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OPTIMISATION MODEL FOR WATER DISTRIBUTION FROM A SYSTEM OF COMBINED RETENTION RESERVOIRS IN THE CASE OF FREE OPTIMISATION TIME

MODEL OPTYMALIZACYJNY DYSTRYBUCJI WODY Z SYSTEMU POŁĄCZONYCH ZBIORNIKÓW RETENCYJNYCH W NIEUSTALONYM HORYZONCIE OPTYMALIZACJI

A b s t r a c t

The paper discusses the problems of optimal water management in a distribution system. The main technical elements of the considered water-economic system include: retention reservoirs, among which water transfer is possible, and a network of connections between the reservoirs and water treatment plants (WTP). The system operation optimisation involves specifying proper water transport routes and the rates of water flow from the retention reservoirs to the WTPs, and the volumes of possible transfers among reservoirs, so as to ensure that total system operation costs as specified by the assumed quality coefficient are minimal. The analytic solution of the formulated optimisation task has been obtained as a result of employing Pontryagin's maximum principle with reference to the assumed quality coefficient. The researchers have assumed fixed initial and end conditions in reservoir level trajectories. Optimisation start and/or end time is free to choose. The solutions obtained gave grounds to develop a simulation computer model representing the system's operation. The analysis of of the results obtained may affect decisions supporting control of water-economic systems existing in reality.

Keywords: *optimisation*, *control*, *a system of retention reservoirs*, *water management*

S t r e s z c z e n i e

W artykule podjęto problematykę optymalnego gospodarowania wodami w systemie dystrybucji. Głównymi elementami technicznymi rozpatrywanego systemu wodno-gospodarczego są zbiorniki retencyjne, pomiędzy którymi możliwy jest przerzut wody, a także sieć połączeń zbiorników ze stacjami uzdatniania wody (SUW). Optymalizacja pracy systemu polega na wyznaczeniu odpowiednich tras transportu wody oraz wielkości natężeń jej przepływów ze zbiorników retencyjnych do SUW i wielkości ewentualnych przerzutów między-zbiornikowych tak, aby łączne koszty eksploatacji systemu określone przyjętym wskaźnikiem jakości były minimalne. Analityczne rozwiązanie sformułowanego zadania optymalizacji otrzymano w wyniku zastosowania zasady maksimum Pontriagina w odniesieniu do przyjętego wskaźnika jakości. Przyjęte zostały sztywne początkowe i końcowe warunki na trajektoriach stanów zbiorników. Czas rozpoczęcia lub zakończenia optymalizacji jest swobodny. Na podstawie uzyskanych rozwiązań stworzono symulacyjny model komputerowy przedstawiający pracę systemu. Analiza otrzymanych wyników będzie mogła mieć wpływ na decyzje wspomagające sterowaniem realnie istniejących systemów wodno-gospodarczych.

Słowa kluczowe: *optymalizacja*, *sterowanie*, *system zbiorników retencyjnych*, *gospodarka wodna*

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1. Introduction

The search for the best optimisation duration may refer to cases of determining the following:

– free start time (undetermined optimal time of optimisation process commencement, for predetermined time of conclusion), (**F**ree **S**tart **T**ime);
– free end time (undetermined optimal time of optimisation process conclusion, for predetermined time of commencement), (**F**ree **E**nd **T**ime).

In the issues related to supporting dispatch decisions concerning control of water outflows from reservoirs, the above-mentioned cases of free optimisation duration with combinations of boundary conditions have broad applications. Due to the numerous boundary condition variants, the moment of optimisation commencement or conclusion has some importance.

The article will demonstrate optimisation variants for operation of a system of reservoirs with free optimisation time:

– FST (**F**ree **S**tart **T**ime),
– FET (**F**ree **E**nd **T**ime),

and the case of boundary conditions in reservoir level trajectories in the form of:

– LCP (**L**eft **C**onditions in level trajectories – **P**redetermined),
– RCP (**R**ight **C**onditions in level trajectories – **P**redetermined).

Therefore, an integral part of the cases discussed will be to determine:

– the vector of best optimisation commencement times $\hat{t}_0^*$, for predetermined optimisation conclusion time T, or
– the vector of best optimisation conclusion times $\hat{\boldsymbol{T}}^*$, for predetermined optimisation commencement time t_0.

Since commencement time and conclusion time may be freely chosen, two characteristic cases occur in this regard, as shown in Fig. 1.

For the vector of predicted inflows to reservoir system, physical sense of quality coefficient (1) combines three tasks:

– ensuring such vector of water outflows from reservoirs $\hat{\boldsymbol{u}}(t)_{(+)}$, $\forall t \in [\hat{\boldsymbol{t}}_0^*, T]$, or $t \in \lfloor t_0, \hat{\boldsymbol{T}}^* \rfloor$ that would minimally diverge from the vector, elements of which are partial demands for water per reservoir in the system $\mathbf{B}(t) \times \mathbf{Y}(t) \times \mathbf{S} \times \mathbf{1}$, $\forall t \in [\hat{\boldsymbol{t}}_0^*, T]$, or $t \in \lfloor t_0, \hat{\boldsymbol{T}}^* \rfloor$;
– accomplishment of objectives specified above at minimum transfer costs of among reservoirs;
– obtaining (at the end of the optimisation horizon T, or $\hat{\boldsymbol{T}}^*$) a filling vector for reservoirs $\hat{\boldsymbol{x}}(T)$, ensuring the required values (right boundary conditions in reservoir level trajectories).

2. Free optimisation time (FT)

We have two cases in this regard. We will search for the following:

– best optimisation process commencement time (free initial time FIT, for predetermined time of optimisation conclusion PET),

– optimal process conclusion time (predetermined initial time PIT, free end time FET.)

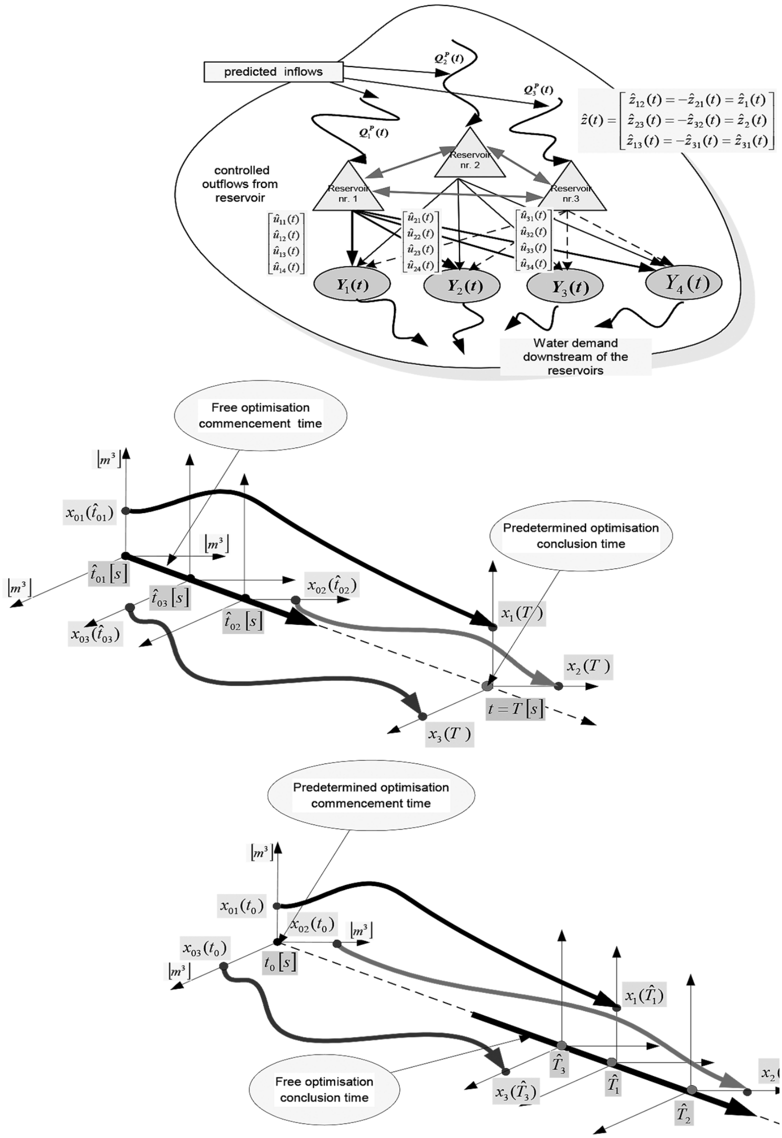

Fig. 1. The water-economic system under discussion

Rys. 1. Rozpatrywany system wodno-gospodarczy

The quality coefficient of the discussed task has the following form:

$$F = 0,5 \cdot \int_{t_0^*,\ (t_0)}^{T,\ (T^*)} \left\{ \begin{aligned} &\left[\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \mathbf{1} - \boldsymbol{u}(t) \right]_{(+)}^{\mathrm{T}} \cdot \\ &\cdot \mathbf{A}_1 \cdot \left[\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \mathbf{1} - \boldsymbol{u}(t) \right]_{(+)} + z^{\mathrm{T}}(t) \cdot \mathbf{A}_2 \cdot z(t) \end{aligned} \right\} dt \tag{1}$$

where:

- $\boldsymbol{t}_0^*$ – vector of free optimisation commencement times for predetermined end time T,
- $\boldsymbol{T}^*$ – vector of free optimisation conclusion times for predetermined commencement time t_0.

Other symbols in coefficient (1) are assumed as follows:

$$\mathbf{B}(t) = \begin{bmatrix} \begin{bmatrix} b_{11}(t) & 0 & 0 & 0 \\ 0 & b_{12}(t) & 0 & 0 \\ 0 & 0 & b_{13}(t) & 0 \\ 0 & 0 & 0 & b_{14}(t) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} b_{21}(t) & 0 & 0 & 0 \\ 0 & b_{22}(t) & 0 & 0 \\ 0 & 0 & b_{23}(t) & 0 \\ 0 & 0 & 0 & b_{24}(t) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} b_{31}(t) & 0 & 0 & 0 \\ 0 & b_{32}(t) & 0 & 0 \\ 0 & 0 & b_{33}(t) & 0 \\ 0 & 0 & 0 & b_{34}(t) \end{bmatrix} \end{bmatrix}$$

Matrix $\mathbf{B}(t)$ is a diagonal block matrix with terms being diagonal matrixes themselves – its elements are functions of reservoirs involvement $i = 1, ..., 3$ in providing for the function of demands $Y_j(t)$, $j = 1, ..., 4$ m^3/s. Matrix $\mathbf{Y}(t)$ is a diagonal block matrix with terms being diagonal matrixes – its elements are demand functions effective in the system $Y_j(t)$, $j = 1, ..., 4$ m^3/s.

$$\mathbf{Y}(t) = \begin{bmatrix} \begin{bmatrix} Y_1(t) & 0 & 0 & 0 \\ 0 & Y_2(t) & 0 & 0 \\ 0 & 0 & Y_3(t) & 0 \\ 0 & 0 & 0 & Y_4(t) \end{bmatrix} & * & * \\ * & \begin{bmatrix} Y_1(t) & 0 & 0 & 0 \\ 0 & Y_2(t) & 0 & 0 \\ 0 & 0 & Y_3(t) & 0 \\ 0 & 0 & 0 & Y_4(t) \end{bmatrix} & * \\ * & * & \begin{bmatrix} Y_1(t) & 0 & 0 & 0 \\ 0 & Y_2(t) & 0 & 0 \\ 0 & 0 & Y_3(t) & 0 \\ 0 & 0 & 0 & Y_4(t) \end{bmatrix} \end{bmatrix}, \quad * = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Control vector (controlled outflows from reservoirs) is a block vector − its elements are vectors with elements constituted by outflows from reservoirs i = 1, ..., 3 to conurbation j = 1, ..., 4.

$$u^T(t) = \left[\left[u_{11}(t) \quad u_{12}(t) \quad u_{13}(t) \quad u_{14}(t) \right] \; \left[u_{21}(t) \quad u_{22}(t) \quad u_{23}(t) \quad u_{24}(t) \right] \; \left[u_{31}(t) \quad u_{32}(t) \quad u_{33}(t) \quad u_{34}(t) \right] \right]$$

Then, matrix $\mathbf{A}_1$ is a positively defined block matrix with terms in a diagonal also consisting of diagonal matrixes – its elements are weight coefficients related to proper control vector elements.

$$\mathbf{A_1} = \begin{bmatrix} \begin{bmatrix} a_{11} & & & \\ & a_{12} & & \\ & & a_{13} & \\ & & & a_{14} \end{bmatrix} & * & * \\ * & \begin{bmatrix} a_{21} & & & \\ & a_{22} & & \\ & & a_{23} & \\ & & & a_{24} \end{bmatrix} & * \\ * & * & \begin{bmatrix} a_{31} & & & \\ & a_{32} & & \\ & & a_{33} & \\ & & & a_{34} \end{bmatrix} \end{bmatrix}$$

Subsequently, a positively defined diagonal matrix $\mathbf{A}_2$

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$ – its elements are weight coefficients connected with the subintegral part of the quality coefficient (1) corresponding to the costs of water transfers among reservoirs, e.g. a_{11} concerns transfer cost $z_1(t)$, that is, from reservoir no. 1 to reservoir no. 2, or vice versa, etc. This approach to the problem with reference to matrix $\mathbf{A}_2$ constitutes certain simplification, since generally the cost of water transfer e.g. from reservoir no. 1 to reservoir no. 2 does not necessarily have to equal to the cost of water transfer from reservoir no. 2 to reservoir no. 1 (gravitational flow and pumping $a_{11}^{res1\rightarrow res2} \neq a_{11}^{res1\leftarrow res2}$).

Then, unit vector $\mathbf{1}^{\mathrm{T}} = \left[[1 \quad 1 \quad 1 \quad 1] \quad [1 \quad 1 \quad 1 \quad 1] \quad [1 \quad 1 \quad 1 \quad 1] \right]$

and vector of transfers among reservoirs:

$$\boldsymbol{z}(t)^{\mathrm{T}} = \left[z_1(t)^{res1\mapsto res2} \quad z_2(t)^{res2\mapsto res3} \quad z_3(t)^{res3\mapsto res1} \right]$$

Equation of level for the system reservoirs:

$$\boldsymbol{f} : \dot{\boldsymbol{x}}(t) = \boldsymbol{Q}^P(t) - \mathbf{S}_1 \cdot \boldsymbol{u}(t) \tag{2}$$

$$\boldsymbol{x}(\#) = \boldsymbol{x}_0 \qquad \boldsymbol{x}(\&) = \boldsymbol{x}_T \tag{3}$$

where: symbol # indicates:

– $\boldsymbol{t}_0^*$ vector of free optimisation commencement times for predetermined end time T, or
– t_0 predetermined optimisation commencement time for vector of free conclusion times $\boldsymbol{T}^*$,

and symbol **&** indicates:

– $\boldsymbol{T}^*$ vector of free optimisation conclusion times for predetermined start time t_0, or
– T predetermined optimisation conclusion time for vector $\boldsymbol{t}_0^*$ of free optimisation commencement times.

The following further symbols are used in the equation of level (2):

$\dot{\boldsymbol{x}}(t)^{\mathrm{T}} = \left[dx_1/dt \quad dx_2/dt \quad dx_3/dt\right]$ vector of reservoir level derivatives [m³/s],

$\boldsymbol{Q}(t)^{\mathrm{T}} = \left[Q_1(t) \quad Q_2(t) \quad Q_3(t)\right]$ vector of predicted inflows to reservoirs [m³/s],

$$\mathbf{S}_1 = \begin{bmatrix} [1 \ 1 \ 1 \ 1] & 0 & 0 \\ 0 & [1 \ 1 \ 1 \ 1] & 0 \\ 0 & 0 & [1 \ 1 \ 1 \ 1] \end{bmatrix}$$

$\mathbf{S}_1$ is a diagonal structural matrix necessary to note the system structure and the system level equation. It determines connections among reservoirs and water consumers.

$\mathbf{S}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ structural matrix necessary to note the connection between reservoirs in the system level equation with reference to transfers among reservoirs,

Matrix **S** is a structural matrix developed as a result of the operation $\mathbf{S} = (\mathbf{S}_1^T \cdot \mathbf{S}_1) * \mathbf{I}$, (symbol *, tabular multiplication).

Optimisation task solution

Hamilton's function for the system of equations (1), (2) takes the following form:

$$H = -f_0 \ + \psi(t)^{\mathrm{T}} \cdot \boldsymbol{f}$$

$$H = -0,5 \cdot \left\{ \begin{array}{l} \left[\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t)\right]_{(+)}^{\mathrm{T}} \cdot \\ \cdot \mathbf{A}_1 \cdot \left[\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t)\right]_{(+)} + \boldsymbol{z}^{\mathrm{T}}(t) \cdot \mathbf{A}_2 \cdot \boldsymbol{z}(t) \end{array} \right\} + \tag{4}$$

$$+ \psi(t)^{\mathrm{T}} \cdot \left[\boldsymbol{Q}^P(t) - \mathbf{S}_1 \cdot \boldsymbol{u}(t) + \mathbf{S}_2 \cdot \boldsymbol{z}(t)\right]$$

The system of equations for Hamilton's function in form of (4) is shown in points $\boxed{\text{A–D}}$:

$\boxed{\mathbf{A.}}$ $\left[\left(\nabla_u H\right)_{\hat{u},\hat{x},\hat{\psi}}\right]^{\mathrm{T}} = \mathbf{0}$ $\qquad \hat{\boldsymbol{u}}(t)_{(+)} = \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \mathbf{1} - \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \hat{\psi}(t)$ (5)

$\boxed{\mathbf{B.}}$ $\left[\left(\nabla_z H\right)_{\hat{u},\hat{z},\hat{x},\hat{\psi}}\right]^{\mathrm{T}} = \mathbf{0}$ $\qquad \hat{\boldsymbol{z}}(t) = \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \hat{\psi}(t)$ (6)

$\boxed{\mathbf{C}.}$ $\left[\left(\nabla_{\eta} H\right)_{\hat{u},\hat{x}}\right]^{\mathrm{T}} = \dot{\hat{\boldsymbol{x}}}(t)$ $\qquad \dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{Q}^{P}(t) - \mathbf{S}_1 \cdot \hat{\boldsymbol{u}}(t)_{(+)} + \mathbf{S}_2 \cdot \hat{\boldsymbol{z}}(t)$ (7)

$\boxed{\mathbf{D}.}$ $\left[-\left(\nabla_{x} H\right)_{\hat{u},\hat{x},\hat{\psi}}\right]^{\mathrm{T}} = \dot{\hat{\psi}}(t)$ $\qquad \dot{\hat{\psi}}(t) = \mathbf{0}_{(3*1)}$ (8)

using equations (5 → 8), we determine the vector of constants

1. For the vector of free optimisation commencement times $\boldsymbol{t}_0^*$

$$\boldsymbol{C}_1 = \left[\left(\mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}}\right) \cdot \left(T - \boldsymbol{t}_0^*\right)\right]^{-1} \cdot$$
$$\cdot \left\{ \boldsymbol{x}(T) - \boldsymbol{x}(\boldsymbol{t}_0^*) - \int_{\boldsymbol{t}_0^*}^{T} \left[\boldsymbol{Q}^{P}(t) - \mathbf{S}_1 \cdot \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \mathbf{1}\right] dt \right\} \tag{9}$$

The vector of constants $\boldsymbol{C}_1$ depends on the vector of free optimisation commencement times, and another equation is needed to calculate it. This second equation is derivable from the following relation:

$$H(\boldsymbol{t}_0^*) - \left(\frac{\partial K[x(\boldsymbol{t}_0^*), \boldsymbol{t}_0^*]}{\partial \boldsymbol{t}_0^*}\right) = 0 \tag{10}$$

In the case discussed, coefficient (1) does not contain the function of initial conditions $K[x(\boldsymbol{t}_0^*), \boldsymbol{t}_0^*]$, therefore Hamiltonian (4) $H(\boldsymbol{t}_0^*) = 0.$

Then, we rearrange the Hamiltonian attempting to determine $\hat{\eta}(\boldsymbol{t}_0^*)$:

$$-0,5 \cdot \left\{ \begin{array}{l} \left[\mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} - \boldsymbol{u}(\boldsymbol{t}_0^*)\right]_{(+)}^{\mathrm{T}} \cdot \\ \cdot \mathbf{A}_1 \cdot \left[\mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} - \boldsymbol{u}(\boldsymbol{t}_0^*)\right]_{(+)} + \boldsymbol{z}^{\mathrm{T}}(\boldsymbol{t}_0^*) \cdot \mathbf{A}_2 \cdot \boldsymbol{z}(\boldsymbol{t}_0^*) \end{array} \right\} +$$
$$+ \hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}} \cdot \left[\boldsymbol{Q}^{P}(\boldsymbol{t}_0^*) - \mathbf{S}_1 \cdot \boldsymbol{u}(\boldsymbol{t}_0^*) + \mathbf{S}_2 \cdot \boldsymbol{z}(\boldsymbol{t}_0^*)\right] = 0$$

$$-0,5 \cdot \left\{ \begin{array}{l} \left[\mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} - \mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} + \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*)\right]_{(+)}^{\mathrm{T}} \cdot \\ \cdot \mathbf{A}_1 \cdot \left[\mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} - \mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} + \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*)\right]_{(+)} + \\ + \left[\mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*)\right]^{\mathrm{T}} \cdot \mathbf{A}_2 \cdot \left[\mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*)\right] \end{array} \right\} +$$
$$+ \hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}} \cdot \left[\begin{array}{l} \boldsymbol{Q}^{P}(\boldsymbol{t}_0^*) - \mathbf{S}_1 \cdot \mathbf{B}(\boldsymbol{t}_0^*) \cdot \mathbf{Y}(\boldsymbol{t}_0^*) \cdot \mathbf{S} \cdot \mathbf{1} + \\ + \mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*) + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \hat{\psi}(\boldsymbol{t}_0^*) \end{array} \right] = 0$$

$$-0,5\cdot\left\{\begin{array}{l}\left[\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]_{(+)}^{\mathrm{T}}\cdot\mathbf{A}_1\cdot\left[\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]_{(+)}+\\+\left[\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]^{\mathrm{T}}\cdot\mathbf{A}_2\cdot\left[\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]\end{array}\right\}+$$

$$+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\begin{bmatrix}\boldsymbol{Q}^P(\boldsymbol{t}_0^*)-\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}+\\+\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)+\\+\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\end{bmatrix}=0$$

$$-0,5\cdot\left\{\left[\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]^{\mathrm{T}}\cdot\mathbf{I}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)+\left[\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right]^{\mathrm{T}}\cdot\mathbf{I}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)\right\}+$$
$$+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\boldsymbol{Q}^P(\boldsymbol{t}_0^*)-\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}+$$
$$+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{T}\cdot\hat{\psi}(\boldsymbol{t}_0^*)+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)=0$$

$$-0,5\cdot\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)-0,5\cdot\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)$$
$$+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\boldsymbol{Q}^P(\boldsymbol{t}_0^*)-\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}+$$
$$+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)+\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}\cdot\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)=0$$

On the left side we divide the equation by $\hat{\psi}(\boldsymbol{t}_0^*)^{\mathrm{T}}$:

$$-0,5\cdot\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)-0,5\cdot\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)$$
$$\boldsymbol{Q}^P(\boldsymbol{t}_0^*)-\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}+$$
$$+\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)+\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\cdot\hat{\psi}(\boldsymbol{t}_0^*)=0$$

$$0,5\cdot\left[\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}+\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\right]\cdot\hat{\psi}(\boldsymbol{t}_0^*)=\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}-\boldsymbol{Q}^P(\boldsymbol{t}_0^*)$$

finally, we obtain:

$$\begin{aligned}\hat{\psi}(\boldsymbol{t}_0^*)=&\left[0,5\cdot\left(\mathbf{S}_1\cdot\mathbf{A}_1^{-1}\cdot\mathbf{S}_1{}^{\mathrm{T}}+\mathbf{S}_2\cdot\mathbf{A}_2^{-1}\cdot\mathbf{S}_2^{\mathrm{T}}\right)\right]^{-1}\cdot\\&\cdot\left[\mathbf{S}_1\cdot\mathbf{B}(\boldsymbol{t}_0^*)\cdot\mathbf{Y}(\boldsymbol{t}_0^*)\cdot\mathbf{S}\cdot\mathbf{1}-\boldsymbol{Q}^P(\boldsymbol{t}_0^*)\right]\end{aligned}\tag{11}$$

By means of comparing equations (9) and (11) it is possible to determine the vector of optimal process commencement times.

Hypothetical scenario of events Example 1.■

In this example, we will assume the following input data with reference to reservoirs and water consumers:

– initial levels of reservoirs change in function of optimisation commencement time: $\boldsymbol{x}_0^{\mathrm{T}} = \left[10 - t_{0_{res1}} \quad 20 - 0.5t_{0_{res2}} \quad 30 + 0.2t_{0_{res3}}\right]$ [m³],

– final levels of reservoirs: $\boldsymbol{x}(T)^{\mathrm{T}} = \left[30 \quad 40 \quad 50\right]$ [m³],

– inflows to reservoirs: $\boldsymbol{Q}(t)^{\mathrm{T}} = \left[(t+1) \quad (0.5t+2) \quad (0.4t+3)\right]$ [m³],

– water demand downstream of the reservoirs:

$$\mathbf{Y}(t) = \begin{bmatrix} [\otimes] & * & * \\ * & [\otimes] & * \\ * & * & [\otimes] \end{bmatrix} \left[\mathrm{m}^3/\mathrm{s}\right], \quad [\otimes] = \begin{bmatrix} 2+0.1t & 0 & 0 & 0 \\ 0 & 3\cdot 1(t) & 0 & 0 \\ 0 & 0 & 3.5+0.2t & 0 \\ 0 & 0 & 0 & 4-0.1t \end{bmatrix}$$

$$* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

functions of reservoir involvement in providing for the demand function:

$$\mathbf{B}(t) = \begin{bmatrix} [\otimes_1] & * & * \\ * & [\otimes_2] & * \\ * & * & [\otimes_3] \end{bmatrix}, \qquad [\otimes_1] = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix} \cdot (t)$$

$$[\otimes_2] = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \cdot (t) \qquad [\otimes_3] = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix} \cdot (t)$$

– matrixes of weight coefficients:

$$\mathbf{A}_1 = \begin{bmatrix} [\oplus] & * & * \\ * & [\oplus] & * \\ * & * & [\oplus] \end{bmatrix}, \quad [\oplus] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– structural matrix:

$$\mathbf{S}_1 = \begin{bmatrix} [\times] & \circ & \circ \\ \circ & [\times] & \circ \\ \circ & \circ & [\times] \end{bmatrix} \qquad [\times] = \left[1 \quad 1 \quad 1 \quad 1\right], \qquad [\circ] = \left[0 \quad 0 \quad 0 \quad 0\right]$$

– structural matrix $\mathbf{S}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

– end time $T = 10$ [s].

Illustration of example no. 1 developed using an original simulation application (Matlab/Simulink application, Fig. 2), is shown in Fig. 3.

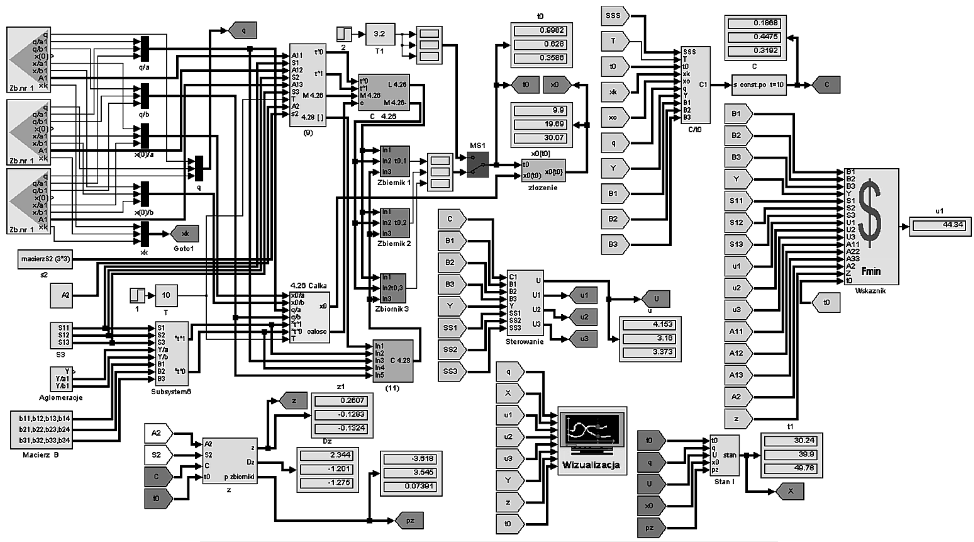

Fig. 2. Analog/digital simulation diagram (Matlab/Simulink)

Rys. 2. Schemat symulacji analogowo/cyfrowej (Matlab/Simulink)

The following are shown in the left row in Fig. 3:

– inflows to the reservoirs (first diagram),
– controlled outflows from the reservoirs for the benefit of successive conurbations (diagrams 2, 3, 4),
– controlled transfers among reservoirs (diagram 5). Black indicates water transfer which is directed (supply reservoir, receiving reservoir) as per the direction assumed in the system level equation. Red indicates those transfers which are directed opposite to the direction in the equation of level.

The right hand row of diagrams shows the following comparisons:

– outflows from reservoirs to successive demand functions.
  Due to various input data as regards reservoirs, inclusion times for reservoirs are not identical. Shaded fields show differences between demand functions $Y_i = 1, ..., 4$ and their provision by controlled outflows from reservoirs (diagrams 6, 7, 8, 9),
– the fifth diagram illustrates reservoir level trajectories with marked optimal times of including successive reservoirs in the system's operation. Inclusion times determine initial, optimum level values for individual reservoirs in the light of (10).

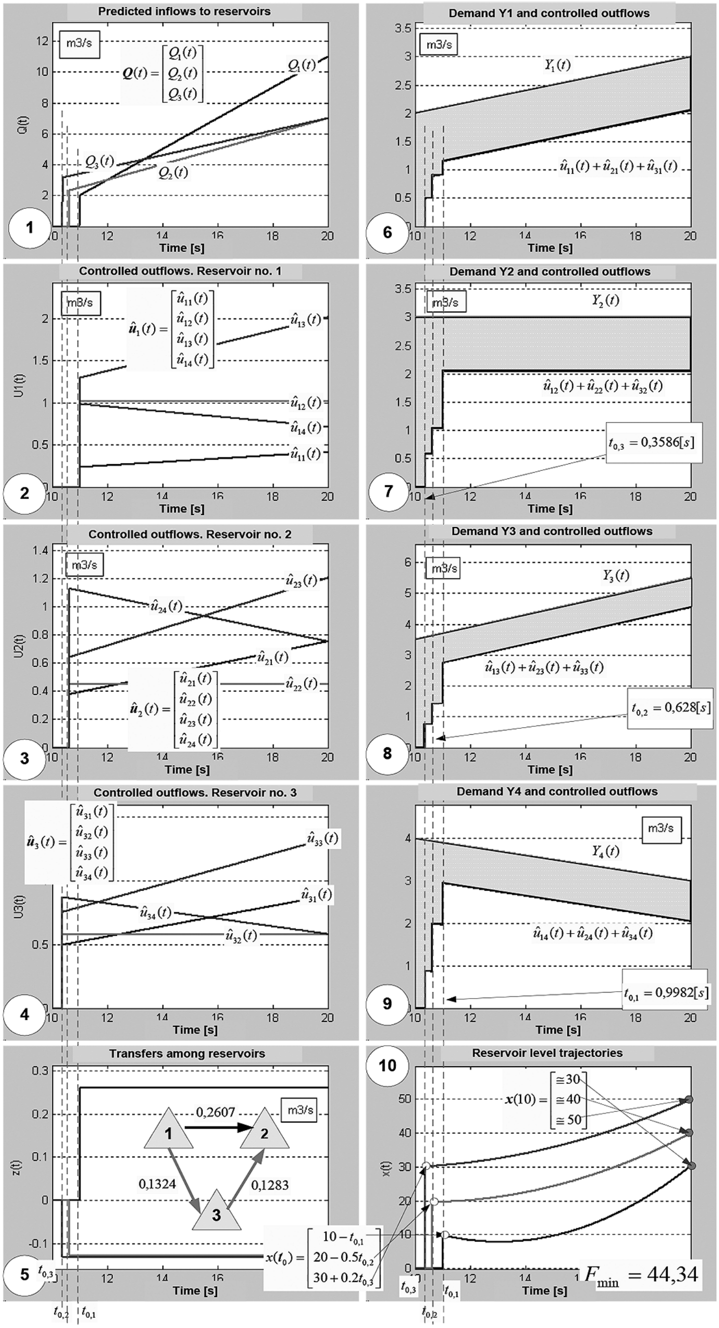

Fig. 3. Simulation results for hypothetical scenario of events ■ *End of example*

Rys. 3. Wyniki symulacji dla hipotetycznego scenariusza zdarzeń ■ *Koniec przykładu*

Conclusions arising from the analysis of the above problem may be formulated as follows:

1. With reference to the issue discussed (free optimisation commencement time), **times of including successive reservoirs in the system's operation are generally different**, dependent on input data assigned to reservoirs (prediction of water inflow to reservoir, reservoir involvement in providing for the demand function vector). Weight coefficients (matrix $\mathbf{A}_1$, $\mathbf{A}_2$) are also of some importance.
2. The determined optimal times for including individual reservoirs in the system's operation directly affect initial levels of reservoirs, which in this context are not rigidly fixed any more. This change implies the fact that aiming at reaching minimum quality coefficient value (1) in a form that is related only to providing for demand function vector, the end levels assumed for the optimisation task will not necessarily be achieved in 100%.
3. Among other things, the role of transfers among reservoirs comes down to minimising the differences between the assumed and achieved end levels of reservoirs.
4. Each change in results obtained (controlled outflows, inclusion times) brings about an increase in the value of coefficient 1.

2. For the vector of free optimisation conclusion times *T*

$$\begin{aligned}\boldsymbol{C}_1 = &\left[\left(\mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}}\right) \cdot \left(\boldsymbol{T}^* - t_0\right)\right]^{-1} \cdot \\ &\left\{ \boldsymbol{x}(T^*) - \boldsymbol{x}(t_0) - \int_{t_0}^{\boldsymbol{T}^*} \left[\boldsymbol{Q}^P(t) - \mathbf{S}_1 \cdot \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \mathbf{1}\right] dt \right\}\end{aligned} \tag{12}$$

The vector of constants $\boldsymbol{C}_1$ depends on the vector of free optimisation conclusion times, and another equation is needed to calculate this. This second equation may be derived using the following relation:

$$H(\boldsymbol{T}^*) - \left(\frac{\partial K[x(\boldsymbol{T}^*), \boldsymbol{T}^*]}{\partial \boldsymbol{T}^*} \right) = 0 \tag{13}$$

In the case discussed, coefficient (1) does not contain function $\partial K\,[x(\boldsymbol{T}^*), \boldsymbol{T}^*]$, and therefore $H(\boldsymbol{T}^*) = 0$. Carrying out much the same sequence of transformations as in the case of free optimisation commencement time, the following relation is revealed:

$$\begin{aligned}\hat{\psi}(\boldsymbol{T}^*) = &\left[0,5 \cdot \left(\mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}}\right)\right]^{-1} \cdot \\ &\cdot \left[\mathbf{S}_1 \cdot \mathbf{B}(\boldsymbol{T}^*) \cdot \mathbf{Y}(\boldsymbol{T}^*) \cdot \mathbf{S} \cdot \mathbf{1} - \boldsymbol{Q}^P(\boldsymbol{T}^*)\right]\end{aligned} \tag{14}$$

Comparing equations (12) and (14), we may calculate the vector of optimal process conclusion times.

Hypothetical scenario of events Example 2.■

Assuming task parameters as in the previous example, modifying only initial and end reservoir filling levels:

- initial levels of reservoirs: $\boldsymbol{x}_0^{\mathrm{T}} = \begin{bmatrix} 30 & 50 & 70 \end{bmatrix}$ [m³],
- end levels of reservoirs change in function of optimisation conclusion time: e.g. $\boldsymbol{x}(T)^{\mathrm{T}} = \begin{bmatrix} (30 + 0.1T^*_{res1}) & (50 - 0.5T^*_{res2}) & (70 + T^*_{res3}) \end{bmatrix}$ [m³],
- inflows to reservoirs: $\boldsymbol{Q}(t)^{\mathrm{T}} = \begin{bmatrix} (0.1t + 3) & (0.5t + 2) & (0.3t + 1) \end{bmatrix}$ [m³],
- water demand downstream of the reservoirs:

$$\mathbf{Y}(t) = \begin{bmatrix} [\otimes] & * & * \\ * & [\otimes] & * \\ * & * & [\otimes] \end{bmatrix} \left[\mathrm{m}^3/\mathrm{s}\right], \quad [\otimes] = \begin{bmatrix} 2 + 0.1t & 0 & 0 & 0 \\ 0 & 3 \cdot 1(t) & 0 & 0 \\ 0 & 0 & 3.5 + 0.2t & 0 \\ 0 & 0 & 0 & 4 + 0.1t \end{bmatrix}$$

functions of reservoir involvement in providing for the demand function:

$$\mathbf{B}(t) = \begin{bmatrix} [\otimes_1] & * & * \\ * & [\otimes_2] & * \\ * & * & [\otimes_3] \end{bmatrix}, \qquad [\otimes_1] = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix} \cdot (t)$$

$$[\otimes_2] = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \cdot (t) \qquad [\otimes_3] = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix} \cdot (t)$$

- matrixes of weight coefficients:

$$\mathbf{A}_1 = \begin{bmatrix} [\oplus] & * & * \\ * & [\oplus] & * \\ * & * & [\oplus] \end{bmatrix}, \quad [\oplus] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- structural matrix:

$$\mathbf{S}_1 = \begin{bmatrix} [\times] & \circ & \circ \\ \circ & [\times] & \circ \\ \circ & \circ & [\times] \end{bmatrix} \qquad [\times] = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \qquad [\circ] = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

- structural matrix $\mathbf{S}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$
- initial time $t_0 = 0$ [s]

Solution:

acc. to (12) $$\boldsymbol{C}_1 = \begin{bmatrix} \left(-0.105T_{res1}^{*3} + 8.3T_{res1}^{*2}\right) \Big/ 28T_{res1}^{*2} \\ \left(-0.775T_{res2}^{*3} + 14.5T_{res2}^{*2}\right) \Big/ 28T_{res2}^{*2} \\ \left(-0.345T_{res3}^{*3} 26.9T_{res3}^{*2}\right) \Big/ 28T_{res3}^{*2} \end{bmatrix},$$

acc. to (14) $$\boldsymbol{C}_1 = \begin{bmatrix} -0.03857T_{res1}^{*} + 0.5214 \\ -0.1386T_{res2}^{*} + 1.136 \\ -0.07286T_{res3}^{*} + 1.593 \end{bmatrix}$$

When comparing sides of the relations above, we come to quadratic equations on account of optimisation conclusion times, which are different for successive reservoirs.

$$\boldsymbol{T}^* = \begin{bmatrix} T_{res1}^{*} = 7.778\,[\mathrm{s}] \\ T_{res2}^{*} = 5.945\,[\mathrm{s}] \\ T_{res3}^{*} = 11.57\,[\mathrm{s}] \end{bmatrix}$$

Next, we replace the values in expression 14 with the times obtained, thus receiving the vector of constant $\boldsymbol{C}_1$:

$$\boldsymbol{C}_1 = \begin{bmatrix} 0.291 \\ 0.4435 \\ 0.7705 \end{bmatrix}$$

The next step involves calculating the optimal control vector (outflows from reservoirs for successive conurbations):

$$\hat{\boldsymbol{u}}(t)_{(+)} = \begin{bmatrix} \begin{bmatrix} 0.4 + 0.020t \\ 0.9 \\ 0.7 + 0.040t \\ 1.2 + 0.075t \end{bmatrix} \\ \begin{bmatrix} 0.8 + 0.04t \\ 0.9 \\ 1.05 + 0.06t \\ 1.6 + 0.1t \end{bmatrix} \\ \begin{bmatrix} 0.8 + 0.04t \\ 1.2 \\ 1.75 + 0.1t \\ 1.2 + 0.075t \end{bmatrix} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} 0.291 \\ 0.291 \\ 0.291 \\ 0.291 \end{bmatrix} \\ \begin{bmatrix} 0.4435 \\ 0.4435 \\ 0.4435 \\ 0.4435 \end{bmatrix} \\ \begin{bmatrix} 0.7705 \\ 0.7705 \\ 0.7705 \\ 0.7705 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0.109 + 0.020t \\ 0.609 \\ 0.409 + 0.040t \\ 0.909 + 0.075t \end{bmatrix} \\ \begin{bmatrix} 0.3565 + 0.04t \\ 0.4565 \\ 0.6065 + 0.06t \\ 1.1565 + 0.1t \end{bmatrix} \\ \begin{bmatrix} 0.0295 + 0.04t \\ 0.4295 \\ 0.9795 + 0.1t \\ 0.4295 + 0.075t \end{bmatrix} \end{bmatrix} \ [\mathrm{m^3/s}]$$

After entering the optimisation conclusion times obtained in the equations, $\boldsymbol{x}(\boldsymbol{T})^{\mathrm{T}} = \left[(30+0.1T^{*}_{res1}) \quad (50-0.5T^{*}_{res2}) \quad (70+T^{*}_{res3}) \right]$ [m3/s] the vector of end conditions is:

$$\boldsymbol{x}(\boldsymbol{T}^{*}) = \begin{bmatrix} (30+0.1T^{*}_{res1}) & (30+0.1\cdot 7,778) & & 30,78 \\ (50-0.5T^{*}_{res2}) = & (50-0.5\cdot 2,91) & \cong & 47,03 \\ (70+T^{*}_{res3}) & (70+11,57) & & 81,57 \end{bmatrix} [\mathrm{m}^3]$$

Whereas, the vector of accomplished end levels obtained as a result of employing specified control is:

$$(T^{*}) = \begin{bmatrix} 30 \\ 50 \\ 70 \end{bmatrix} + \int_{0}^{\substack{T^{*}_{\mathrm{res1}} = 7.778 \\ T^{*}_{\mathrm{res2}} = 5.945 \\ T^{*}_{\mathrm{res3}} = 11.57}} \left\{ \begin{bmatrix} (0.1t+3) \\ (0.5t+2) \\ (0.3t+1) \end{bmatrix} - \begin{bmatrix} [1 \;\; 1 \;\; 1 \;\; 1] & 0 & 0 \\ 0 & [1 \;\; 1 \;\; 1 \;\; 1] & 0 \\ 0 & 0 & [1 \;\; 1 \;\; 1 \;\; 1] \end{bmatrix} * \\ * \begin{bmatrix} \begin{bmatrix} 0.109+0.020t \\ 0.609 \\ 0.409+0.040t \\ 0.909+0.075t \end{bmatrix} \\ \begin{bmatrix} 0.3565+0.04t \\ 0.4565 \\ 0.6065+0.06t \\ 1.1565+0.1t \end{bmatrix} \\ \begin{bmatrix} 0.0295+0.04t \\ 0.4295 \\ 0.9795+0.1t \\ 0.4295+0.075t \end{bmatrix} \end{bmatrix} \right\} = \begin{bmatrix} 31,07 \\ 48.62 \\ 54,05 \end{bmatrix} [\mathrm{m}^3]$$

While analysing the above problem, the conclusions arising may be formulated as follows:

1. With reference to the discussed issue (free optimisation conclusion time), **times of withdrawing successive reservoirs from operation in the system are generally different**, dependent on reservoir system input data (prediction of water inflow to reservoir, reservoir involvement in providing for the demand function vector).
2. The determined optimal times of withdrawing individual reservoirs from operation in the system directly affect the reservoir end levels, which in this context are no longer rigidly fixed (as was the case in the task with free optimisation commencement time). As we know, in principle the quality coefficient value (1) is related to providing for the demand function vector, and as a consequence the end levels determined in function of optimisation conclusion time will not necessarily be achieved in 100%.
3. Among other things, the role of transfers among reservoirs comes down to minimising the differences between the assumed and achieved reservoir end levels.
4. Each change in results obtained (controlled outflows, inclusion times) brings about an increase in the value of coefficient 1.

Illustration (Fig. 5) presents diagrams obtained as a result of simulating example no. 2 using the Matlab/Simulink application (Fig. 4).

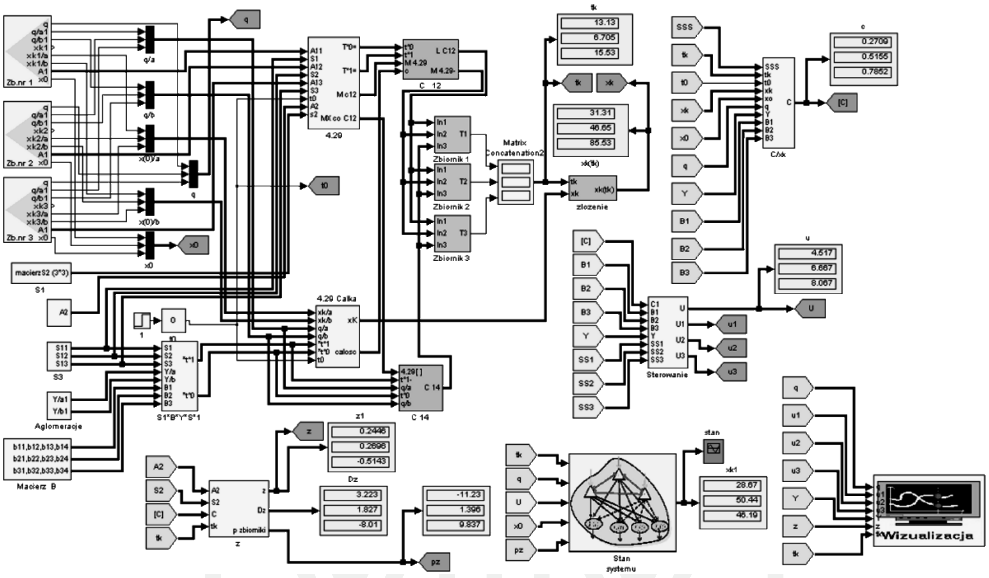

Fig. 4. Analog/digital simulation diagram (Matlab/Simulink)

Rys. 4. Schemat symulacji analogowo/cyfrowej (Matlab/Simulink)

The left row in Fig. 5 shows:
- inflows to the reservoirs (diagram 1),
- controlled outflows from the reservoirs for the benefit of successive conurbations (diagrams 2, 3, 4),
- controlled transfers among reservoirs (diagram 5). Black indicates water transfer which is directed (supply reservoir, receiving reservoir) as per the direction assumed in the system level equation. Red indicates those transfers which are directed opposite to the direction in the equation of level.

The right row of the diagrams shows the following comparisons:
- The sums of outflows from reservoirs to successive conurbations.
  Due to various input data as regards reservoirs, inclusion times for reservoirs are not identical. Shaded fields show differences between demand functions Y_i =1, ..., 4 and their provision by controlled outflows from reservoirs (diagrams 6, 7, 8, 9).
- The 10th diagram illustrates reservoir level trajectories with marked optimal times of withdrawing successive reservoirs from operation in the system. Withdrawal times determine final, optimum level values for individual reservoirs.

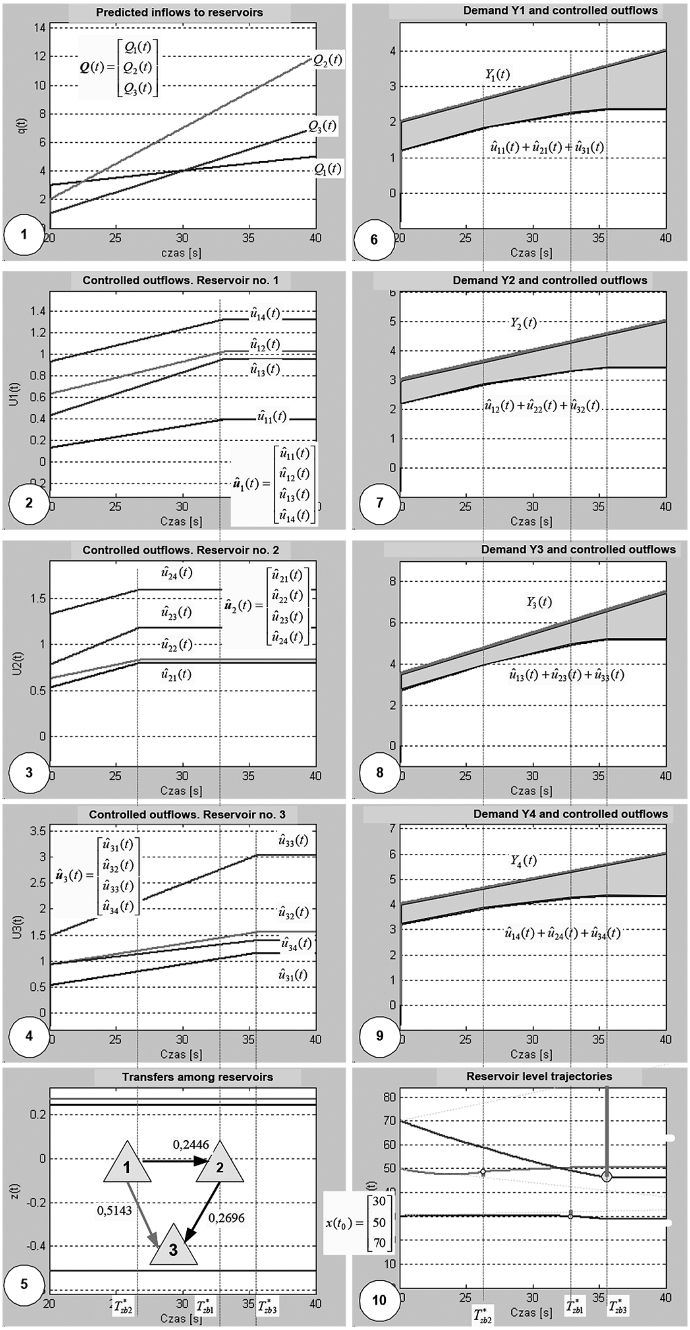

Fig. 5. Simulation results for hypothetical scenario of events ■ *End of example*

Rys. 5. Wyniki symulacji dla hipotetycznego scenariusza zdarzeń ■ *Koniec przykładu*

3. Summary

This article discusses control issues for a system of retention reservoirs which supply certain group of consumers with water. The issues presented may constitute cases frequently used in practice.

Predetermined initial and end conditions in level trajectories constitute a highly practical optimisation task, which may form a basis for the functioning of complex reservoir system control.

Additionally, undetermined optimisation commencement/conclusion time (commencing the work of successive reservoirs within the system, concluding the work of the entire reservoir system) considerably extends the group of possible cases.

The numerical examples presented in this article refer to specific system structure (reservoirs, consumers, connections); whereas the analytical formulae obtained as a result of solving the tasks presented are general. They are valid in virtually any dimensionality with reference to the system elements and connections between them, which allows analysis of the work of freely configured water distribution systems.

The role of transfers among reservoirs is to:
– improve meeting requirements related to end conditions,
– improve water distribution from the system (water deliveries to consumers), that is, minimise losses referred to the required customers' needs.

The researchers carried out simulation of an example system operation using original computer applications, based on hypothetical data, which form the so-called scenarios of events within an analysed system. Further, simulation results were discussed. As a consequence, various control variants may be proposed (depending on the scenario of events), which have one common characteristic: treating prediction in a deterministic way.

In each case, simulation results include:
– vector of best optimisation commencement/conclusion times,
– vector of outflows from reservoirs for the entire determined optimisation horizon,
– vector of transfers among reservoirs,
– vector of end filling levels in the system reservoirs, which guarantees required water volume in the system reservoirs at the end of the time horizon under consideration.

Analysis of a suitably large set of hypothetical situations allows effective determination of the system structure and the variability range of its main parameters: $\mathbf{B}(t)$, $\mathbf{Y}(t)$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\boldsymbol{x}(T)$. Obtained solutions: the vector of control operations $\hat{\boldsymbol{u}}(t)_{(+)}$, the vector of transfers among reservoirs $\hat{\boldsymbol{z}}(t)$, and as a consequence quality coefficient (F) are the functions of these parameters.

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