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RESONANCE PHENOMENA IN MICRO/NANOELECTROMECHANICAL SYSTEMS

ZJAWISKA REZONANSOWE W UKŁADACH MIKRO/NANOELEKTROMECHANICZNCH

A b s t r a c t

In the paper, some aspects of nonlinearity of micro/nanoelectromechanical systems (MEMS/NEMS) are presented. Because of great values of strains of micro/nanobeams the nonlinear description is necessary. Particularly, the nonlinear inertia term is added to equation relating to motion of the beam. Numerical calculations of resonance curves and instability regions are given. Results are presented on graphs.

Keywords: *parametric resonance, stability, MEMS, NEMS, nonlinearities*

S t r e s z c z e n i e

W artykule przedstawiono pewne aspekty nieliniowości w mikro/nanoukładach elektromechanicznych (MEMS/NEMS). Ze względu na duże odkształcenia mikro/nanobelek nieliniowy opis jest konieczny. W szczególności do równania ruchu belki wprowadzono wyraz opisujący nieliniową bezwładność. Podano wyniki obliczeń numerycznych dla krzywych rezonansowych oraz obszarów niestateczności. Rezultaty przedstawiono na wykresach.

Słowa kluczowe: *rezonans parametryczny, stateczność, MEMS, NEMS, nieliniowość*

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1. Introduction

Microelectromechanical and nanoelectromechanical systems (MEMS and NEMS) are applied in different types of detectors and sensors e.g. in mass sensors. These systems frequently work in parametric resonance states. Great values of strain and of quality factor are characteristic features of considered mechanical systems. Therefore the nonlinear description of the systems is necessary. Response of the system depends on the types of nonlinearities.

This paper bases on references [1–4] where some mechanical micro/nanosystems connected with excitation electric systems are described. In the papers [1, 2] a parametrical excited microelectromechanical oscillator is analysed. To describe this oscillator, a Mathieu equation with nonlinearity is adopted and a perturbation method of solution is used. Two kinds of nonlinearities are taken into account: nonlinear elasticity and nonlinear excitation caused by an electric field. The system without damping and with small damping is considered. The system is used for example as mass sensor. In [3] the similar system is described but it is subject to harmonic forcing or to parametric excitation. Theoretical and experimental investigations are presented. The Duffing equation and the Mathieu equation with nonlinearity are used. In [4] a microbeam which is forced by an electric field is described. Discretization of the equation of motion of the beam and saving only the first ordinary differential equation lead to the same equation as adopted in [1–3] for lumped-mass systems.

The paper, a nonlinear inertia force is taken into account and its effect on the motion of system is presented. The nonlinear inertia force is particularly important if we consider mass sensors. The nonlinearity changes the value of excitation frequency for which a transition between stable and unstable solutions occurs.

2. Equation of motion of the system with additional nonlinearity

First we consider MEMS oscillator presented in Fig. 1, [1]. The system consists of: A, B – non-interdigitated comb-drive actuators, C – flexures, D – backbone. The oscillator is excited by an electric signal. The equation of motion of the system presented in Fig. 1 has the following form (cf. [1] equation (1))

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+k_1x+k_3x^3+(r_1x+r_3x^3)V_A^2(1+\cos\theta t)=0, \tag{1}$$

where m is the mass of an oscillator (a shuttle, a backbone [1, 2, 6]), c is the damping coefficient, k_1 and k_3 are respectively the linear and cubic nonlinear mechanical elastic coefficients, r_1 and r_3 are respectively the linear and cubic nonlinear electrostatic stiffness of the non-interdigitated comb-fingers, finally the excitation voltage $V(t)$ applied to the system is: $V(t)=V_A\sqrt{1+\cos\theta t}$, V_A and θ are positive constants.

The values of coefficients of equation (1), measured by different methods, are the following (cf. [1])

$$k_1 = 2.85\frac{\mu\mathrm{N}}{\mu\mathrm{m}},\quad r_1 = 2.96\cdot 10^{-3}\frac{\mu\mathrm{N}}{\mathrm{V}^2\mu\mathrm{m}},\quad c = 3.88\cdot 10^{-8}\frac{\mathrm{kg}}{\mathrm{s}},\quad m = 9.93\cdot 10^{-11}\,\mathrm{kg},$$
$$k_3 = 0.075\frac{\mu\mathrm{N}}{\mu\mathrm{m}^3},\quad r_3 = -2.1\cdot 10^{-4}\frac{\mu\mathrm{N}}{\mathrm{V}^2\mu\mathrm{m}^3},\quad \omega_0^2 = \frac{k_1}{m}. \tag{2}$$

Nonlinearities in equation (1) arise from the excitation (term with r_3) and from the property of the system (term with k_3).

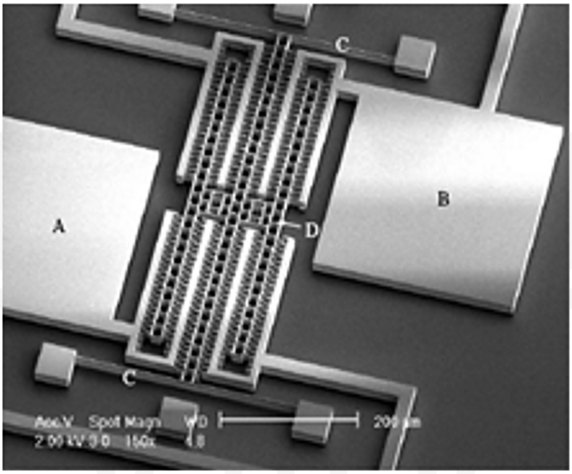

Fig. 1. Parametrically excited MEMS oscillator – electron microscope image [1]

Equation (1) is written in the form (cf. [5, 6])

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \left[\frac{k_1}{m} + \frac{r_1}{m}V_A^2(1+\cos\theta t)\right]x + \left[\frac{k_3}{m} + \frac{r_3}{m}V_A^2(1+\cos\theta t)\right]x^3 = 0. \tag{3}$$

Next we consider a microbeam presented in Fig. 2, [4]. The microbeam of length l is actuated by three capacitors. Two ends of the microbeam are fixed. Equation of motion of the microbeam which is treated as Bernoulli beam is given in [4] equation (1). Limiting considerations to the first vibration mode of the microbeam one can prove (cf. [5] eq. (3), (7) and [4] eq.(21)) that the identical equation as (3) describes deflection of the center of the microbeam.

In agreement with considerations of V.V. Bolotin [7], one can add to equation (3) an additional nonlinear term – the so-called nonlinear inertia term. The nonlinear inertia term has the following form (cf. Appendix)

$$2\kappa(x^2\ddot{x} + x\dot{x}^2), \tag{4}$$

where κ is a coefficient of nonlinear inertia; for two-articulated joint beam $2\kappa = \frac{\pi^4}{2l^2}\left(\frac{1}{3} - \frac{5}{8\pi^2}\right)$, for fixed-fixed beam $2\kappa = \frac{8\pi^4}{9l^2}\left(\frac{2}{3} + \frac{5}{16\pi^2}\right)$.

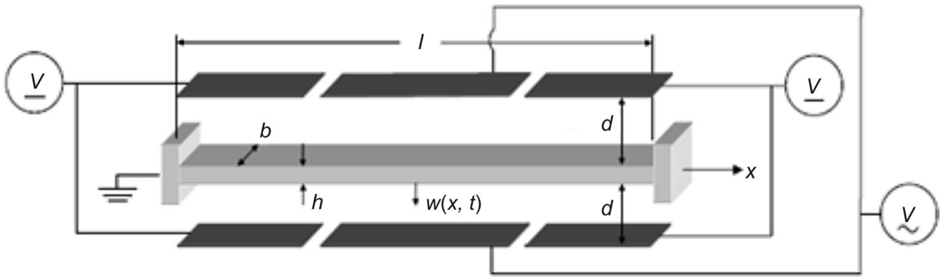

Fig. 2. Parametrically excited microbeam actuated by electric field [4]. The letter x in this figure denotes a coordinate of the cross-section of the beam

Therefore the initial equation of motion (3) is replaced with the equation

$$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\left[\frac{k_1}{m}+\frac{r_1}{m}V_A^2(1+\cos\theta t)\right]x+\left[\frac{k_3}{m}+\frac{r_3}{m}V_A^2(1+\cos\theta t)\right]x^3+2\kappa(x^2\ddot{x}+x\dot{x}^2)=0. \quad (5)$$

Introducing the nondimensional time $\tau=\frac{\theta}{2}t$ we get the following form of equation with nonlinear inertia term

$$\frac{d^2x}{d\tau^2}+\alpha\frac{dx}{d\tau}+(\beta+2\delta\cos 2\tau)x+(\delta_3+\delta_3'\cos 2\tau)x^3+2\kappa\left[x^2\frac{d^2x}{d\tau^2}+x\left(\frac{dx}{d\tau}\right)^2\right]=0, \quad (6)$$

where (cf. [2, 3])

$$\alpha=\frac{2c}{\theta m},\ \beta=\left(\omega_0^2+\frac{r_1}{m}V_A^2\right)\frac{4}{\theta^2},\ 2\delta=\frac{r_1}{m}V_A^2\frac{4}{\theta^2},\ \delta_3=\left(\frac{k_3}{m}+\frac{r_3}{m}V_A^2\right)\frac{4}{\theta^2},\ \delta_3'=\frac{4}{\theta^2}\frac{r_3}{m}V_A^2. \quad (7)$$

3. Solution method, resonance curves

We look for vibration amplitudes in the steady state of the main parametric resonance on the base of Floquet theorem – at the boundaries of stable and unstable solutions, the solutions are periodic. Then solutions may be represented with Fourier series. If we confine ourselves to the first term of the series, we get

$$x=a\sin\tau+b\cos\tau. \quad (8)$$

We employ the harmonic balance method equating the coefficients at sin τ and cos τ to zero and neglecting higher harmonics. Finally we get

$$\begin{gathered}\left[-1+\beta-\delta+\left(\frac{3}{4}\delta_3-\kappa\right)A^2\right]a-\alpha b-\frac{1}{2}\delta_3'a^3=0,\\ \alpha a+\left[-1+\beta+\delta+\left(\frac{3}{4}\delta_3-\kappa\right)A^2\right]b+\frac{1}{2}\delta_3'b^3=0,\end{gathered} \quad (9)$$

where the square of vibration amplitude $A^2 = a^2 + b^2$. It is a system of two algebraic nonlinear equations of the third order for unknown a and b. We look for non-zero solutions of (9) ($a \neq 0$, $b \neq 0$) because only in this case $A \neq 0$.

To solve the system of equations (9) we put $b = az$, where z is an unknown, we get

$$a^2 = \frac{1-\beta+\delta+\alpha z}{\left(\frac{3\delta_3}{4}-\kappa\right)(1+z^2)-\delta_3'/2} \tag{10}$$

and next for z we get the following algebraic equation of the fourth order

$$a_4 z^4 + [a_3 + a_2(1-\beta+\delta)]z^3 + 2\alpha a_1 z^2 + [a_3 + a_2(1-\beta-\delta)]z + \alpha(a_1 - a_2) = 0, \tag{11}$$

where

$$a_1 = \frac{3}{4}\delta_3 - \kappa, \quad a_2 = \frac{1}{2}\delta_3', \quad a_3 = 2\delta a_1, \quad a_4 = \alpha(a_1 + a_2). \tag{12}$$

The Ferrari method was used to solve this equation. Only real, not equal to zero roots of this equation are important, which inserted into (10) give $a^2 > 0$. If this requirement is fulfilled the amplitude of vibration is calculated as

$$A = |\,a\,|\sqrt{1+z^2}. \tag{13}$$

Results of numerical calculations are presented in the Fig. 3. The resonance curve, without taking into account the nonlinear inertia, is drawn by a dotted line. The resonance curve, taking into account the nonlinear inertia, is drawn by a dashed line. The difference is visible. The values of amplitudes are different but the frequency of transition between zero solution region to nonzero solutions region are the same.

4. Regions of unstable solutions

Inserting the solution (8) to equation (6) and neglecting nonlinear terms one obtains (cf. [7, 8]) the formulae for the boundaries of the first, more dangerous region of dynamic instability in the form

$$\frac{\theta}{2\Omega} = \sqrt{1 \pm \sqrt{\mu^2 - \mu^{*2}}}, \tag{14}$$

where μ^* is the critical value of parameter of excitation, cf. [7, 8]

$$\mu^* = \frac{c}{\Omega m}, \quad \Omega = \omega_0\sqrt{1-\frac{P_t}{P_*}} = \omega_0\sqrt{1+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}}. \tag{15}$$

Therefore

$$\frac{\theta}{2\Omega} = \sqrt{1 \pm \sqrt{\left(\frac{r_1 V_A^2}{2\Omega^2 m}\right)^2 - \left(\frac{c}{\Omega m}\right)^2}}$$

or (16)

$$\frac{\theta}{2\omega_0}=\sqrt{1+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}}\sqrt{1\pm\sqrt{\left(\frac{r_1V_A^2}{2\Omega^2m}\right)^2-\left(\frac{c}{\Omega m}\right)^2}}=\sqrt{1+\frac{r_1V_A^2}{k_1^2}\pm\sqrt{\frac{(r_1V_A^2)^2}{4k_1^2}-\frac{c^2}{k_1m}\left(1+\frac{r_1V_A^2}{k_1}\right)}}.$$

In the papers [1] and [5] the boundaries of the first instability region are obtained on the ground of the perturbation method and given by the formulae

$$f_1=\frac{\omega_0}{2\pi}\left[2+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}-\sqrt{\left(\frac{r_1V_A^2}{2\omega_0^2m}\right)^2-\frac{c^2}{m^2\omega_0^2}}\right],$$
$$f_2=\frac{\omega_0}{2\pi}\left[2+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}+\sqrt{\left(\frac{r_1V_A^2}{2\omega_0^2m}\right)^2-\frac{c^2}{m^2\omega_0^2}}\right], \tag{17}$$

where f_1 and f_2 are frequencies.

The results obtained by these two different methods are presented on the graphs in Fig. 4. The regions of instability are almost identical and the value of critical amplitude of voltage for which the unstable solutions occur are the same.

5. Conclusions

In cited papers the resonance curves are given for two special cases: 1. for nonlinear elasticity without damping or 2. for damping without nonlinear elasticity. In this paper the viscous damping together with two types of nonlinearities: the nonlinear elasticity and nonlinear inertia are taken into consideration. The nonlinear inertia term was introduced by analogy to considerations connected with beam. The solution for amplitude of vibrations is obtained in half-analytic form. The value of nonlinear inertia coefficient κ has an effect on the value of vibration amplitude.

6. Appendix

We quote the considerations of V.V. Bolotin [7] which concern two articulated-joint beams excited by axial force of the form $P(t) = P_0 + P_t \cos\theta t$. Limiting considerations to the first vibration mode of the beam, the time dependence of deflection $x(t)$ of the beam is described by the Mathieu equation

$$\frac{d^2x}{dt^2}+2\varepsilon\frac{dx}{dt}+\omega_0^2\left(1-\frac{P_0+P_t\cos\theta t}{P_*}\right)x=0 \tag{18}$$

or

$$\frac{d^2x}{dt^2}+2\varepsilon\frac{dx}{dt}+\Omega^2(1-2\mu\cos\theta t)x=0, \tag{19}$$

where $\quad \varepsilon = \frac{c}{2m},\ \omega_0^2 = \frac{\pi^4}{l^3}\frac{EI}{m} \equiv \frac{k_1}{m},\ P_* = \frac{\pi^2}{l^2}EI = \frac{k_1}{\pi^2}l,\ \Omega = \omega_0\sqrt{1-\frac{P_t}{P_*}},\ \mu = \frac{P_t}{2(P_* - P_0)};$

ω_0 is natural circular frequency of the beam, P_* is the value of the first Euler critical force for this beam. The replacing elastic constant k_1 is introduced which models the beam by a mass m on the spring with elastic constant $k_1 = \frac{\pi^4}{l^3}EI$.

Comparing (18) and (19) with equation (3) one gets

$$P_0 = P_t = -\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}P_*, \quad \mu = -\frac{r_1 V_A^2}{2\Omega^2 m'}, \quad \Omega = \sqrt{\frac{k_1}{m} + \frac{r_1}{m}V_A^2}. \tag{20}$$

According to [7] one can add to equation (19) a term of nonlinear inertia (nonlinearity of geometric nature). An inertia force connected with longitudinal displacement u of concentrated mass M_L has the form: $-M_L\ddot{u}$ and modify the longitudinal force in eq. (18). The longitudinal force is now $P(t) = P_0 + P_t\cos\theta t - M_L\ddot{u}$, where $u = \frac{1}{2}\int_0^l\left(\frac{\partial w}{\partial z}\right)^2 dz$.

Confining ourselves to the first vibration mode one gets $\ddot{u} = \frac{\pi^2}{2l}(x\ddot{x} + \dot{x})$, where $x(t)$ is the time part of transverse displacement of the beam.

Equation (19), including (20), takes the following form

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \left[\frac{k_1}{m} + \frac{r_1}{m}V_A^2(1+\cos\theta t) + \frac{k_1}{mP_*}M_L\ddot{u}\right]x = 0. \tag{21}$$

The microbeam considered in the paper [4] has no concentrated mass on its end. Nonlinear inertia is connected only with distributed mass of the beam. In [7] is demonstrated that taking into account the distributed mass is equivalent to adding the concentrated mass $M_L = \left(\frac{1}{3} - \frac{5}{8\pi^2}\right)m$, where m is the mass of the beam. Finally, the inertia term is written in the form (4).

It seems that the inertia forces cannot be neglected because of large deflections which appear in MEMS and NEMS which are taken into consideration.

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