_k\rho(1+\tau)]$
$a_{65} = 0$	$b_{65} = (\tau^3 + 6\tau^2 + 4\tau + 1) / (2h_k\rho)$
$a_{66} = (\tau^2 + \tau - 1) / [24h_k\sigma(1+\tau)]$	$b_{66} = (\tau^2 + \tau - 1)(1 + 2\tau) / [24h_k\rho(1+\tau)\tau]$

Table 2

Expressions for the coefficients t_i, $i = 0(1)3$, z_j $j = 1(1)6$ in Eqs. (2.27) and (2.28)

$t_0 = -(1+\tau)(27\tau^5 + 133\tau^4 + 155\tau^3 - 10\tau^2 - 62\tau - 18) / [60\delta\sigma(2+\tau)]$
$t_1 = -(3\tau^6 + 60\tau^5 + 302\tau^4 + 555\tau^3 + 422\tau^2 + 140\tau + 18) / [60\sigma(2+\tau)(1+\tau)\delta]$
$t_2 = \tau(27\tau^6 + 190\tau^5 + 508\tau^4 + 735\tau^3 + 628\tau^2 + 270\tau + 42) / [60\delta\sigma(2+\tau)(1+\tau)]$
$t_3 = -\tau(12\tau^6 + 65\tau^5 + 103\tau^4 + 90\tau^3 + 103\tau^2 + 65\tau + 12) / [120\delta\sigma(2+\tau)]$
$z_1 = (6\tau^6 + 15\tau^5 - \tau^4 - 28\tau^3 - \tau^2 + 15\tau + 6) / [6\delta\rho(1+\tau)^2]$
$z_2 = -\tau(6\tau^6 + 15\tau^5 - \tau^4 - 28\tau^3 - \tau^2 + 15\tau + 6) / [6\delta\rho(1+\tau)^2]$
$z_3 = -(27\tau^7 + 70\tau^6 + 20\tau^5 - 52\tau^4 + 83\tau^3 + 100\tau^2 + 25\tau - 3) / [30\delta\rho(1+2\tau)(1+\tau)^2]$
$z_4 = -\tau(3\tau^7 - 25\tau^6 - 100\tau^5 - 83\tau^4 + 52\tau^3 - 20\tau^2 - 70\tau - 27) / [30\delta\rho(2+\tau)(1+\tau)^2]$
$z_5 = -(48\tau^6 + 157\tau^5 + 133\tau^4 - 21\tau^3 + 83\tau^2 + 107\tau + 33) / [15\delta\rho(2+\tau)(1+\tau)]$
$z_6 = \tau(33\tau^6 + 107\tau^5 + 83\tau^4 - 21\tau^3 + 133\tau^2 + 157\tau + 48) / [15\delta\rho(1+2\tau)(1+\tau)]$

Table 3

Solution errors obtained for example 1[*]

n	λ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$	Θ_0	Θ_2	τ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$
8	1e08	2.40e-11	2.40e-11	---	---	0.9980	4.52e-12	4.52e-12
16	1e08	5.32e-13	5.32e-13	5.5	5.5	0.9991	6.00e-14	5.98e-14
32	1e08	1.58e-14	1.58e-14	5.1	5.1	0.9997	9.70e-16	9.71e-16

Table 4

Solution errors obtained for example 2[*]

n	λ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$	Θ_0	Θ_2	τ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$
8	0	1.97e-07	8.09e-08	---	---	0.985	1.90e-08	3.12e-08
16	0	4.34e-09	1.78e-09	5.5	5.5	0.991	8.82e-10	9.61e-10
32	0	8.12e-11	3.34e-11	5.7	5.7	0.996	8.52e-12	1.17e-11
8	1	7.56e-05	1.45e-03	---	---	1.160	1.67e-05	2.75e-04
16	1	7.80e-06	3.81e-04	3.3	2.0	1.110	3.46e-07	5.25e-05
32	1	7.50e-07	8.86e-05	3.4	2.1	1.040	6.70e-08	1.84e-05
8	2	5.64e-05	5.23e-04	---	---	0.910	1.22e-05	8.53e-04
16	2	3.94e-06	3.75e-05	3.8	3.9	0.960	1.21e-06	1.68e-05
32	2	2.65e-07	2.49e-06	3.9	3.8	0.790	8.67e-08	1.98e-06

Table 5

Solution errors obtained for example 3[*]

n	λ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$	Θ_0	Θ_2	τ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$
8	1e03	1.53e-09	8.31e-08	---	---	0.96	1.05e-10	2.72e-08
16	1e03	2.77e-11	1.51e-09	5.8	5.9	0.98	2.29e-12	3.79e-10
32	1e03	4.69e-13	2.53e-11	5.9	5.9	0.99	4.42e-14	5.84e-12

Table 6

Solution errors obtained for example 4[*]

n	λ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$	Θ_0	Θ_2	τ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u^{(2)}}^{(\infty)}$
8	4	2.40e-10	3.90e-09	---	---	0.995	2.99e-11	4.66e-09
16	4	5.32e-12	8.57e-11	5.5	5.5	0.997	6.49e-13	1.05e-10
32	4	1.06e-13	1.60e-12	5.7	5.8	0998	2.55e-14	2.07e-12

[*] Column 3–6 refer to uniform meshes, column 7–9 refer to geometric meshes.

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