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SOME REMARKS ON NON-SEPARABLE GAPS IN $P(\omega)/FIN$

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Abstract

The Hausdorff gap is the well known example of a non-separable, increasingly ordered gap in $P(\omega)/fin$. In this paper new construction of a non-separable gap in $P(\omega)/fin$ is presented.

Keywords: Boolean algebra, Cech-Stone compactification, gap

Streszczenie

W artykule została przedstawiona nowa konstrukcja nierozdzielalnej luki w $P(\omega)/fin$.

Słowa kluczowe: algebra Boole'a, uzwarcenie Cecha-Stone'a, luka

The author is responsible for the language in all paper.

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Boolean algebra $P(\omega)/fin$ plays an important role in the foundations of mathematics. Many mathematical problems can be reduced to questions on properties of $P(\omega)/fin$. Notion, which is frequently used in investigation concerning $P(\omega)/fin$ is the notion of gap (cf. [1], [4]).

Let us begin by reviewing some basic facts and definitions. By ω the set of all natural numbers is denoted. The symbol *fin* stands for the ideal of all finite subsets of ω. The ideal determines the following equivalence relation:

$$\text{For } A, B \subseteq \omega,\ A =_* B \text{ if and only if } A \div B \in \textit{fin}.$$

$P(\omega)/fin$ is its factor algebra. An order in $P(\omega)/fin$ is defined as usual, namely:

$$A \subseteq_* B \ \text{ iff } A \backslash B \in \textit{fin}.$$

Let λ, κ be cardinals. A gap of type $\lambda, \kappa)$ in the $P(\omega)/fin$ is a pair:

$$(\{A_\gamma: \gamma < \lambda\}, \{B_\beta: \beta < \kappa\})$$

of subsets of $P(\omega)/fin$ such that $A_\gamma \cap B_\beta =_* \varnothing$. If for every $\gamma_1 < \gamma_2 < \lambda$, $\beta_1 < \beta_2 < \kappa$ $A_{\gamma 1} \subseteq_* A_{\gamma 2}$ and $B_{\beta 1} \subseteq_* B_{\beta 2}$, the gap is said to be increasingly ordered. An element $C \subseteq \omega$ **fills** (separates) the gap if $A_\gamma \subseteq_* C$ and $B_\beta \cap C =_* \varnothing$ for every $\gamma < \lambda$, $\beta < \kappa$. If there is no such an element, the gap is called non-separable. One can ask gaps of what type exist in $P(\omega)/fin$.

A research concerning gaps in $(\omega)/fin$ is an important and deep line of investigation. Let us recall basic facts. It is easily proved that there are no non-separable gaps of type (ω, ω). On the other hand Hausdorff constructed a non-separable gap of type (ω_1, ω_1) (cf. [2]). This gap, say $\mathcal{L} = (\{X_\alpha: \alpha < \omega_1\}, \{Y_\beta: \beta < \omega_1\})$, is increasingly ordered and $\{\gamma < \beta: \max X_\gamma \cap Y_\beta < k\}$ is finite for every $\beta < \omega_1$ and $k \in \omega$.

Under CH (the Continuum Hypothesis), there exist only gaps of type (ω_1, ω_1). If $2^\omega > \omega_1$ and MA (the Martin Axiom) holds the each increasingly ordered gap f type $\lambda, \kappa)$ with $\lambda, \kappa < 2^\omega$, $\lambda \neq \omega_1$ or $\kappa \neq \omega_1$ is separated ([5]).

The smallest cardinal number for which there exists a non-separable gap in $P(\omega)/fin$ is the bounding number $\mathbf{b}$ (cf. [6]). Remind that $\mathbf{b}$ is the size of the smallest unbounded family in ω^ω equipped with the following order: for $f, g \in \omega^\omega$, $f \leq_* g$ iff $\{n: f(n) > g(n)\} \in \textit{fin}$.

We present another construction of an (unordered) gap of type $(2^\omega, 2^\omega)$.
The set F consist of all finite sequences $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_n)$ such that:

$$\varepsilon_0 = 0,\ \varepsilon_{2n+1} = 2 \text{ and } \varepsilon_{2n+2} \in \{0, 1\},\ n \in \omega.$$

Let

$$F_n = \{\underline{\varepsilon} \in F: \ell(\underline{\varepsilon}) \leq 2n\}$$

and

$$F_e = \{\underline{\varepsilon} \in F: \ell(\underline{\varepsilon}) = 2n \text{ for some } n \in \omega\}.$$

Divide ω into two disjoint, infinite subsets X and Y and fix two functions f and g such that:

$$\textbf{(*)}\ f: F \to X \text{ is a bijection and if } \underline{\varepsilon} \subseteq \underline{\rho} \text{ then } f(\underline{\varepsilon}) \leq f(\underline{\rho}).$$

$$\textbf{(**)}\ g: F_e \times F_e \to Y \text{ is an injection and if } \underline{\varepsilon}^1 \subseteq \underline{\rho}^1, \underline{\varepsilon}^2 \subseteq \underline{\rho}^2 \text{ then}$$

$$g(\underline{\varepsilon}^1,\underline{\varepsilon}^2) \le g(\underline{\rho}^1, \underline{\rho}^2).$$

We define two families of finite subsets $\{A(\underline{\varepsilon}): \underline{\varepsilon} \in \mathcal{F}\}$, $\{B(\underline{\varepsilon}): \underline{\varepsilon} \in \mathcal{F}\}$ by induction on the lenght of $\underline{\varepsilon}$.

For $\underline{\varepsilon}$ such that $\ell(\underline{\varepsilon}) = 1$ or $\ell(\underline{\varepsilon}) = 2$ put $A(\underline{\varepsilon}) = B(\underline{\varepsilon}) = \emptyset$.

Assume that $\ell(\underline{\varepsilon}) = 3,4$. Then:

$$A_{(0,2,0)} = \{f((0)), f((0,2)), f((0,2,0))\},\ A_{(0,2,1)} = \{f((0)), f((0,2)), f((0,2,1))\},$$

$$B_{(0,2,0)} = \{f((0,2,1))\},\ B_{(0,2,1)} = \{f((0,2,0))\},$$

$$A_{(0,2,0,2)} = A_{(0,2,0)} \cup \{f((0,2,0,2))\},\ A_{(0,2,1,2)} = A_{(0,2,1)} \cup \{f((0,2,1,2))\}$$

$$B_{(0,2,0,2)} = B_{(0,2,0)} \cup \{g((0,2,1,2), (0,2,0,2))\},\ B_{(0,2,1,2)} = B_{(0,2,1)} \cup \{g((0,2,1,2), (0,2,0,2))\}$$

Assume inductively that for $n \ge 2$, we have defined families $\{A(\underline{\varepsilon}): \underline{\varepsilon} \in \mathcal{F}_n\}$ and $\{B(\underline{\varepsilon}): \underline{\varepsilon} \in \mathcal{F}_n\}$ satisfying the following conditions:

1. $A(\underline{\varepsilon}) \cap B(\underline{\varepsilon}) = \emptyset$ for every $\underline{\varepsilon} \in \mathcal{F}_n$.
2. If $\underline{\varepsilon}, \underline{\rho} \in \mathcal{F}_n$ and $\underline{\varepsilon}(k) \ne \underline{\rho}(k)$, for some $k \le 2n$, then

$$A(\underline{\varepsilon}) \cap B(\underline{\rho}) \ne \emptyset \text{ and } B(\underline{\varepsilon}) \cap A(\underline{\rho}) \ne \emptyset.$$

3. If $\underline{\varepsilon}, \underline{\rho} \in \mathcal{F}_n$ and $\underline{\varepsilon} \subseteq \underline{\rho}$, then $A(\underline{\varepsilon}) \subseteq A(\underline{\rho})$ and $B(\underline{\varepsilon}) \subseteq B(\underline{\rho})$.
4. If $\underline{\varepsilon}, \underline{\rho} \in \mathcal{F}_n$ and $\underline{\varepsilon}(k) \ne \underline{\rho}(k)$, let $\ell = \min\{k: \underline{\varepsilon}(k) \ne \underline{\rho}(k)\}$. Then $\max A(\underline{\varepsilon}) \cap B(\underline{\rho}) = f(\underline{\varepsilon}|_{\ell})$, $\max B(\underline{\varepsilon}) \cap B(\underline{\rho}) = f(\underline{\varepsilon}|_{\ell-1})$, $\max A(\underline{\varepsilon}) \cap A(\underline{\rho}) = f(\underline{\varepsilon}|_{\ell-1})$.

For $\underline{\varepsilon} \in \mathcal{F}_n$ put:

$$A(\underline{\varepsilon} \wedge 0) = A(\underline{\varepsilon}) \cup \{f(\underline{\varepsilon} \wedge 0)\},\ A(\underline{\varepsilon} \wedge 1) = A(\underline{\varepsilon}) \cup \{f(\underline{\varepsilon} \wedge 1)\},$$

$$B(\underline{\varepsilon} \wedge 0) = B(\underline{\varepsilon}) \cup \{f(\underline{\varepsilon} \wedge 1)\},\ B(\underline{\varepsilon} \wedge 1) = B(\underline{\varepsilon}) \cup \{f(\underline{\varepsilon} \wedge 1)\},$$

$$A(\underline{\varepsilon} \wedge 02) = A(\underline{\varepsilon} \wedge 0) \cup \{f(\underline{\varepsilon} \wedge 02)\},\ A(\underline{\varepsilon} \wedge 12) = A(\underline{\varepsilon} \wedge 1) \cup \{f(\underline{\varepsilon} \wedge 12)\},$$

$$B(\underline{\varepsilon} \wedge 02) = B(\underline{\varepsilon} \wedge 0) \cup \{g(\underline{\varepsilon} \wedge 02, \underline{\varepsilon} \wedge 12)\},\ B(\underline{\varepsilon} \wedge 12) = B(\underline{\varepsilon} \wedge 1) \cup \{g(\underline{\varepsilon} \wedge 02, \underline{\varepsilon} \wedge 12)\}.$$

It is obvious that the family $\mathcal{F}_{n+1}$ satisfies conditions (1) and (3).

For (2), let $\underline{\rho}, \underline{\varepsilon} \in \mathcal{F}_{n+1}$. If $\ell(\underline{\rho}) = \ell(\underline{\varepsilon})$ or $\ell(\underline{\rho}) = 2n + 1$, $\ell(\underline{\varepsilon}) = 2n + 2$, the condition follows from the definition. Suppose that $\ell(\underline{\rho}) = k < 2n + 1 \le \ell(\underline{\varepsilon})$. Let $\boldsymbol{l} = \min\{k: \underline{\varepsilon}_k \ne \underline{\rho}_k\}$. Then $\underline{\sigma} = \underline{\varepsilon}|\boldsymbol{l} = \underline{\rho}|\boldsymbol{l}$ and $\emptyset \ne A(\underline{\sigma})\wedge\underline{\rho}_l) \cap B(\underline{\sigma} \wedge \underline{\varepsilon}_l) = A(\underline{\rho}) \cap B(\underline{\varepsilon})$. (The remaining cases can be checked in the same way.)

To check the assumption (4), note that $A(\underline{\varepsilon} \wedge i) \cap B(\underline{\varepsilon} \wedge j) = A(\underline{\varepsilon} \wedge i2) \cap B(\underline{\varepsilon} \wedge j2)$, for $i, j \in \{0, 1\}$, $i \ne j$. Since f satisfies the condition (*), it follows that $\max A(\underline{\varepsilon} \wedge i) \cap B(\underline{\varepsilon} \wedge j) = f(\underline{\varepsilon} \wedge i)$. Moreover $A(\underline{\varepsilon} \wedge i) \cap A(\underline{\varepsilon} \wedge j) = A(\underline{\varepsilon} \wedge i2) \cap A(\underline{\varepsilon} \wedge j2) = A(\underline{\varepsilon})$, thus

$$\max A(\underline{\varepsilon} \wedge i) \cap A(\underline{\varepsilon} \wedge j) = f(\underline{\varepsilon}).$$

If $\ell(\underline{\rho}) = k < 2n + 1 \le \ell(\underline{\varepsilon})$ and $\underline{\sigma} = \underline{\varepsilon}|\boldsymbol{l} = \underline{\rho}|\boldsymbol{l}$, $\underline{\rho}_l \ne \underline{\varepsilon}_l$ then

$$\max A(\underline{\sigma} \wedge \rho_l) \cap B(\underline{\sigma} \wedge \varepsilon_l) = \max A(\underline{\rho}) \cap B(\underline{\varepsilon}) = f(\underline{\rho}|\boldsymbol{l}).$$

(The remaining cases can be checked in the same way.) This finishes the inductive construction.

Let X be the family of all sequences x: $\omega \to \{0, 1, 2\}$ which satisfy the conditions:

$$x(0) = 0, x(2n + 1) = 2, x(2n + 2) \in \{0, 1\}.$$

Then

$$\underline{A}(x) = \bigcup_{n\in\omega} A(x|n), \underline{B}(x) = \bigcup_{n\in\omega} B(x|n)$$

are infinite subsets of ω.

It is easy to check that for $x, y \in X$, if $x \neq y$ then:

$$\underline{A}(x) \cap \underline{B}(x) = \emptyset, \underline{A}(x) \cap \underline{B}(y) \neq \emptyset, \underline{A}(x) \cap \underline{A}(y) \in \textbf{\textit{fin}} \text{ and } \underline{B}(x) \cap \underline{B}(y) \in \textbf{\textit{fin}}.$$

Theorem 1 *The gap* $\mathcal{L} = (\{\underline{A}(x): x \in X\}, \{\underline{B}(x): x \in X\})$ *satisfies the following condition: for every uncountable set* $Y \subseteq X$, $\mathcal{L}_Y = (\{\underline{A}(x): x \in Y\}, \{\underline{B}(x): x \in Y\})$ *is non-separable.*

Proof. Suppose that for $Y = \{x_\alpha : \alpha < \kappa\} \subseteq X$, $\omega < \kappa \leq 2^\omega$, there exists a C which separeates the gap $\mathcal{L}_Y$. Let $s_\alpha = \underline{A}(x_\alpha)\backslash C$, $t_\alpha = \underline{B}(x_\alpha) \cap C$.

Then s_α, t_α are finite subsets of ω and since $\underline{A}(x_\alpha) \cap \underline{B}(x_\alpha) = \emptyset$, it follows that $s_\alpha \cap t_\alpha = \emptyset$. Δ-lemma implies that there exist an uncountable set $\Gamma \subseteq \kappa$ $\Gamma \subset \kappa$ and finite sets s, t such that for all $\alpha \in \gamma$, $s_\alpha = s$ and $t_\alpha = t$.

If $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$ then $\emptyset = s_\alpha \cap t_\beta = s \cap t = s_\alpha \cap t_\alpha = \emptyset$, a contradiction. This finishes the proof.

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