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ON NONLOCAL EVOLUTION FUNCTIONAL-DIFFERENTIAL PROBLEM IN A BANACH SPACE

NIELOKALNE EWOLUCYJNE FUNKCJONALNO-RÓŻNICZKOWE ZAGADNIENIE W PRZESTRZENI BANACHA

Abstract

The aim of this paper is to prove two theorems on the existence and uniqueness of mild and classical solutions of a nonlocal semilinear functional-differential evolution Cauchy problem in a Banach space. The method of semigroups, the Banach fixed-point theorem and the Bochenek theorem (see [3]) about the existence and uniqueness of the classical solution of the first order differential evolution problem in a not necessarily reflexive Banach space are used to prove the existence and uniqueness of the solutions of the considered problem. The results are based on publications [1 — 8].
Keywords: evolution problem, functional-differential problem, nonlocal problem

Streszczenie

W artykule udowodniono dwa twierdzenia o istnieniu i jednoznaczności rozwiązań całkowych i klasycznych nielokalnego semiliniowego funkcjonalno-różniczkowego ewolucyjnego zagadnienia Cauchy'ego w dowolnej przestrzeni Banacha. W tym celu zastosowano metodę półgrup, twierdzenie Banacha o punkcie stałym i twierdzenie Bochenka [3] o istnieniu i jednoznaczności klasycznego rozwiązania ewolucyjnego zagadnienia różniczkowego pierwszego rzędu w niekoniecznie refleksywnej przestrzeni Banacha. Artykuł bazuje na publikacjach [1 — 8].
Słowa kluczowe: zagadnienie ewolucyjne, zagadnienie funkcjonalno-różniczkowe, zagadnienie nielokalne

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1. Preliminaries

In this paper, we prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution nonlocal Cauchy problem using the method of semigroups, the Banach fixed-point theorem and the Bochenek theorem (see [3]) about the existence and uniqueness of the classical solution of the linear first-order differential evolution problem in a not necessarily reflexive Banach space.

Let E be a real Banach space with norm $\|\cdot\|$ and let $A : E \to E$ be a closed densely defined linear operator. For an operator A, let $\mathcal{D}(A)$, $\rho(A)$ and A^* denote its domain, resolvent set and adjoint, respectively.

For Banach space E, $\mathcal{C}(E)$ denote the set of closed linear operators from E into itself.

We will need the class $G(\tilde{M},\beta)$ of operators A satisfying the conditions:

There exist constants $\tilde{M} > 0$ and $\beta \in \mathbb{R}$ such that

(C_1) $A \in \mathcal{C}(E)$, $\overline{\mathcal{D}(A)} = E$ and $(\beta, +\infty) \subset \rho(-A)$,

(C_2) $\left\|(A+\xi)^{-k}\right\| \leqslant \tilde{M}(\xi-\beta)^{-k}$ for each $\xi > \beta$ and $k = 1, 2, \ldots$

We will use the assumption:

Assumption (Z). The adjoint operator A^* is densely defined in E^*, i.e. $\overline{\mathcal{D}(A^*)} = E^*$.

It is known (see [5], p. 485 and [7], p. 20) that for $A \in G(\tilde{M},\beta)$ there exists exactly one strongly continuous semigroup $T(t) : E \to E$ for $t \geqslant 0$ such that $-A$ is its infinitesimal generator and

$$\|T(t)\| \leqslant \tilde{M}e^{\beta t} \quad \text{for } t \geqslant 0.$$

Throughout this paper, we assume (C_1), (C_2) and assumption (Z).

In this paper, we assume that $t_0 > 0$, $a > 0$,

$$\begin{aligned} \mathcal{J} := [t_0, t_0+a],\ \Delta &:= \{(t,s) : t_0 \leqslant s \leqslant t \leqslant t_0+a\}, \\ M &:= \sup_{t\in[0,a]} \|T(t)\|, \\ X &:= \mathcal{C}(\mathcal{J}, E) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} F_1 &: \mathcal{J} \times E^{m+1} \to E,\ F_2 : \Delta \times E^2 \to E,\ \tilde{G} : X \to E, \\ f &: \Delta \times E \to E,\ \sigma_i : \mathcal{J} \to \mathcal{J}\ (i = 1, \ldots, m) \end{aligned}$$

are given functions satisfying some assumptions.

The functional-differential evolution nonlocal Cauchy problem considered here is of the form

$$\begin{aligned} u'(t) \quad &+ \quad Au(t) = F_1(t, u(t), u(\tilde{\sigma}_1(t)), \ldots, u(\tilde{\sigma}_m(t))) + \\ &+ \quad \int_{t_0}^{t} F_2(t, s, u(s), \int_{t_0}^{s} f(s, \tau, u(\tau)) d\tau) ds, \ t \in \mathcal{J} \setminus \{t_0\}, \qquad (1.2) \end{aligned}$$

$$u(t_0) + \tilde{G}(u) = u_0, \qquad (1.3)$$

where $u_0 \in E$.

To study problem (1.2)–(1.3) we will need some information related to the following linear problem:

$$\begin{aligned} u'(t) + Au(t) \quad &= \quad k(t), \quad t \in \mathcal{J} \setminus \{t_0\}, \qquad (1.4) \\ u(t_0) \quad &= \quad x \qquad (1.5) \end{aligned}$$

and the following definition:

A function $u : \mathcal{J} \to E$ is said to be a classical solution of problem (1.4)–(1.5) if

(i) u is continuous and continuously differentiable on $\mathcal{J} \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = k(t) \quad$ for $t \in \mathcal{J} \setminus \{t_0\}$,

(iii) $u(t_0) = x$.

To study problem (1.2)–(1.3) we will also need the following theorem:

Theorem 1.1 (see [3]). *Let $k : \mathcal{J} \to E$ be Lipschitz continuous on $\mathcal{J}$ and $x \in \mathcal{D}(A)$.*

Then u given by the formula

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)k(s) ds, \quad t \in \mathcal{J} \qquad (1.6)$$

is the unique classical solution of the Cauchy problem (1.4)–(1.5).

2. On mild solution

A function $u : \mathcal{J} \to X$ satisfying the integral equation

$$\begin{aligned} u(t) \quad &= \quad T(t - t_0)u_0 - T(t - t_0)\tilde{G}(u) + \\ &+ \quad \int_{t_0}^{t} T(t - s) F_1(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s))) ds + \\ &+ \quad \int_{t_0}^{t} T(t - s) \Big(\int_{t_0}^{s} F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu) d\tau \Big) ds, \ t \in \mathcal{J}, \end{aligned}$$

is said to be a mild solution of the nonlocal Cauchy problem (1.2)–(1.3).

Theorem 2.1. *Assume that*

(*i*) *for all* $z_i \in E$ $(i = 0, 1, \ldots, m)$, *the function*

$$\mathcal{J} \ni t \mapsto F_1(t, z_0, z_1, \ldots, z_m) \in E \quad \text{is continuous,}$$

for all $z_i \in E$ $(i = 1, 2)$, *the function*

$$\Delta \ni (t, s) \mapsto F_2(t, s, z_1, z_2) \in E \quad \text{is continuous,}$$

for all $z \in E$, *the function*

$$\Delta \ni (t, s) \mapsto f(t, s, z) \quad \text{is continuous,}$$

$$\tilde{G} : \ X \to E, \ \sigma_i \in \mathcal{C}(\mathcal{J}, \mathcal{J}) \ (i = 1, \ldots, m) \text{ and } u_0 \in E.$$

(*ii*) *there are constants* $L_i > 0$ $(i = 1, 2, 3, 4)$ *such that*

$$\|F_1(t, z_0, z_1, \ldots, z_m) - F_1(t, \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_m)\| \leqslant$$
$$\leqslant L_1 \sum_{i=0}^{m} \|z_i - \tilde{z}_i\| \ \text{ for } t \in \mathcal{J}, \ z_i, \ \tilde{z}_i \in E \ (i = 1, \ldots, m); \tag{2.1}$$

$$\|F_2(t, s, z_1, z_2) - F_2(t, s, \tilde{z}_1, \tilde{z}_2)\| \leqslant L_2 \sum_{i=1}^{2} \|z_i - \tilde{z}_i\|$$
$$\text{for } (t, s) \in \Delta, \ z_i, \tilde{z}_i \in E, \ (i = 1, 2); \tag{2.2}$$
$$\|f(t, s, z) - f(t, s, \tilde{z})\| \leqslant L_3 \|z - \tilde{z}\|$$
$$\text{for } (t, s) \in \Delta, \ z, \tilde{z} \in E; \tag{2.3}$$
$$\left\|\tilde{G}(w) - \tilde{G}(\tilde{w})\right\| \leqslant L_4 \|w - \tilde{w}\| \ \text{ for } w, \tilde{w} \in X; \tag{2.4}$$

(*iii*) $M[L_1 a(m + 1) + L_2 a^2(1 + L_3 a) + L_4] < 1$.

Then the nonlocal problem (1.2)–(1.3) *has a unique mild solution in* $\mathcal{J}$.

Proof. Introduce an operator $\mathfrak{F}$ on X by the formula

$$\begin{aligned}(\mathfrak{F}w)(t) \ &:= \ T(t - t_0)u_0 - T(t - t_0)\tilde{G}(w) + \\ &+ \int_{t_0}^{t} T(t - s)F_1(s, w(s), w(\sigma_1(s)), \ldots, w(\sigma_m(s)))ds + \\ &+ \int_{t_o}^{t} T(t - s)\Big(\int_{t_0}^{s} F_2(s, \tau, w(\tau), \int_{t_0}^{\tau} f(\tau, \mu, w(\mu))d\mu)d\tau\Big)ds\end{aligned} \tag{2.5}$$

for $w \in X$ and $t \in \mathcal{J}$.

It is easy to see that

$$\mathfrak{F}:\ X \to X. \tag{2.6}$$

Now, we will show that $\mathfrak{F}$ is a contraction on X. For this purpose, observe that from (2.5), (1.1) and (2.1)–(2.4),

$$\begin{aligned}
&\|(\mathfrak{F}w)(t) - (\mathfrak{F}\tilde{w})(t)\| \leqslant ML_4\|w-\tilde{w}\| + \qquad (2.7)\\
+\ & ML_1\int_{t_0}^{t}\Big(\|w(s)-w(\tilde{s})\| + \sum_{i=1}^{m}\|w(\sigma_i(s))-\tilde{w}(\sigma_i(s))\|\Big)ds +\\
+\ & ML_2\int_0^t\Big(\int_0^s\big(\|w(\tau)-\tilde{w}(\tau)\| +\\
+\ & \int_{t_0}^{\tau}\|f(\tau,\mu,w(\mu))-f(\tau,\mu,\tilde{w}(\mu))\|\,d\mu\big)d\tau\Big)ds \leqslant\\
\leqslant\ & ML_4\|w-\tilde{w}\| + ML_1a(m+1)\|w-\tilde{w}\| +\\
+\ & ML_2\int_0^t\Big(\int_{t_0}^{s}[\|w(\tau)-\tilde{w}(\tau)\| + L_3\int_{t_0}^{\tau}\|w(\mu)-\tilde{w}(\mu)\|\,d\mu]d\tau\Big)ds \leqslant\\
\leqslant\ & q\|w-\tilde{w}\| \quad \text{for } w,\tilde{w}\in X,
\end{aligned}$$

where

$$q := M\big(L_1a(m+1) + L_2a^2(1+L_3a) + L_4\big).$$

Then, by (2.7) and by assumption (iii),

$$\|\mathfrak{F}w - \mathfrak{F}\tilde{w}\| \leqslant q\|w-\tilde{w}\| \quad \text{for } w,\tilde{w}\in X \text{ with } 0<q<1. \tag{2.8}$$

Consequently, from (2.6) and (2.8), operator $\mathfrak{F}$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space X there is only one fixed point of $\mathfrak{F}$ and this point is the mild solution of the nonlocal Cauchy problem (1.2)–(1.3). So, the proof of Theorem 2.1 is complete. □

3. On classical solution

A function $u : \mathcal{J} \to E$ is said to be a classical solution of the nonlocal Cauchy problem (1.2)–(1.3) on $\mathcal{J}$ if :

(i) u is continuous on $\mathcal{J}$ and continuously differentiable on $\mathcal{J}\setminus\{t_0\}$,

(ii) $u'(t) + Au(t) = F_1\big(t, u(t), u(\sigma_1(t)), \ldots, u(\sigma_m(t))\big)) +$
$+\int_{t_0}^{t} F_2(t,s,u(s),\int_{t_0}^{s} f(s,\tau,u(\tau))d\tau)ds,\ t\in\mathcal{J}\setminus\{t_0\}$,

(iii) $u(t_0) + \tilde{G}(u) = u_0$.

Theorem 3.1. *Suppose that assumptions* (*i*)*–*(*iii*) *of Theorem 2.1 are satisfied. Then the nonlocal Cauchy problem* (1.2)–(1.3) *has a unique mild solution on* $\mathcal{J}$, *denoted by* u. *Assume, additionally, that:*

(*i*) $u_0 \in \mathcal{D}(A)$ *and* $\tilde{G}(u) \in \mathcal{D}(A)$;

(*ii*) *there are constants* $C_i > 0$ $(i = 1, 2)$ *such that*

$$\left\|F_1(t, z_0, z_1, \ldots, z_m) - F_1(\tilde{t}, z_0, z_1, \ldots, z_m)\right\| \leqslant C_1 \left|t - \tilde{t}\right| \\ \text{for } t, \tilde{t} \in \mathcal{J},\ z_i \in E\ (i = 0, 1, \ldots, m) \tag{3.1}$$

and

$$\left\|F_2(t, s, z_1, z_2) - F_2(\tilde{t}, s, z_1, z_2)\right\| \leqslant C_2 \left|t - \tilde{t}\right| \\ \text{for } (t, s) \in \Delta,\ (\tilde{t}, s) \in \Delta,\ z_i \in E\ (i = 1, 2); \tag{3.2}$$

(*iii*) *there is a constant* $c > 0$ *such that*

$$\left\|u(\sigma_i(t)) - u(\sigma_i(\tilde{t}))\right\| \leqslant c \left\|u(t) - u(\tilde{t})\right\| \\ \text{for } t, \tilde{t} \in \mathcal{J}\ (i = 0, 1, \ldots, m). \tag{3.3}$$

Then u *is the unique classical solution of the nonlocal Cauchy problem* (1.2)–(1.3) *on* $\mathcal{J}$.

Proof. Since all the assumptions of Theorem 2.1 are satisfied, the nonlocal Cauchy problem (1.2)–(1.3) possesses a unique mild solution which, according to the assumption, is denoted by u.

Now we will show that u is the unique classical solution of the problem (1.2)–(1.3) on $\mathcal{J}$. To this end, introduce

$$N_1 := \max_{s \in \mathcal{J}} \left\|F_1\big(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s))\big)\right\| \tag{3.4}$$

and

$$N_2 := \max_{(\xi,\eta) \in \Delta} \left\|F_2(\xi, \eta, u(\eta), \int_{t_0}^{\eta} f(\eta, \mu, u(\mu)) d\mu\right\|, \tag{3.5}$$

and observe that

$$\begin{aligned} & u(t+h)-u(t)= \\ = \; & T(t-t_0)\big(T(h)-I\big)u_o - T(t-t_0)\big(T(h)-I\big)\tilde{G}(u)+ \\ + \; & \int_{t_0}^{t_0+h} T(t+h-s)F_1\big(s,u(s),u(\sigma_1(s)),\dots,u(\sigma_m(s))\big)ds+ \\ + \; & \int_{t_0}^{t} T(t-s)\Big(F_1\big(s+h,u(s+h),u(\sigma_1(s+h)),\dots,u(\sigma_m(s+h))\big)- \\ & \qquad - F_1\big(s,u(s),u(\sigma_1(s)),\dots,u(\sigma_m(s))\big)\Big)ds+ \\ + \; & \int_{t_0}^{t_0+h} T(t+h-s)\Big(\int_{t_0}^{s} F_2\big(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu)d\mu\big)d\tau\Big)ds+ \\ + \; & \int_{t_0}^{t} T(t-s)\Big(\int_{t_0}^{s} \big(F_2\big(s+h,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu\big)- \\ & \qquad - F_2\big(s,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu\big)d\tau\Big)ds+ \\ + \; & \int_{t_0}^{t} T(t-s)\Big(\int_{s}^{s+h} F_2\big(s+h,\tau,u(\tau),\int_{t_0}^{\tau} f(\tau,\mu,u(\mu))d\mu\big)d\tau\Big)ds \end{aligned} \tag{3.6}$$

for $t\in[t_0,t_0+a)$, $h>0$ and $t+h\in(t_0,t_0+a]$.

Consequently by (3.6), (1.1) and (3.1)–(3.5),

$$\begin{aligned} & \|u(t+h)-u(t)\| \leqslant hM\|Au_0\| + hM\left\|A\tilde{G}(u)\right\| + \\ + \; & hMN_1 + ahML_1 + ML_1\int_{t_0}^{t}\Big(\|u(s+h)-u(s)\| + \\ + \; & \sum_{i=1}^{m}\|u(\sigma_i(s+h))-u(\sigma_i(s))\|\Big)ds + a^2ML_2h + 2aMN_2h \leqslant \\ \leqslant \; & Ch + ML_1(1+mc)\int_{t_0}^{t}\|u(s+h)-u(s)\|\,ds \end{aligned} \tag{3.7}$$

for $t\in[t_0,t_0+a)$, $h>0$ and $t+h\in(t_0,t_0+h]$, where

$$C := M\Big(\|Au_0\| + \left\|A\tilde{G}(u)\right\| + N_1 + aL_1 + a^2L_2 + 2aN_2\Big).$$

From (3.7) and Gronwall's inequality,

$$\|u(t+h)-u(t)\| \leqslant Ce^{aML_1(1+mc)}h$$

for $t\in[t_0,t_0+h]$, $h>0$ and $t+h\in(t_0,t_0+a]$.

Hence u is Lipschitz continuous on $\mathcal{J}$.

The Lipschitz continuity of u on $\mathcal{J}$ and inequalities (3.1), (2.1), (3.2) imply that the function

$$\begin{aligned}\mathcal{J} \ni t \mapsto k(t) \quad &:= \quad F_1\big(t, u(t), u(\sigma_1(t)), \ldots, \sigma_m(t))\big) + \\ &+ \quad \int_{t_0}^{t} F_2\big(t, s, u(s), \int_{t_0}^{s} f(s, \tau, u(\tau))d\tau\big)ds \in E\end{aligned}$$

is Lipschitz continuous on $\mathcal{J}$. This property of $t \mapsto k(t)$ together with assumptions of Theorem 3.1 imply, by Theorem 1.1, by Theorem 2.1 and by the definition of the mild solution from Section 2, that the linear Cauchy problem

$$\begin{aligned} v'(t) + Av(t) &= k(t), \quad t \in \mathcal{J} \setminus \{t_0\}, \\ v(t_0) &= u_0 - \tilde{G}(u) \end{aligned}$$

has a unique classical solution v such that

$$\begin{aligned} v(t) \quad &= \quad T(t - t_0)u_0 - T(t - t_0)\tilde{G}(u) + \int_{t_0}^{t} T(t - s)k(s)ds = \\ &= \quad T(t - t_0)u_0 - T(t - t_0)\tilde{G}(u) + \\ &+ \quad \int_{t_0}^{t} T(t - s)F_1\big(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_m(s))\big)ds + \\ &+ \quad \int_{t_0}^{t} T(t - s)\Big(\int_{t_0}^{s} F_2\big(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu\big)d\tau\Big)ds = \\ &= \quad u(t), \quad t \in \mathcal{J}. \end{aligned}$$

Consequently, u is the unique classical solution of the nonlocal Cauchy problem (1.2)–(1.3) on$\mathcal{J}$. Therefore, the proof of Theorem 3.1 is complete.

□

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