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KOŁODZIEJ'S SUBSOLUTION THEOREM FOR UNBOUNDED PSEUDOCONVEX DOMAINS

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Abstract. In this paper we generalize Kołodziej's subsolution theorem to bounded and unbounded pseudoconvex domains, and in that way we are able to solve complex Monge–Ampère equations on general pseudoconvex domains. We then give a negative answer to a question of Cegrell and Kołodziej by constructing a compactly supported Radon measure μ that vanishes on all pluripolar sets in $\mathbb{C}^n$ such that $\mu(\mathbb{C}^n) = (2\pi)^n$, and for which there is no function u in $\mathcal{L}_+$ such that $(dd^c u)^n = \mu$. We end this paper by solving a Monge–Ampère type equation. Furthermore, we prove uniqueness and stability of the solution.

1. Introduction. The idea of subsolutions plays a prominent role in the theory of partial differential equations. Let us consider an equation $Pu = f$ with given data f, where P is certain differential operator. Since we do not *a priori* know that a solution u exists, it is natural to say that v is a subsolution to $Pu = f$ if $f \leq Pv$. The motivation is that if u is a solution and v is a subsolution in this sense, and a comparison principle is valid within the given function class, then $v \leq u$. One would expect that if an equation has a subsolution, then there also exists a solution. This idea originated from potential theory and Oscar Perron's 1923 work [**32**], where he studied the Laplace equation and solved it in certain cases using the upper envelope of subharmonic functions. His work was then continued by Brelot, Carathéodory, Wiener, among others [**6, 8, 36–38**]. This is what is sometimes today referred to as the Perron–Wiener–Brelot approach (see e.g. Chapter 6 in [**2**]).

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In the complex analytic setting, we are instead interested in the complex Monge–Ampère equation, which is a non-linear, complex analytic generalization of the Laplace equation in $\mathbb{R}^2$ to $\mathbb{C}^n$. Let us now give a brief background of the setting, and for further information about pluripotential theory we refer the reader to [**17, 21, 22, 28**]. Let ∂, $\bar\partial$ be the usual differential operators, $d = (\partial + \bar\partial)$ and $d^c = i\,(\bar\partial - \partial)$. For smooth functions we then put the following definition

$$(dd^c u)^n := \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n \text{ times}} = 4^n\, n!\, \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) dV,$$

where dV is the usual volume form on $\mathbb{C}^n$. It is not possible to define the complex Monge–Ampère operator $(dd^c\,\cdot\,)^n$ in a suitable way on all plurisubharmonic functions and still have the range contained in the class of non-negative Radon measures (see e.g. [**34**]). In [**11**], Cegrell introduced a subset $\mathcal{E}$ of non-positive plurisubharmonic functions for which the complex Monge–Ampère operator is well defined (see Section 2 for the definition of $\mathcal{E}$).

A major breakthrough in the theory of complex Monge–Ampère equations is the following celebrated theorem:

Kołodziej's subsolution theorem ([**24**]). *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and $f \in C(\partial\Omega)$. Let $u \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$, $\lim_{z\to w} u(z) = f(w)$, for all $w \in \partial\Omega$. If μ is a non-negative and finite measure such that $\mu \leq (dd^c u)^n$, then there exists a uniquely determined bounded plurisubharmonic function v such that $(dd^c v)^n = \mu$ and $\lim_{z\to w} v(z) = f(w)$, for all $w \in \partial\Omega$.*

Under some additional assumptions on μ this was proved earlier by Kołodziej himself in [**23**]. The above theorem was later generalized in [**1**] to a larger function class, and a larger set of measures. In [**29**], the assumption of hyperconvexity is relaxed to pseudoconvexity. But here the set $\mathcal{E}$ need to be replaced, since its underlying domain is assumed to be hyperconvex. One way of handle this situation is motivated by that being in $\mathcal{E}$ is a local property ([**5**]); therefore, $\mathcal{E}$ may be substituted by the following set:

$$\mathcal{D}(\Omega) := \big\{u \in \mathcal{PSH}(\Omega) : \text{ for all } B(z_0,r) \Subset \Omega \text{ there exists a constant } C = C(B(z_0,r),u) \text{ such that } u \in \mathcal{E} + C\big\}.$$

An alternative (global) approach to the set $\mathcal{D}(\Omega)$ is possible by following [**16**], but we omit it here.

The next step is to consider unbounded domains in $\mathbb{C}^n$. Example 3.1 is due to Jarnicki and Zwonek [**20**], and it shows that there exists an unbounded hyperconvex domain in $\mathbb{C}^n$ that is not biholomorphically equivalent to any bounded pseudoconvex domain in $\mathbb{C}^n$. This shows that the complex Monge–Ampère equation on unbounded domains is considerably different from the one considered on bounded domains. In an unpublished preprint ([**25**], partially published in [**27**]), Kołodziej proved the following: *Given two entire locally bounded, entire plurisubharmonic functions v and w satisfying $w \leq v$, $(dd^c v)^n \leq (dd^c w)^n$ and $\lim_{|z|\to\infty}(v(z) - w(z)) = 0$, one can solve the Monge–Ampère equation $(dd^c u)^n = \mu$ for any measure μ with*

$$(dd^c v)^n \leq \mu \leq (dd^c w)^n.$$

Furthermore, the solution u is unique among functions satisfying $w \leq u \leq v$. In Theorem 3.2, we generalize this theorem to unbounded pseudoconvex domains, and $\mathcal{D}(\Omega)$. As an immediate consequence we get the following subsolution theorem.

Theorem 3.3. *Let Ω be a bounded or unbounded pseudoconvex domain, and let $u \in \mathcal{D}(\Omega)$ be such that the smallest maximal plurisubharmonic majorant $\tilde{u}$ of u exists. Then for any non-negative Radon measure μ that satisfies $\mu \leq (dd^c u)^n$ there exists $w \in \mathcal{D}(\Omega)$ such that*

$$(dd^c w)^n = \mu \qquad \text{and} \qquad u \leq w \leq \widetilde{u} \quad \text{on } \Omega. \tag{1.1}$$

Furthermore, if μ vanishes on pluripolar sets, then the solution w of (1.1) is uniquely determined.

In connection with the above subsolution theorem it is worth mentioning that in [**18**] (see also [**19**]), Guan proved a related theorem, where he assumes that Ω is unbounded and smoothly bounded, and that there exist a strictly plurisubharmonic subsolution in $C^2(\bar{\Omega})$ with classical boundary values equal to given smooth boundary data.

In Section 4 we give an example that answers Cegrell and Kołodziej's question in [**14**]. More precisely, we give an example of a compactly supported Radon measure μ with $\mu(\mathbb{C}^n) = (2\pi)^n$, that vanishes on pluripolar sets in $\mathbb{C}^n$, for which there is no function u in $\mathcal{L}_+$ such that $(dd^c u)^n = \mu$ (Example 4.5). Here $\mathcal{L}_+$ is the Lelong class given by

$$\mathcal{L}_+ := \{u \in \mathcal{PSH}(\mathbb{C}^n) : \exists C = C(u) \in \mathbb{R} \;\; |u(z) - \log(1 + |z|)| \leq C\}.$$

We end this paper by solving a Monge–Ampère type equation, and proving the uniqueness and stability of the solution.

Theorem 5.1. *Let Ω be a bounded or unbounded pseudoconvex domain. Let $\varphi \in \mathcal{D}(\Omega)$ be such that the measure $\mu = (dd^c\varphi)^n$ vanishes on pluripolar sets, and assume that the smallest maximal plurisubharmonic majorant $\tilde{\varphi}$ of φ exists. Assume also that $F(x, z) \geq 0$ is a $dx \times d\mu$-measurable function on*

$\mathbb{R} \times \Omega$ that is continuous in the x variable. If there exists a bounded function g such that

$$0 \leq F(x,z) \leq g(z),$$

then there exists a function $u \in \mathcal{D}(\Omega)$ that satisfies

$$(dd^c u)^n = F(u(z),z)\,\mu.$$

Furthermore, if F is a nondecreasing function in the first variable, then the solution u is uniquely determined. Assume that $0 \leq f, f_j \leq 1$ are measurable functions such that $\{f_j\,\mu\}$ converges to $\{f\,\mu\}$ in weak topology, as j tends to $+\infty$, and for each j let u_j and u be solutions of*

$$(dd^c u_j)^n = F(u_j(z),z)f_j(z)\,\mu, \qquad \text{and} \qquad (dd^c u)^n = F(u(z),z)f(z)\,\mu\,.$$

Then $\{u_j\}$ converges in capacity to u, as j tends to $+\infty$.

Theorem 5.1 generalizes numerous corresponding results, including in [**3, 7, 9, 15, 17, 26**]. We refer to Chapter 7.2 in [**17**] for a historical account.

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2. Preliminaries. Let us first introduce some notation that will simplify the exposition of this paper.

Definition 2.1. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. Let $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ be the set of bounded plurisubharmonic functions φ defined on Ω, such that

$$\lim_{\Omega\ni z\to\xi} \varphi(z) = 0 \quad \text{for every } \ \xi \in \partial\Omega, \qquad \text{and} \qquad \int_\Omega (dd^c\varphi)^n < \infty.$$

Definition 2.2. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. Let $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ be the set of plurisubharmonic functions φ defined on Ω, such that for each $z_0 \in \Omega$ there exist a neighborhood ω of z_0 in Ω, and a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_0$ that converges pointwise to φ on ω, as j tends to $+\infty$, and

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty.$$

Definition 2.3. Let $\Omega \subseteq \mathbb{C}^n$ be an arbitrary open set in $\mathbb{C}^n$. We put:

$$\mathcal{D}(\Omega) := \big\{u \in \mathcal{PSH}(\Omega) : \text{ for all } \ B(z_0,r) \Subset \Omega \text{ there exists a constant } C = C(B(z_0,r),u) \text{ such that } u \in \mathcal{E} + C\big\}.$$

Proposition 2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be an arbitrary open set in $\mathbb{C}^n$. Then*
1) If $u, v \in \mathcal{D}(\Omega)$, then $u + v \in \mathcal{D}(\Omega)$.
2) If $u \in \mathcal{D}(\Omega)$ and $v \in \mathcal{PSH}(\Omega)$, then $\max(u, v) \in \mathcal{D}(\Omega)$.
3) In particular, if $u \in \mathcal{D}(\Omega)$ and $v \in \mathcal{PSH}(\Omega)$, $u \leq v$, then $v \in \mathcal{D}(\Omega)$.

Let Ω be a bounded or unbounded set in $\mathbb{C}^n$. A sequence of bounded domains $\{\Omega_j\}$ such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$, and $\cup_{j=1}^{\infty} \Omega_j = \Omega$, will be referred to as a *fundamental sequence* for Ω. Recall that a domain Ω is pseudoconvex if and only if there exists a fundamental sequence for Ω consisting of bounded hyperconvex domains.

Definition 2.5. Let Ω be an arbitrary open set in $\mathbb{C}^n$, $\{\Omega_j\}$ a fundamental sequence for Ω, and u a plurisubharmonic function defined on Ω. Let us now define

$$u^j := \sup\left\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq u \text{ on } \mathcal{C}\Omega_j\right\},$$

where $\mathcal{C}\Omega_j$ denotes the complement of Ω_j in Ω, and

$$\tilde{u} := \left(\lim_{j \to +\infty} u^j\right)^* .$$

Here $(w)^*$ denotes the upper semicontinuous regularization of w.

Remark.

1. If $u \in \mathcal{PSH}(\Omega)$ is bounded above, then $u^j \in \mathcal{PSH}(\Omega)$ and $u^j = u$ on $\mathcal{C}\Omega_j$. Definition 2.5 implies that $\{u^j\}$ is an increasing sequence, and therefore $\lim_{j\to\infty} u^j$ exists q.e. (quasi-everywhere) on Ω. Hence, the function $\tilde{u}$ is plurisubharmonic on Ω.
2. If Ω is a bounded hyperconvex domain, and $u \in \mathcal{E}$, then by [**11**] $\tilde{u} \in \mathcal{E}$, since $u \leq \tilde{u} \leq 0$, and $\tilde{u}$ is the smallest maximal plurisubharmonic majorant of u (*cf.* [**4, 5**] or [**13**]). Following [**12**] we set

$$\mathcal{N} := \{u \in \mathcal{E} : \tilde{u} = 0\} \quad \text{and} \quad \mathcal{F} := \left\{u \in \mathcal{N} : \int_\Omega (dd^c u)^n < \infty\right\}.$$

3. If Ω is a bounded (but not hyperconvex) open set, then (2) holds with $\mathcal{E}$ replaced with $\mathcal{D}$.
4. If Ω is unbounded, then the function $\tilde{u}$ may not exist (see Example 2.6).

Example 2.6. Let $\Omega = \mathbb{C}^n$, $\Omega_j = B(0, j)$ and $u(z) = |z|^2$. Then by Definition 2.5 $u^j(z) = \max(j^2, |z|^2)$. Hence, $\tilde{u} = +\infty$. □

Definition 2.7. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. We say that a plurisubharmonic function u defined on Ω belongs to the class $\mathcal{N}(\Omega, H) (= \mathcal{N}(H))$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{N}$ such that

$$H \geq u \geq \varphi + H .$$

Let Ω be a bounded and open set in $\mathbb{C}^n$. Recall that the Bedford–Taylor capacity of $X \subset \Omega$ is defined as

$$C_n(X,\Omega) = C_n(X) = \sup\{\int_X (dd^c u)^n; u \in \mathcal{PSH}(\Omega), -1 \le u \le 0\}.$$

A sequence $\{u_j\}$ of functions defined on the bounded and open set Ω is said to converge in capacity to u if for any $t > 0$ and $K \Subset \Omega$

$$\lim_{j\to+\infty} C_n(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

For an unbounded open set Ω we define *the convergence in capacity* as follows. Let $\{\Omega_k\}$ be a fundamental sequence for Ω. Then $\{u_j\}$ is said to converge in capacity to u if for any $t > 0$, $k \in \mathbb{N}$ and $K \Subset \Omega$ there holds

$$\lim_{j\to+\infty} C_n(K \cap \{|u - u_j| > t\}, \Omega_k) = 0.$$

Note that this definition is well posed, and it does not depend on the given fundamental sequence, because for any $K \Subset \Omega_1 \subset \Omega_2$

$$A \cdot C_n(K, \Omega_1) \le C_n(K, \Omega_2) \le B \cdot C_n(K, \Omega_1),$$

where $A, B > 0$ are constants.

3. Subsolution principle. The following example is due to Jarnicki and Zwonek [**20**], and it shows that there exists an unbounded hyperconvex domain in $\mathbb{C}^n$ that is not biholomorphically equivalent to any bounded pseudoconvex domain in $\mathbb{C}^n$.

Example 3.1. We construct an unbounded hyperconvex domain in $\mathbb{C}^n$ that is not biholomorphically equivalent to any bounded pseudoconvex domain in $\mathbb{C}^n$. Let A be the Cantor set in $\mathbb{R} \subset \mathbb{C}$. It is well known that the set $\mathbb{C} \setminus A$ is compact and regular with respect to the Laplace equation (see e.g. [**35**]). Furthermore, the analytic capacity of A is zero, so every bounded holomorphic function on $\mathbb{C}\setminus A$ extends holomorphically to $\mathbb{C}$ and therefore must be constant, by the Liouville theorem.

Let us now define D as $\mathbb{C} \setminus A$. Then D is an unbounded domain that is not biholomorphically equivalent to any bounded domain in $\mathbb{C}$, because only bounded holomorphic functions on D are the constant functions.

Set $\Omega = D^n = D \times \cdots \times D$. Then Ω is the required example, i.e. it is an unbounded hyperconvex domain in $\mathbb{C}^n$ that is not biholomorphically equivalent to any bounded pseudoconvex domain in $\mathbb{C}^n$. We shall prove this by induction. The case $n = 1$ is already concluded above. Assume now that any bounded holomorphic function on D^{n-1} must be constant and take a bounded holomorphic function φ defined on Ω. By the assumption, for any $z \in D$ and $z' \in D^{n-1}$ functions $\varphi(\cdot, z)$ and $\varphi(z', \cdot)$ are bounded and therefore constant.

Note that for any $(a',a),(b',b) \in \Omega$ we have $\varphi(a',a) = \varphi(a',b) = const$ and also $\varphi(a',b) = \varphi(b',b) = const'$. Therefore, $\varphi(a',a) = \varphi(b',b)$. □

We now prove a generalization of Kołodziej's Theorem 1.1 in [**25**].

THEOREM 3.2. *Let Ω be a bounded or unbounded pseudoconvex domain. Let $v,u \in \mathcal{D}(\Omega)$ be two functions satisfying $u \le v$, and $(dd^c v)^n \le (dd^c u)^n$. Then for any non-negative Radon measure μ with*

$$(dd^c v)^n \le \mu \le (dd^c u)^n,$$

there exists a function $w \in \mathcal{D}(\Omega)$ such that

$$(dd^c w)^n = \mu \qquad \text{and} \qquad u \le w \le v \qquad \text{on } \Omega.$$

PROOF. Take a fundamental sequence $\{\Omega_j\}$ for Ω consisting of bounded hyperconvex domains, and set $u^j = u|_{\Omega_j}$, $v^j = v|_{\Omega_j}$. First we are going to construct plurisubharmonic functions w^j defined on Ω_j such that

$$(dd^c w^j)^n = \mu|_{\Omega_j}, \quad u^j \le w^j \le v^j \quad \text{and } w^j \le \widetilde{u^j} \quad \text{on } \Omega_j, \tag{3.1}$$

where $\widetilde{u^j}$ is the smallest maximal plurisubharmonic majorant of u^j in Ω_j as defined in Definition 2.5.

We shall follow the construction form [**1**]. By using the Cegrell–Lebesgue decomposition theorem for positive measures (see e.g. [**11**]) on μ, we obtain $\mu = \mu_r + \mu_s$. Here μ_r is the regular part that vanishes on pluripolar sets, and μ_s is the singular part that is carried by a pluripolar set. More precisely, with the notations $\mu^j := \mu|_{\Omega_j}$, $\mu_r^j := \mu_r|_{\Omega_j}$, $\mu_s^j := \mu_s|_{\Omega_j}$ we get

$$\mu^j = f^j (dd^c \phi^j)^n + \mu_s^j \qquad \text{on } \Omega_j,$$

where $f^j \ge 0$, $f^j \in L^1((dd^c\phi^j)^n)$, and $\phi^j \in \mathcal{E}_0(\Omega_j)$. Now by applying the Radon–Nikodym theorem, we obtain

$$f^j (dd^c \phi^j)^n = \tau^j \chi_{\{u^j > -\infty\}} (dd^c u^j)^n \quad \text{and} \quad \mu_s^j = \tau^j \chi_{\{u^j = -\infty\}} (dd^c u^j)^n \quad \text{on } \Omega_j,$$

where $0 \le \tau^j \le 1$ are Borel functions. For each $k \in \mathbb{N}$, let μ_k^j be the measure defined on Ω_j by $\mu_k^j = \min(f^j, k)(dd^c\phi^j)^n$. Hence, $\mu_k^j \le (dd^c(k^{\frac{1}{n}}\phi^j))^n$; therefore, by Kołodziej's subsolution theorem there exists a uniquely determined function $\psi_k^j \in \mathcal{E}_0(\Omega_j)$ such that

$$\left(dd^c \psi_k^j\right)^n = \mu_k^j \qquad \text{on } \Omega_j.$$

Choose an increasing sequence of simple functions $\{g_k^j\}_{k=1}^{\infty}$, $\operatorname{supp} g_k^j \Subset \Omega_j$, that converges to $g^j = \chi_{\{u^j=-\infty\}}\tau^j$ on Ω_j, as $k \to +\infty$, and let $w^{g_k^j} \in \mathcal{F}(\Omega_j)$ (see Theorem 4.8 in [**1**]) be such that

$$(dd^c w^{g_k^j})^n = g_k^j (dd^c u^j)^n \qquad \text{on } \Omega_j.$$

For any given maximal plurisubharmonic function H defined on Ω_j, define

$$W_k^j(z) := \sup\Big\{\varphi(z) : \varphi \in \mathcal{E}(\Omega_j), \left(dd^c\psi_k^j\right)^n \leq (dd^c\varphi)^n \text{ and } \varphi \leq \min(w^{g_k^j}, H)\Big\}.$$

By Theorem 4.12 in [**1**],

$$(dd^c W_k^j)^n = \left(dd^c\psi_k^j\right)^n + g_k^j(dd^c u^j)^n,$$

and therefore

$$(dd^c W_{k+1}^j)^n = \left(dd^c\psi_{k+1}^j\right)^n + g_{k+1}^j(dd^c u^j)^n \geq \left(dd^c\psi_k^j\right)^n$$

and

$$W_{k+1}^j \leq \min(w^{g_{k+1}^j}, H) \leq \min(w^{g_k^j}, H),$$

which means that $W_{k+1}^j \leq W_k^j$. Then the function defined on Ω_j by

$$U(\mu^j, (dd^c u^j)^n, H) := \lim_{k\to\infty} W_k^j \tag{3.2}$$

is plurisubharmonic, satisfies

$$(dd^c U(\mu^j, (dd^c u^j)^n, H))^n = \mu^j\,, \quad \text{and} \tag{3.3}$$
$$u^j + H \leq U(\mu^j, (dd^c u^j)^n, H) \leq H \quad \text{on } \Omega_j\,.$$

Furthermore, the function defined in (3.2) has the following properties:

1) If $H_1 \leq H_2$, then $U(\mu, (dd^c u)^n, H_1) \leq U(\mu, (dd^c u)^n, H_2)$.
2) If $\mu_1 \leq \mu_2$, then $U(\mu_1, (dd^c u)^n, H) \geq U(\mu_2, (dd^c u)^n, H)$; and
3) The solution $U(\mu, (dd^c u)^n, H)$ does not depend on the measure $(dd^c u)^n$, provided $\mu \leq (dd^c u)^n$.

To simplify the notation, set

$$w^j := U\left(\mu^j, (dd^c u^j)^n, \widetilde{u^j}\right),$$

where $\widetilde{u^j}$ is the smallest maximal plurisubharmonic majorant of u^j in Ω_j as defined in Definition 2.5. Using this notation, from (3.3) we derive:

$$\left(dd^c w^j\right)^n = \mu^j = \mu|_{\Omega_j} \qquad \text{and} \qquad w^j \leq \widetilde{u^j} \quad \text{on } \Omega_j.$$

We were set out to prove that the construction of w^j satisfies (3.1), thus it remains to prove that $u^j \leq w^j \leq v^j$ on Ω_j. Since $(dd^c u^j)^n \geq (dd^c\psi_k^j)^n$,

$u^j \le w^{g^j_k}$, and $u^j \le \widetilde{u^j}$ we have that $u^j \le w^j$ on Ω_j. To see that $w^j \le v^j$, let us first note that from properties (1) and (3) above there follows

$$\begin{aligned} w^j = U(\mu^j, (dd^c u^j)^n, \widetilde{u^j}) &= U(\mu^j, (dd^c w^j)^n, \widetilde{u^j}) \le U((dd^c v^j)^n, (dd^c w^j)^n, \widetilde{u^j}) \\ &= U((dd^c v^j)^n, (dd^c v^j)^n, \widetilde{u^j}) \le U((dd^c v^j)^n, (dd^c v^j)^n, \widetilde{v^j}). \end{aligned}$$

Since

$$v^j \le U\left((dd^c v^j)^n, (dd^c v^j)^n, \widetilde{v^j}\right) \le \widetilde{v^j},$$

then $v^j \in \mathcal{N}(\Omega_j, \widetilde{v^j})$ (see e.g. [**33**] or [**17**]), and thus the uniqueness part of Theorem 3.7 in [**1**] yields $v^j = U((dd^c v^j)^n, (dd^c v^j)^n, \widetilde{v^j})$. Hence, $w^j \le v^j$ on Ω_j.

Next, we shall prove that $\{w^j\}$ is an increasing sequence. We get $u^j = u^{j+1} \le w^{j+1}$ on Ω_j, whence $\widetilde{u^j} \le \widetilde{w^{j+1}|_{\Omega_j}}$. This together with property (1) above yields

$$w^{j+1} = U\left(\mu^j, (dd^c u^j)^n, \widetilde{w^{j+1}|_{\Omega_j}}\right) \ge U\left(\mu^j, (dd^c u^j)^n, \widetilde{u^j}\right) = w^j \quad \text{ on } \Omega_j\,.$$

Thus, $\{w_j\}$ is an increasing sequence. To complete this proof, let us define w on Ω as follows:

$$w := \left(\lim_{j\to+\infty} w^j\right)^*.$$

Then, $w \in \mathcal{D}(\Omega)$, $u \le w \le v$ and $(dd^c w)^n = \mu$ on Ω. □

Remark. It should be emphasized that Theorem 3.2 yields new results even for *bounded* hyperconvex domains, since the function v in not necessarily bounded from above.

Now we shall proceed to prove Kołodziej's subsolution theorem for plurisubharmonic functions in bounded or unbounded pseudoconvex domains.

Theorem 3.3. *Let Ω be a bounded or unbounded pseudoconvex domain, and let $u \in \mathcal{D}(\Omega)$ be such that the smallest maximal plurisubharmonic majorant $\tilde{u}$ of u exists. Then for any non-negative Radon measure μ that satisfies $\mu \le (dd^c u)^n$ there exists $w \in \mathcal{D}(\Omega)$ such that*

$$(dd^c w)^n = \mu \qquad \text{and} \qquad u \le w \le \widetilde{u} \quad \text{on } \Omega. \tag{3.4}$$

Furthermore, if μ vanishes on pluripolar sets, then the solution w of (3.4) is uniquely determined.

Proof. The existence part follows immediately from the proof of Theorem 3.2. Let $\{\Omega_j\}$ be a fundamental sequence for Ω. To prove uniqueness, assume that there exist functions $w, v \in \mathcal{D}(\Omega)$ with

$$(dd^c v)^n = (dd^c w)^n = \mu\,, \qquad u \le w \qquad \text{on } \Omega,$$

and that for all $j \in \mathbb{N}$ it holds that $w \leq \widetilde{u}$ and $v \leq \widetilde{u}$ on Ω_j. Then $v|_{\Omega_j}, w|_{\Omega_j} \in \mathcal{N}(\Omega_j, \widetilde{u})$. Therefore by the uniqueness part of Theorem 3.7 in [**1**] we get $v = w$, since μ vanishes on pluripolar sets.

By the same reasoning as above, we conclude that the solution w does not depend on the fundamental sequence $\{\Omega_j\}$. □

Remark. The assumption about the existence of a majorant $\tilde{u}$ in Theorem 3.3 is necessary if we want to have the subsolution theorem in the present form. To see it, just take $\mu = 0$, then the solution for this homogeneous Dirichlet problem gives us the required majorant.

Remark. Note that Theorem 3.3 holds if Ω is a bounded pseudoconvex domain and u is a plurisubharmonic function bounded from above.

Next we shall make a simple observation that we are later going to use together with Theorem 3.3 in the proof of Theorem 5.1. It is a stability theorem for the complex Monge–Ampère operator in pseudoconvex domains.

Theorem 3.4. *Let Ω be a bounded or unbounded pseudoconvex domain, and let Ω_j be a fundamental sequence for Ω consisting of bounded hyperconvex domains. Let $v \in \mathcal{D}(\Omega)$ be such that the measure $\mu = (dd^c v)^n$ vanishes on all pluripolar sets, and that the smallest maximal plurisubharmonic majorant $\tilde{v}$ of v exists. Let $0 \leq f, f_j \leq 1$ be measurable functions such that $\{f_j\, \mu\}$ converges to $f\, \mu$ in weak* topology. If $u, u_j \in \mathcal{D}(\Omega)$ are functions satisfying*

1. $(dd^c u_j)^n = f_j\, \mu$,
2. $(dd^c u)^n = f\, \mu$,
3. $v \leq u_j \leq \tilde{v}|_{\Omega_j}$, *and* $v \leq u \leq \tilde{v}|_{\Omega_j}$ *on* Ω_j,

then $\{u_j\}$ converges to u in capacity.

Proof. This follows immediately from Theorem 7.12 from [**17**], since the functions u_j, u are all in $\mathcal{N}(\Omega_j, \tilde{v}|_{\Omega_j})$. □

Corollary 3.5. *Let Ω be a bounded or unbounded pseudoconvex domain. Assume also that $v \in \mathcal{D}(\Omega)$ is such that the measure $(dd^c v)^n$ vanishes on pluripolar sets, and that the smallest maximal plurisubharmonic majorant $\tilde{v}$ of v exists. Define*

$$\mathcal{S}(v) := \{\varphi \in \mathcal{D}(\Omega) : (dd^c\varphi)^n \leq (dd^c v)^n, v \leq \varphi \leq \tilde{v} \text{ on } \Omega\}.$$

The weak convergence and the convergence in capacity are equivalent in $\mathcal{S}(v)$.

Proof. This follows from the proof of the Corollary in [**15**, p. 723] (or as the proof of Corollary 7.15 in [**17**]). □

4. Entire radially symmetric plurisubharmonic functions. In this section we answer (in Example 4.5) Cegrell and Kołodziej's question by constructing a compactly supported Radon measure μ that vanishes on all pluripolar sets in $\mathbb{C}^n$ such that $\mu(\mathbb{C}^n) = (2\pi)^n$ and for which there is no function u in $\mathcal{L}_+$ such that $(dd^c u)^n = \mu$. Here $\mathcal{L}_+$ is defined as in (4.1) below. In this section $\mathcal{PSH}^R(\mathbb{C}^n)$ denotes the set of functions defined on $\mathbb{C}^n$ that are radially symmetric and plurisubharmonic.

We need Monn's following result (cf. [**30, 31**]).

Theorem 4.1. *Let μ be a rotation invariant Radon measure defined on $\mathbb{C}^n$, and let $F(t) = \frac{1}{(2\pi)^n}\mu(B_t)$. Then there exists a function $u = u(\mu) \in \mathcal{PSH}^R(\mathbb{C}^n)$ such that $(dd^c u)^n = \mu$. Furthermore,*

$$u(z) - u(w) = \int_{|w|}^{|z|} \frac{1}{t} F^{\frac{1}{n}}(t)dt,$$

and the solution is unique (up to a constant) in the family of entire radially symmetric, plurisubharmonic functions.

Remark. Note that in general the solution in Theorem 4.1 is not unique. One way to see this is the following: If $(dd^c u)^n = \mu$, then $(dd^c(u+v))^n = \mu$, for any entire pluriharmonic function v.

Let us recall the Lelong classes.

$$\mathcal{L} := \{u \in \mathcal{PSH}(\mathbb{C}^n) : \exists\, C = C(u) \in \mathbb{R} \;\; u(z) \le \log(1+|z|) + C\},$$

and

$$\mathcal{L}_+ := \{u \in \mathcal{PSH}(\mathbb{C}^n) : \exists\, C = C(u) \in \mathbb{R} \;\; |u(z) - \log(1+|z|)| \le C\}. \tag{4.1}$$

From Theorem 4.1 we have the following corollaries:

Corollary 4.2. *Let $u \in \mathcal{PSH}^R(\mathbb{C}^n)$ and let $F(t) = \frac{1}{(2\pi)^n}(dd^c u)^n(B_t)$. Then $u \in \mathcal{L}$ if and only if,*

$$\int_1^\infty \frac{1}{t}(F^{\frac{1}{n}}(t) - 1)dt < \infty. \tag{4.2}$$

Proof. Theorem 4.1 yields

$$\int_1^r \frac{1}{t}(F^{\frac{1}{n}}(t) - 1)dt = u(r) - u(1) - \ln r,$$

so condition (4.2) is equivalent to the fact that $u \in \mathcal{L}$. □

Corollary 4.3. *Let $u \in \mathcal{PSH}^R(\mathbb{C}^n)$ and let $F(t) = \frac{1}{(2\pi)^n}(dd^c u)^n(B_t)$. Then $u \in \mathcal{L}_+$ if and only if,*

$$\int_0^1 \frac{1}{t} F^{\frac{1}{n}}(t)dt < \infty, \tag{4.3}$$

and

$$\left|\int_1^\infty \frac{1}{t}(F^{\frac{1}{n}}(t)-1)dt\right| < \infty. \tag{4.4}$$

PROOF. Condition (4.3) is equivalent with the fact that $u(0) > -\infty$, i.e. u is locally bounded. Furthermore, condition (4.4) is equivalent with the fact that u has logarithmic growth at infinity. □

COROLLARY 4.4. *Let μ be a regular, rotation invariant measure on $\mathbb{C}^n$. If supp$\mu \subset B_R$, then for $|z| > R$*

$$u(\mu)(z) = \frac{\mu(\mathbb{C}^n)^{\frac{1}{n}}}{2\pi}\log|z| + u(\mu)(R) - \frac{\mu(\mathbb{C}^n)^{\frac{1}{n}}}{2\pi}\log R.$$

In particular:

- *$u(\mu) \in \mathcal{L}$ if and only if $\mu(\mathbb{C}^n) \le (2\pi)^n$, and*
- *$u(\mu) \in \mathcal{L}_+$ if and only if $\mu(\mathbb{C}^n) = (2\pi)^n$ and $\int_0^1 \frac{1}{t}\mu(B_t)^{\frac{1}{n}}(t)dt < \infty$.*

Next we answer Cegrell and Kołodziej's question from [**14**] by constructing an example of a regular compactly supported measure μ, $\mu(\mathbb{C}^n) = (2\pi)^n$ that vanishes on pluripolar sets in $\mathbb{C}^n$ for which there is no $u \in \mathcal{L}_+$ such that $(dd^cu)^n = \mu$.

EXAMPLE 4.5. Let us define a function f by

$$f(t) = \begin{cases} n!2^{n-1}(\ln 2)^n t^{-2n}(-\ln t)^{-n-1} & \text{if } 0 < t < \frac{1}{2}, \\ 0 & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Then $\mu = f dV_{2n}$ is a compactly supported Radon measure with $\mu(\mathbb{C}^n) = (2\pi)^n$, and vanishes on all pluripolar sets in $\mathbb{C}^n$. Furthermore,

$$\mu(B_t) = \begin{cases} (2\pi)^n\left(-\frac{\ln 2}{\ln t}\right)^n & \text{if } 0 < t < \frac{1}{2}, \\ (2\pi)^n & \text{if } t \ge \frac{1}{2}\,. \end{cases}$$

Hence,

$$\int_0^{\frac{1}{2}} \frac{1}{t}\mu(B_t)^{\frac{1}{n}}(t)dt = 2\pi\ln 2\int_0^{\frac{1}{2}} \frac{-1}{t\ln t}dt = \infty.$$

Thus, there is no radially symmetric function $u \in \mathcal{L}_+$ such that $(dd^cu)^n = \mu$ (Corollary 4.4).

Next we shall prove that there exists no function $v \in \mathcal{L}_+$ with $(dd^cv)^n = \mu$. For the contrary, suppose that such function v exists, and let $U(n)$ denote the unitary group in $\mathbb{C}^n$. Construct a function v^R defined on $\mathbb{C}^n$ by

$$v^R(z) = \sup\{v(T(z)); T \in U(n)\}^*\,.$$

This construction implies that $v^R \in \mathcal{PSH}^R(\mathbb{C}^n)$, and $v^R \in \mathcal{L}_+$, since $u \circ T \in \mathcal{L}_+$ for any $T \in U(n)$. The fact that for any $T \in U(n)$ we have $(dd^c u \circ T)^n = \mu$ yields $(dd^c v^R)^n \geq \mu$. Therefore,

$$\int_0^{\frac{1}{2}} \frac{1}{t} \left((dd^c v^R)^n(B_t)\right)^{\frac{1}{n}}(t)dt \geq \int_0^{\frac{1}{2}} \frac{1}{t}\mu(B_t)^{\frac{1}{n}}(t)dt = \infty,$$

which is impossible since $v^R \in \mathcal{L}_+$.

Note that the function f could easily be modified to be smooth outside the origin, compactly supported, and such that $\mu = f dV_{2n}$ has all the desired properties. □

5. Monge–Ampère type equation.

THEOREM 5.1. *Let Ω be a bounded or unbounded pseudoconvex domain. Let $\varphi \in \mathcal{D}(\Omega)$ be such that the measure $\mu = (dd^c\varphi)^n$ vanishes on pluripolar sets, and assume that the smallest maximal plurisubharmonic majorant $\tilde{\varphi}$ of φ exists. Assume also that $F(x, z) \geq 0$ is a $dx \times d\mu$-measurable function on $\mathbb{R} \times \Omega$ that is continuous in the x variable. If there exists a bounded function g such that*

$$0 \leq F(x, z) \leq g(z),$$

then there exists a function $u \in \mathcal{D}(\Omega)$ that satisfies

$$(dd^c u)^n = F(u(z), z)\,\mu.$$

Furthermore, if F is a nondecreasing function in the first variable, then the solution u is uniquely determined. Assume that $0 \leq f, f_j \leq 1$ are measurable functions such that $\{f_j\,\mu\}$ converges to $\{f\,\mu\}$ in weak topology, as j tends to $+\infty$, and for each j let u_j and u be solutions of*

$$(dd^c u_j)^n = F(u_j(z), z) f_j(z)\,\mu, \qquad \text{and} \qquad (dd^c u)^n = F(u(z), z) f(z)\,\mu.$$

Then $\{u_j\}$ converges in capacity to u, as j tends to $+\infty$.

PROOF. *Part I. Existence of a solution.* By Theorem 3.3, there exists a unique function ψ such that

$$(dd^c\psi)^n = g\,\mu \qquad \text{and } c\varphi \leq \psi \leq c\widetilde{\varphi},$$

where $c = (\sup_\Omega g)^{\frac{1}{n}}$. The smallest maximal plurisubharmonic majorant $\tilde{\psi}$ of ψ exists, since $\widetilde{\varphi}$ exists by assumption. Set

$$K = \{\phi \in \mathcal{D}(\Omega) : \psi \leq \phi \leq \tilde{\psi} \text{ on } \Omega\}.$$

The set K is convex, and compact in the L^1_{loc} topology. Let us define a map $\mathcal{T} : K \to K$ so that if

$$(dd^c v)^n = F(u(z), z)\,\mu\,, \text{ then } \mathcal{T}(u) = v.$$

Furthermore, if $u \in K$, then $F(u(z), z)\,\mu \leq (dd^c\psi)^n$. Therefore, Theorem 3.3 yields that there exists a uniquely determined function $v \in \mathcal{D}(\Omega)$ satisfying $(dd^c v)^n = F(u(z), z)\,\mu$, and $\psi \leq v \leq \widetilde{\psi}$ on Ω. Thus, $v \in K$. In other words, $\mathcal{T}$ is well defined. Next, we shall prove that $\mathcal{T}$ is continuous, and then the Schauder–Tychonoff fixed point theorem concludes the existence part of the proof. Assume that $\{u_j\}$ is a sequence in K that converges to a function u, as $j \to +\infty$. By [**15**] there exists a subsequence (still denoted by $\{u_j\}$) converging to u in $L^1_{loc}(\mu)$. Theorem 3.4, applied to the measure $g\,\mu$ implies that the sequence $\{v_j\}$ defined by $v_j = \mathcal{T}(u_j)$ converges in capacity to some function $v \in K$. Since $v_j, v \in K$ we can use [**10**] to get $\{(dd^c v_j)^n\}$ tends in the weak* topology to $(dd^c v)^n$ as $j \to +\infty$. Hence,

$$(dd^c v)^n = \lim_{j\to+\infty} (dd^c v_j)^n = \lim_{j\to+\infty} F(u_j(z), z)\,\mu = F(u(z), z)\,\mu = (dd^c\mathcal{T}(u))^n,$$

which implies that $v = \mathcal{T}(u)$ by Theorem 3.3. Thus,

$$\lim_{j\to+\infty} \mathcal{T}(u_j) = \mathcal{T}(u),$$

i.e. $\mathcal{T}$ is continuous.

Part II. Uniqueness of a solution. Assume that F is a function that is nondecreasing in the first variable, and assume that there exist functions $u, v \in \mathcal{D}(\Omega)$ such that

$$(dd^c u)^n = F(u(z), z)\,\mu \text{ and } (dd^c v)^n = F(v(z), z)\,\mu.$$

Let $\{\Omega_j\}$ be a fundamental sequence for Ω composed of bounded hyperconvex domains, and set $u^j = u|_{\Omega_j}$ and $v^j = v|_{\Omega_j}$. On the set $\{z \in \Omega_j : u^j(z) < v^j(z)\}$, we have

$$(dd^c u^j)^n = F(u^j(z), z)\,\mu \leq F(v^j(z), z)\,\mu = (dd^c v^j)^n.$$

Using a suitable comparison principle (see e.g. [**17**]) yields that

$$\int_{\{u^j<v^j\}} (dd^c v^j)^n \leq \int_{\{u^j<v^j\}} (dd^c u^j)^n.$$

Hence, $(dd^c u^j)^n = (dd^c v^j)^n$ on $\{z \in \Omega_j : u^j(z) < v^j(z)\}$. In a similar manner, we get $(dd^c u^j)^n = (dd^c v^j)^n$ on $\{z \in \Omega_j : u^j(z) > v^j(z)\}$. Furthermore, on $\{u^j = v^j\}$ we have

$$(dd^c u^j)^n = F(u^j(z), z)\,\mu = F(v^j(z), z)\,\mu = (dd^c v^j)^n.$$

Hence, $(dd^c u^j)^n = (dd^c v^j)^n$ on Ω_j. Thus, $u^j = v^j$ on Ω_j and there follows $u = v$ on Ω.

Part III. Stability of solutions. Assume that F is a function that is nondecreasing in the first variable and let $0 \leq f, f_j \leq 1$ be measurable functions such that $\{f_j\,\mu\}$ converges to $f\,\mu$ in weak* topology, as $j \to +\infty$. The first and second part of the theorem yield that there exist uniquely determined functions u_j such that

$$(dd^c u_j)^n = F(u_j(z), z) f_j(z)\mu$$

and $\psi \leq u_j \leq \tilde{\psi}$ on Ω. Therefore there exists a subsequence, still denoted by $\{u_j\}$, that converges to u in the weak topology. Furthermore, $\psi \leq u \leq \tilde{\psi}$ on Ω. Corollary 3.5 yields $u_j \to u$ in capacity, and then [**10**] implies that the sequence $\{(dd^c u_j)^n\}$ tends to $(dd^c u)^n$ in the weak* topology, as $j \to +\infty$. Passing to subsequence, still denoted by $\{f_j\}$, we may assume that $\{f_j\}$ converges to f pointwise a.e. w.r.t. $[\mu]$. The dominated convergence theorem gives us

$$(dd^c u)^n = \lim_{j\to\infty} (dd^c u_j)^n = \lim_{j\to\infty} F(u_j(z), z) f_j(z)\,\mu = F(u(z), z) f(z)\,\mu.$$

Hence, u is a solution to

$$(dd^c u)^n = F(u(z), z) f(z)\mu\,.$$

Since this argument works for any subsequence taken from the original sequence, we conclude that $\{u_j\}$ converges in capacity to u, as $j \to +\infty$. $\square$

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