UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.12.003.1121 FASCICULUS L, 2012

ON COMPUTATION OF SKEW-SYMMETRIC GENERATOR FOR AN ORTHOGONAL MATRIX

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Abstract. In this paper, we constructively prove that for any matrix A over a field of characteristic 0 and its eigenvalue $\lambda \neq 0$ there exists a diagonal matrix D with diagonal coefficients ± 1 such that DA has no eigenvalue λ. Hence and by the canonical result on Cayley transformation, for each orthogonal matrix U one can find a diagonal matrix D and a skew-symmetric matrix S such that $U = D(S-I)^{-1}(S+I)$.

1. Introduction. Let S be a skew-symmetric matrix of dimension n over the field of real numbers and let I be the identity matrix of the same dimension. It is known that the Cayley transformation $U = (S-I)^{-1}(S+I)$ transforms a skew-symmetric matrix S into an orthogonal matrix U which has no eigenvalue equal to 1. Since there exist orthogonal matrices which have 1 in their spectrum, the Cayley transformation is not a bijection between the set of all skew-symmetric matrices and the set of all orthogonal matrices. Thus this is not a way to generate all orthogonal matrices. A. Osborne and H. Liebeck proved that for any orthogonal matrix U there exists a diagonal matrix D with diagonal coefficients ± 1 such that DU does not have 1 as an eigenvalue (it follows directly from Lemma in [**1**]).

In this paper we present an algorithm which, for a given orthogonal matrix U, enables a construction of a diagonal matrix D with coefficients on diagonal ± 1 such that DU has no eigenvalue 1. Hence, via Cayley transformation one can find a skew-symmetric matrix S such that $U = D(S-I)^{-1}(S+I)$. Of course, if we find an appropriate matrix D then $S = (DU-I)^{-1}(DU+I)$. Therefore, to compute D is the most difficult part of the task. From Lemma

2010 *Mathematics Subject Classification.* 15B10.

Key words and phrases. Orthogonal matrix, Cayley transformation, characteristic polynomial.

in [**1**] it easily follows that we are always able to find a matrix D by checking all $2^n - 1$ matrices whose diagonal entries are equal to ± 1 (we count without identity matrix). We provide some quicker method based on Theorem 1, which is a modified version of the Lemma.

2. Main theorem. Let A be a square matrix of dimension n over a field of characteristic 0, and λ its eigenvalue. We denote by $k_a(A, \lambda)$ the algebraic multiplicity of λ. We always write D for a diagonal matrix with diagonal coefficients ± 1. We denote by D_i the diagonal matrix where the i-th element on the diagonal is equal to -1 and the others are equal to 1.

THEOREM 1. *If $\lambda \neq 0$ and $k_a(A, \lambda) = k$, then there exists a diagonal matrix D_i such that $k_a(D_i A, \lambda) < k$.*

To prove Theorem 1 we need the following description of characteristic polynomials. Consider the characteristic polynomial of A given by $W(t) = \det(tI - A)$, where I is the identity matrix and t is a variable. It is well known (see Theorem 7.1.2 in [**2**]) that

$$W(t) = t^n + \sum_{k=1}^{n} (-1)^k M_k t^{n-k}, \tag{$\star$}$$

where M_k is the sum of all $(k) \times (k)$ minors obtained by crossing out $n - k$ columns and $n - k$ rows of the same indices. In particular, $M_1 = \operatorname{tr}(A)$ and $M_n = \det(A)$. This form of characteristic polynomial will be crucial in our proof.

PROOF OF THEOREM 1. For the contrary, let us suppose that there exists a matrix A with an eigenvalue $\lambda \neq 0$ such that $k_a(D_i A, \lambda) \geq k := k_a(A, \lambda)$ for all D_i, $i = 1, \dots, n$. Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{ni} & \dots & a_{nn} \end{bmatrix}.$$

Then

$$D_i A = \begin{bmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{i1} & \dots & -a_{ii} & \dots & -a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{ni} & \dots & a_{nn} \end{bmatrix}.$$

Let $W_i(t) = \det(tI - D_iA)$ be the characteristic polynomial of D_iA. Since the coefficients in the i-th row of D_iA have signs opposite to the coefficients in the i-th row of A, every minor which occurs in the sum M_k and is not obtained by crossing out the i-th row and column, changes its sign. Hence and by ($\star$), for $i = 1, \ldots, n$, we get

$$W_i(t) = W(t) + 2[a_{ii}t^{n-1} - M_2^i t^{n-2} + \ldots (-1)^{k+1} M_k^i t^{n-k} + \ldots$$
$$+(-1)^{n+1}\det(A)] = W(t) + 2\sum_{k=1}^{n}(-1)^{k+1}M_k^i t^{n-k},$$

where M_k^i is the sum of all minors from M_k that contain elements of the i-th row of the matrix A. It is obvious that each minor from M_k is a summand of exactly k of the sums M_k^i. We observe that

$$tW'(t) - nW(t) = \sum_{k=1}^{n}(-1)^{k+1}kM_k t^{n-k} = \sum_{k=1}^{n}\sum_{i=1}^{n}(-1)^{k+1}M_k^i t^{n-k}.$$

Putting

$$F(t) = \sum_{i=1}^{n} W_i(t),$$

we thus get

$$F(t) = nW(t) + 2\sum_{k=1}^{n}\sum_{i=1}^{n}(-1)^{k+1}M_k^i t^{n-k} = nW(t) + 2[tW'(t) - nW(t)]$$
$$= -nW(t) + 2tW'(t).$$

We assumed that λ is a root of multiplicity $\geq k$ of each polynomial W_i, whence λ is a root of $F(t)$ of multiplicity at least k. Consequently

$$W(t) = Q(t)(t-\lambda)^k$$

with $Q(\lambda) \neq 0$. Hence $W'(t) = Q'(t)(t-\lambda)^k + kQ(t)(t-\lambda)^{k-1}$ and thus

$$F(t) = (t-\lambda)^{k-1}[-nQ(t)(t-\lambda) + 2tQ'(t)(t-\lambda) + 2tkQ(t)].$$

Put

$$S(t) := -nQ(t)(t-\lambda) + 2tQ'(t)(t-\lambda) + 2tkQ(t).$$

Since $S(\lambda) = 2\lambda kQ(\lambda) \neq 0$, $S(t)$ is not divisible by $t - \lambda$, whence $F(t)$ is not divisible by $(t-\lambda)^k$, and thus λ is a root of $F(t)$ of multiplicity $< k$. This contradiction completes the proof. $\square$

3. Computation of a skew-symmetric generator for an orthogonal matrix. Let U be a matrix of dimension n over a field of characteristic 0, and let $\lambda \neq 0$ be an eigenvalue of U. Now we aim to find a diagonal matrix D such that DU does not have an eigenvalue λ. We know from Theorem 1 that there exist a diagonal matrix D_{i_1} such that $k_a(D_{i_1}U,\lambda) < k_a(U,\lambda)$. To find D_{i_1} we have to check at most n matrices. If $k_a(D_{i_1}U,\lambda) \geqslant 1$, we repeat the reasoning with U replaced by $D_{i_1}U$. In the j-th step we check $n-1$ matrices of the type D_i. If we encounter the least optimistic situation when multiplicity is decreasing by 1 in each step, we have to check $n+(k-1)(n-1) = k(n-1)+1$ matrices of the type D_i with $k = k_a(U,\lambda)$. Since $k \leq n$, we have to check at most $k(n-1)+1 \leq n^2-n+1$. Let $l \leq k$ be the number of steps in the procedure. Then $D = D_{i_1}D_{i_2}\dots D_{i_l}$ is the diagonal matrix looked for.

This procedure seems to be more effective than the method based on the original Lemma from [**1**], where we have to check at most 2^n-1 diagonal matrices.

Assume now that U is an orthogonal matrix over the field of real numbers. If we apply the algorithm described above to U and an eigenvalue 1, we can find a diagonal matrix D such that $k_a(DU,1) = 0$. Then we can use the Cayley transformation to get a skew-symmetric generator for U, i.e. a skew-symmetric matrix S such that $U = D(S-I)^{-1}(S+I)$. We provide an example which illustrates how this algorithm works.

Example 1. Let

$$A := \frac{1}{2601}\begin{bmatrix} -1951 & 192 & -568 & -756 & -1424 \\ 192 & 801 & -744 & -2016 & 1212 \\ -568 & -744 & 2201 & -972 & 316 \\ -756 & -2016 & -972 & 135 & 1080 \\ -1424 & 1212 & 316 & 1080 & 1415 \end{bmatrix}.$$

Then $\operatorname{Spec}(A) = \{-1,1\}$ and $k_a(A,1) = 3$. We obtain $\operatorname{Spec}(D_1A) = \big\{-1,1, \frac{1951-20\sqrt{7397}i}{2601}, \frac{1951+20\sqrt{7397}i}{2601}\big\}$ and $k_a(D_1A,1) = 2$. Repeating the procedure for D_1A, we get $k_a(D_2D_1A,1) = 1$. In the next step, we check that $1 \notin \operatorname{Spec}(D_3D_2D_1A)$. Then the Cayley transformation of $D_3D_2D_1A$ gives us $S = (D_3D_2D_1A-I)^{-1}(D_3D_2D_1A+I)$, which is a skew-symmetric generator of A. Finally, we get

$$S = \frac{1}{13}\begin{bmatrix} 0 & 0 & 0 & -18 & -32 \\ 0 & 0 & 0 & -8 & 6 \\ 0 & 0 & 0 & -6 & -2 \\ 18 & 8 & 6 & 0 & 0 \\ 32 & -6 & 2 & 0 & 0 \end{bmatrix}.$$

4. Open problem. It is easy to check that the argument used in the proof of Theorem 1 will not work over a field of positive characteristic p for an eigenvalue of algebraic multiplicity divisible by p. Some basic calculations show that the theorem is valid for matrices of dimension 3 and 4 over a field of characteristic $p = 3$. However, in other cases we formulate it as a conjecture:

CONJECTURE 1. *Let A be a matrix over a field of characteristic $p \neq 2$ and $\lambda \neq 0$ its eigenvalue. Then there exists a diagonal matrix D_i such that $k_a(D_iA, \lambda) < k_a(A, \lambda)$.*

References

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Received July 5, 2012

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