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UNIQUENESS OF SOLUTIONS TO INVERSE PARABOLIC SEMILINEAR PROBLEMS UNDER NONLOCAL CONDITIONS WITH INTEGRALS

JEDNOZNACZNOŚĆ ROZWIĄZAŃ ODWROTNYCH PARABOLICZNYCH SEMILINIOWYCH ZAGADNIENIŃ Z NIELOKALNYMI WARUNKAMI Z CAŁKAMI

Abstract

The uniqueness of classical solutions to inverse parabolic semilinear problems together with nonlocal initial conditions with integrals, for the operator $\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial}{\partial x_j} \right) + v(x,t) - \frac{\partial}{\partial t}$, $x = (x_1, \dots, x_n)$, in the cylindrical domain $D := D_0 \times (t_0, t_0 + T) \subset \mathfrak{R}^{n+1}$, where $t_0 \in \mathfrak{R}$, $0 < T < \infty$ are studied. The result consists in the introduction of nonlocal conditions with integrals.

Keywords: inverse problems, parabolic problems, semilinear equation, nonlocal condition with integral, cylindrical domain, uniqueness of solutions

Streszczenie

W artykule studiowana jest jednoznaczność klasycznych rozwiązań odwrotnych parabolicznych semiliniowych zagadnień z nielokalnymi początkowymi warunkami z całkami dla operatora

$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial}{\partial x_j} \right) + v(x,t) - \frac{\partial}{\partial t}$, $x = (x_1, \dots, x_n)$, w walcowym obszarze $D := D_0 \times (t_0, t_0 + T) \subset \mathfrak{R}^{n+1}$,

gdzie $t_0 \in \mathfrak{R}$, $0 < T < \infty$. Wynik polega na tym, że zostały wprowadzone warunki nielokalne z całkami.

Słowa kluczowe: zagadnienia odwrotne, zagadnienia paraboliczne, równanie semiliniowe, nielokalny warunek z całką, obszar walcowy, jednoznaczność rozwiązań

1. Introduction

In this paper, we prove two theorems on the uniqueness of classical solutions to inverse parabolic semilinear problems, for the equation:

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u(x,t)}{\partial x_j} \right) + v(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t} = f(x,t,u(x,t)), \quad (1)$$

$$(x,t) \in D := D_0 \times (t_0, t_0 + T) \subset \mathfrak{R}^{n+1},$$

where $t_0 \in \mathfrak{R}$, $0 < T < \infty$. The coefficients $a_{ij}(i, j=1, \dots, n)$ and the function f are given. By the solution of the inverse problem, for equation (1), we mean a pair of functions (u, v) satisfying equation (1) and suitable conditions. The nonlocal initial condition considered in the paper is of the form:

$$u(x, t_0) + \frac{h(x)}{T} \int_{t_0}^{t_0+T} u(x, \tau) d\tau = f_0(x), \quad x \in D_0,$$

where $|h(x)| \leq 1$ for $x \in D_0$.

The obtained result is a continuation of the results given by Rabczuk in [6], by Beznohchenko and Prilenko in [1], by Chabrowski in [4], by Brandys in [2] and by the first author in [2] and [3].

2. Preliminaries

The notation, definitions and assumptions from this section are valid throughout this paper.

We will need the set $\mathfrak{R}_- := (-\infty, 0)$.

Let t_0 be a real finite number, $0 < T < \infty$ and $x = (x_1, \dots, x_n) \in \mathfrak{R}^n$.

Define the domain (see [2] or [3])

$$D := D_0 \times (t_0, t_0 + T),$$

where D_0 is an open and bounded domain in \mathfrak{R}^n such that the boundary ∂D_0 satisfies the following conditions:

If $n \geq 2$ then ∂D_0 is a union of a finite number of surface patches of class C^1 , which have no common interior points but have common boundary points.

If $n \geq 3$ then all the edges of ∂D_0 are sums of a finite numbers of $(n-2)$ – dimensional surface patches of class C^1 .

Assumption (A_1) .

$$a_{ij}, \frac{\partial a_{ij}}{\partial x_s} \in C(\bar{D}, \mathfrak{R}) \quad (i, j, s = 1, \dots, n), \text{ where } a_{ij} = a_{ij}(x, t) \text{ for } (x, t) \in \bar{D} \quad (i, j = 1, \dots, n); a_{ji}(x, t) = a_{ij}(x, t)$$

for $(x, t) \in D$ $(i, j = 1, \dots, n)$ and $\sum_{i,j=1}^n a_{ij}(x, t) \lambda_i \lambda_j \geq 0$ for arbitrary $(x, t) \in D$ and $(\lambda_1, \dots, \lambda_n) \in \mathfrak{R}^n$.

Assumption (A_2) .

(i) $f: \bar{D} \in \mathfrak{R} \ni (x, t, z) \rightarrow f(x, t, z) \in \mathfrak{R}, f \in C(\bar{D} \times \mathfrak{R}, \mathfrak{R}), f(x, t, 0) \neq 0$ for $(x, t) \in D$,

$$\frac{\partial f}{\partial z} \in C(\bar{D} \times \mathfrak{R}, \mathfrak{R}) \text{ and } \frac{\partial f(x, t, z)}{\partial z} > 0 \text{ for } (x, t) \in \bar{D}, z \in \mathfrak{R};$$

(ii) $f_1: \partial D_0 \times [0, T] \rightarrow \mathfrak{R};$

(ii') $k \in C(\partial D_0 \times [0, T], \mathfrak{R})$ and $k(x, t) \leq 0$ for $(x, t) \in \partial D_0 \times [0, T];$

(iii) $f_0: D_0 \rightarrow \mathfrak{R}.$

Assumption (A_3) . $h \in C(\bar{D}_0, \mathfrak{R})$ and $|h(x)| \leq 1$ for $x \in D_0$.

Let $C^{2,1}(\bar{D}, \mathfrak{R})$ be the space of all $w \in C(\bar{D}, \mathfrak{R})$ such that $\frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j} \in C(\bar{D}, \mathfrak{R})$ for $i, j = 1, \dots, n$ and $\frac{\partial w}{\partial t} \in C(\bar{D}, \mathfrak{R})$.

The symbol L is reserved for the operator given by the formula:

$$(Lw)(x, t) := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial w(x, t)}{\partial x_j} \right) \quad (2)$$

for $w \in C^{2,1}(\bar{D}, \mathfrak{R}), (x, t) \in \bar{D}$.

By n_x , where $x \in \partial D_0$, we denote the interior normal to ∂D_0 at x . Shortly, we denote, also, n_x by n .

Let $u \in C^{2,1}(\bar{D}, \mathfrak{R}), x_0 \in \partial D_0$ and $t \in [t_0, t_0 + T]$. The expression:

$$\frac{du(x, t)}{d\nu(x_0, t)} := \sum_{i=1}^n \frac{\partial u(x_0, t)}{\partial x_i} \sum_{j=1}^n a_{ij}(x_0, t) \cos(n_{x_0}, x_j) \quad (3)$$

is called the transversal derivative of the function u at the point (x_0, t) . Shortly, we denote, also

$$\frac{du(x, t)}{d\nu(x_0, t)} \text{ by } \frac{d}{d\nu} u(x_0, t) \text{ or by } \frac{du}{d\nu_{x_0}}.$$

For the given functions $a_{ij}(i, j = 1, \dots, n)$ satisfying Assumption (A_1) and for the given functions f, f_1, f_0 and h satisfying Assumptions (A_2) (i) – (iii) and (A_3) , the first Fourier's inverse semilinear problem in D together with a nonlocal initial condition with integral consists in finding a pair of functions $u \in C^{2,1}(\bar{D}, \mathfrak{R}), v \in C(\bar{D}, \mathfrak{R})$ satisfying the equation

$$(Lu)(x,t) + v(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t} = f(x,t,u(x,t)) \text{ for } (x,t) \in D, \quad (4)$$

the nonlocal initial condition:

$$u(x,t_0) + \frac{h(x)}{T} \int_{t_0}^{t_0+T} u(x,\tau) d\tau = f_0(x) \text{ for } x \in \bar{D}_0, \quad (5)$$

the boundary condition:

$$u(x,t) = f_1(x,t) \text{ for } x \in \partial D_0 \times [t_0, t_0+T] \quad (6)$$

and the condition:

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} v(x,t) u^2(x,t) dx \right] dt = C, \quad (7)$$

where C is a negative constant independent of u and v .

A pair (u,v) of functions possessing the above properties is called a solution of the first Fourier's inverse semilinear problem (4)–(7) in D .

Remark 2.1. The assumption that $f(x,t,0) \neq 0$ for $(x,t) \in D$ (see Assumption $(A_2)(i)$) implies that $u = 0$ cannot satisfy equation (4). Consequently, the above assumption implies that only $u \neq 0$ is considered in the paper.

If condition (6) from the first Fourier's inverse semilinear nonlocal problem (4)–(7) is replaced by the condition

$$\frac{d}{d\nu_x} u(x,t) + k(x,t)u(x,t) = f_1(x,t) \text{ for } (x,t) \in \partial D_0 \times [t_0, t_0+T], \quad (8)$$

where k is the given function satisfying Assumption $(A_2)(ii')$ then problem (4), (5), (8) and (7) is said to be the mixed inverse semilinear problem in D together with a nonlocal initial condition with integral. A pair of functions $u \in C^{2,1}(\bar{D}, \mathfrak{R})$, $v \in (\bar{D}, \mathfrak{R}_-)$ satisfying equation (4) and conditions (5), (8), (7) is called a solution of the mixed inverse semilinear problem (4), (5), (8) and (7) in D .

Assumption (A_4) . For every two solutions (u_1, v_1) and (u_2, v_2) of problem (4) – (7) or of problem (4), (5), (8) and (7) the following formulas hold:

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} v_i(x,t) u_j^2(x,t) dx \right] dt = C \quad (i, j = 1, 2; i \neq j).$$

Remark 2.2. The reason for which Assumption (A_4) is introduced is that the considered problems are inverse.

Assumption (A_5) . For each two solutions (u_1, v_1) and (u_2, v_2) of problem (4)–(7) or of problem (4), (5), (8) and (7) the following inequality:

$$\left[\frac{1}{T} \int_{t_0}^{t_0+T} (u_1(x, \tau) - u_2(x, \tau)) d\tau \right]^2 \leq [u_1(x, t_0 + T) - u_2(x, t_0 + T)]^2 \quad \text{for } x \in D_0$$

is satisfied.

Remark. 2.3. The reason for which Assumption (A_5) is introduced is that the considered problems are nonlocal.

3. Theorems about uniqueness

In this section, we shall prove two theorems about the uniqueness of solutions of inverse parabolic semilinear problems together with nonlocal initial conditions.

Theorem 3.1. *Suppose that coefficients a_{ij} ($i, j=1, \dots, n$) of the differential equation satisfy Assumption (A_1) and the functions f, f_1, f_0 and h satisfy Assumptions (A_2) (i) – (iii) and (A_3) . Then, the first Fourier's inverse semilinear problem (4) – (7) admits at most one solution in D in the class of the solutions satisfying Assumptions (A_4) and (A_5) .*

Proof. Suppose that (u_1, v_1) and (u_2, v_2) are two solutions of problem (4) – (7) in D and let

$$w := u_1 - u_2 \text{ in } \bar{D}. \quad (9)$$

Then, the following formulas hold:

$$(Lw)(x, t) + v_1(x, t)u_1(x, t) - v_2(x, t)u_2(x, t) - \frac{\partial w(x, t)}{\partial t} \quad (10)$$

$$= f(x, t, u_1(x, t)) - f(x, t, u_2(x, t)) \quad \text{for } (x, t) \in \bar{D},$$

$$w(x, t_0) + \frac{h(x)}{T} \int_{t_0}^{t_0+T} w(x, \tau) d\tau = 0 \quad \text{for } x \in \bar{D}_0, \quad (11)$$

$$w(x, t) = 0 \quad \text{for } (x, t) \in \partial D_0 \times [t_0, t_0 + T], \quad (12)$$

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} v_i(x, t) u_i^2(x, t) dx \right] dt = C \quad (i=1, 2). \quad (13)$$

From the assumption that $u, u_2 \in C^{2,1}(\bar{D}, \mathfrak{R})$ from the second and fourth part of Assumption $(A_2)(i)$ and from the mean value theorem, there exists $\theta \in (0, 1)$ such that:

$$\begin{aligned} & f(x, t, u_1(x, t)) - f(x, t, u_2(x, t)) \\ &= w(x, t) \frac{\partial f(x, t, u_2(x, t) + \theta w(x, t))}{\partial z} \quad \text{for } (x, t) \in \bar{D}. \end{aligned} \quad (14)$$

By (14), (10), by Assumption (A_1) , by (2) and by [5] (Section 17.11),

$$\begin{aligned} & \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \\ &= - \int_{t_0}^{t_0+T} \left[\int_{\partial D_0} w \sum_{i=1}^n \cos(n, x_i) \sum_{j=1}^n a_{ij} \frac{\partial w}{\partial x_j} d\sigma_x \right] dt \\ & \quad - \int_{t_0}^{t_0+T} \left[\int_{D_0} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dx \right] dt \\ & \quad - \int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial x_i} w dx \right] dt + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1^2 dx \right] dt \\ & \quad + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_2^2 dx \right] dt \\ & \quad - \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1 u_2 dx \right] dt \\ & \quad - \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_1 u_2 dx \right] dt, \end{aligned} \quad (15)$$

where $d\sigma_x$ is a surface element in \mathfrak{R}^n .

From (15), (12), from the last part of Assumption (A_1) and from the inequalities

$$- \int_{t_0}^{t_0+T} \left[\int_{D_0} v_i u_1 u_2 dx \right] dt \leq - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_i (u_1^2 + u_2^2) dx \right] dt \quad (i=1, 2) \quad (16)$$

we have

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \leq - \int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial t} w dx \right] dt \quad (17)$$

$$\begin{aligned}
& + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1^2 dx \right] dt + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_2^2 dx \right] dt \\
& - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 (u_1^2 + u_2^2) dx \right] dt \\
& - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 (u_1^2 + u_2^2) dx \right] dt.
\end{aligned}$$

Using integration by parts, we obtain:

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial t} w dx \right] dt = \frac{1}{2} \int_{D_0} w^2(x, t_0 + T) dx - \frac{1}{2} \int_{D_0} w^2(x, t_0) dx. \quad (18)$$

Formulas (17), (18) and (11) imply the inequality:

$$\begin{aligned}
& \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \quad (19) \\
& \leq -\frac{1}{2} \int_{D_0} w^2(x, t_0 + T) dx + \frac{1}{2} \int_{D_0} h^2(x) \left[\frac{1}{T} \int_{t_0}^{t_0+T} w(x, \tau) d\tau \right]^2 dx \\
& + \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1^2 dx \right] dt - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_2^2 dx \right] dt \\
& + \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_2^2 dx \right] dt - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_1^2 dx \right] dt.
\end{aligned}$$

From (19) and (13), and Assumptions (A_4) and (A_5) , we have:

$$\begin{aligned}
& \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \quad (20) \\
& \leq -\frac{1}{2} \int_{D_0} w^2(x, t_0 + T) [1 - h^2(x)] dx.
\end{aligned}$$

By (20) and by Assumption (A_3) we obtain:



$$\int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x,t,u_2+\theta w)}{\partial z} dx \right] dt \leq 0.$$

From the above inequality and from the last part of Assumption $(A_2)(i)$:

$$w^2 \leq 0 \quad \text{in } D$$

and therefore:

$$w = 0 \quad \text{in } D.$$

The above formula implies that:

$$u_1 = u_2 \quad \text{in } D.$$

Consequently, by (10):

$$(v_1 - v_2) u_1 = 0 \quad \text{in } D.$$

Therefore, from Remark 2.1, we have that:

$$v_1 = v_2 \quad \text{in } D.$$

The proof of Theorem 3.1 is thereby complete.

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1, concerning to the coefficients a_{ij} ($i, j=1, \dots, n$) and the functions f, f_1, f_0 and h are satisfied and that the function k satisfies Assumption $(A_2)(ii')$. Then, the mixed inverse semilinear problem (4), (5), (8) and (7) admits at most one solution in D in the class of the solutions satisfying Assumptions (A_4) and (A_5) .*

Proof. Suppose that (u_1, v_1) and (u_2, v_2) are two solutions of problem (4), (5), (8) and (7) in D and let

$$w := v_1 - v_2 \quad \text{in } \bar{D}. \quad (21)$$

Then, the following formulas hold:

$$(Lw)(x,t) + v_1(x,t)u_1(x,t) - v_2(x,t)u_2(x,t) - \frac{\partial w(x,t)}{\partial t} \quad (22)$$

$$= f(x,t,u_1(x,t)) - f(x,t,u_2(x,t)) \quad \text{for } (x,t) \in \bar{D},$$

$$w(x,t_0) + \frac{h(x)}{T} \int_{t_0}^{t_0+T} w(x,\tau) d\tau = 0 \quad \text{for } x \in \bar{D}_0, \quad (23)$$

$$\frac{d}{d\nu_x} w(x,t) + k(x,t)w(x,t) = 0 \quad \text{for } (x,t) \in \partial D_0 \times [t_0, t_0 + T], \quad (24)$$

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} v_i(x,t) u_i^2(x,t) dx \right] dt = C \quad (i=1,2). \quad (25)$$

Applying a similar argument as in the proof of Theorem 3.1 and using the definition of $\frac{\partial u}{\partial v_x}$ (see (3)), we have:

$$\begin{aligned} & \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x,t, u_2 + \theta w)}{\partial z} dx \right] dt \quad (26) \\ &= - \int_{t_0}^{t_0+T} \left[\int_{\partial D_0} w \frac{d}{\partial v_x} w d\sigma_x \right] dt \\ & - \int_{t_0}^{t_0+T} \left[\int_{D_0} \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dx \right] dt \\ & - \int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial x_i} w dx \right] dt \\ & + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1^2 dx \right] dt + \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_2^2 dx \right] dt \\ & - \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1 u_2 dx \right] dt \\ & - \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_1 u_2 dx \right] dt. \end{aligned}$$

From (26), (24), and as in the proof of Theorem 3.1, we obtain:

$$\begin{aligned} & \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x,t, u_2 + \theta w)}{\partial z} dx \right] dt \quad (27) \\ & \leq \int_{t_0}^{t_0+T} \left[\int_{\partial D_0} k w^2 d\sigma_x \right] dt - \frac{1}{2} \int_{D_0} w^2(x, t_0 + T) [1 - h^2(x)] dx \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_1^2 dx \right] dt - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_1 u_2^2 dx \right] dt \\
& + \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_2^2 dx \right] dt - \frac{1}{2} \int_{t_0}^{t_0+T} \left[\int_{D_0} v_2 u_1^2 dx \right] dt.
\end{aligned}$$

By (27), by Assumption $(A_2)(ii')$ and (A_3) , and by (25) and Assumption (A_4) , we obtain the inequality:

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \leq 0.$$

From the above inequality and from the last part of Assumption $(A_2)(i)$,

$$w^2 \leq 0 \quad \text{in } D$$

and, therefore, the same argument as in the proof of Theorem 3.1 implies that the proof of Theorem 3.2 is complete.

4. Physical interpretation of the nonlocal condition (5)

Theorems 3.1 and 3.2 can be applied to description of physical problems in the heat conduction theory, for which we cannot measure the temperature at the initial instant, but we can measure the temperature in the form of the nonlocal condition (5).

Also, observe that the nonlocal condition (5) considered in Theorem 3.1 and 3.2 is more general than the classical initial condition and the integral periodic condition and the integral antiperiodic condition. Namely, if the function h from condition (5) satisfies the relation:

$$h(x) = 0 \quad \text{for } x \in \overline{D}_0$$

then condition (5) is reduced to the initial condition:

$$u(x, t_0) = f_0(x) \quad \text{for } x \in \overline{D}_0.$$

Instead, if the function h and f in (5) satisfy the conditions:

$$h(x) = -1 \quad [h(x) = 1] \quad \text{for } x \in \overline{D}_0,$$

$$f_0(x) = 0 \quad \text{for } x \in \overline{D}_0$$

then condition (5) is reduced, respectively, to the integral periodic [antiperiodic] initial condition:

$$u(x, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} u(x, \tau) d\tau \quad [u(x, t_0) = -\frac{1}{T} \int_{t_0}^{t_0+T} u(x, \tau) d\tau] \quad \text{for } x \in \bar{D}_0$$

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