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ON BASIC PROPERTIES OF δ-PRIME AND δ-SEMIPRIME RINGS

O PODSTAWOWYCH WŁASNOŚCIACH PIERŚCIENI δ-PIERWSZYCH I δ-PÓŁPIERWSZYCH

A b s t r a c t

We provide a self-contained discussion of the notions of δ-primeness and δ-semiprimeness for associative rings, possibly without identity. Some of the facts and properties presented in the article seem less known and quite difficult to find in standard reference sources.

Keywords: associative ring, derivation, δ-ideal, δ-prime ring, δ-semiprime ring, δ-prime radical, δ-nilpotent element.

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S t r e s z c z e n i e

Praca jest „samowystarczalnym" omówieniem pojęć δ-pierwszości i δ-półpierwszości, rozważanych w algebrze nieprzemiennej. Wszystkie udowodnione w pracy twierdzenia stosują się i do pierścieni z jedynką, i do pierścieni bez jedynki. Część faktów i własności przedstawionych w pracy wydaje się mało znana i raczej trudna do odszukania w standardowej literaturze.

Słowa kluczowe: pierścień łączny, różniczkowanie, δ-ideał, pierścień δ-pierwszy, pierścień δ-półpierwszy, radykał δ-pierwszy, element δ-nilpotentny.

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1. Preliminaries and introduction

Throughout the article, R is an associative ring and $\delta : R \longrightarrow R$ is a derivation. We do not assume that R has an identity.

Definition 1.1. *A map $\delta : R \longrightarrow R$ is said to be a derivation, if it is additive and satisfies the Leibniz rule*

$$\forall\, a,b \in R:\ \delta(ab) = \delta(a)b + a\delta(b).$$

Notice that the zero map is a derivation of the ring R. We define

$$\delta^n = \begin{cases} \mathrm{id}_R, & \text{if } n = 0, \\ \underbrace{\delta \circ \ldots \circ \delta}_{n}, & \text{if } n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

The center of the ring R will be denoted by $\mathrm{Z}(R)$, i.e.,

$$\mathrm{Z}(R) = \{a \in R:\ ab = ba \text{ for all } b \in R\}.$$

Let us remark that $\mathrm{Z}(R)$ is a subring of R. For any elements $a,b \in R$ we define $[a,b] = ab - ba$. By "ideal of the ring R" we always mean a left, right, or two-sided ideal.

Prime rings and, more generally, semiprime rings are fundamental objects of study in noncommutative algebra. For a long time the research has also been focused on various extensions of these classes of rings. Taking into account the analogues of prime and semiprime rings defined by means of ideals that are invariant with respect to either a single derivation or a family of derivations, yields important examples of such extensions. The analogues are referred to as δ-(semi)prime rings and Δ-(semi)prime rings, respectively. They still attract interest of algebraists.

The article does not bring new results. Our first purpose is to collect and systematize basic facts about δ-prime rings and δ-semiprime rings. Some of these facts seem a bit less known. The second purpose is to provide complete and self-contained proofs for all the presented theorems (the proofs are very often omitted in reference sources). The proofs we provide are mostly modifications of corresponding "nondifferential" proofs given in the classical monographs [4, 6, 7]. Two features of the article seem worth emphasizing: all the proofs are valid for rings without identity and a brief introduction to δ-nilpotent elements is included.

The article is organized as follows. In Section 2 we collect some useful facts and examples concerning δ-ideals. In Section 3 we discuss various characterizations of δ-prime rings and δ-prime ideals. Section 4 is devoted to strongly nilpotent elements and δ-nilpotent elements. Finally, in Section 5 we deal with characterizations of δ-semiprime rings.

2. δ-ideals

We begin with a few standard definitions.

Definition 2.1. *A set $S \subseteq R$ is called δ-stable, if $\delta(S) \subseteq S$. An ideal I of the ring R is said to be a δ-ideal, if it is δ-stable.*

Definition 2.2. *The two-sided ideal of R generated by the set $\{[a,b] : a,b \in R\}$ is called the commutator ideal. This ideal is denoted by* $\mathrm{C}(R)$.

Definition 2.3. *For a set $S \subseteq R$ we define*

- *the left annihilator* $\mathrm{ann}_\ell(S) = \{a \in R : ab = 0 \text{ for all } b \in S\}$,
- *the right annihilator* $\mathrm{ann}_r(S) = \{a \in R : ba = 0 \text{ for all } b \in S\}$.

Notice that if δ is the zero derivation, then every ideal of the ring R is a δ-ideal. Moreover, $\mathrm{ann}_\ell(S)$ is a left ideal of R, $\mathrm{ann}_r(S)$ is a right ideal of R, and R is commutative if and only if $\mathrm{C}(R) = \{0\}$.

Before we turn to more interesting observations, let us state an obvious but useful formula.

Lemma 2.4. *If $a,b \in R$, then $\delta([a,b]) = [\delta(a),b] + [a,\delta(b)]$.*

Take now a closer look at $\mathrm{C}(R)$, $\mathrm{Z}(R)$ and annihilators.

Proposition 2.5. *The commutator ideal $\mathrm{C}(R)$ is a δ-ideal and the center $\mathrm{Z}(R)$ is a δ-stable set. Moreover, if $S \subseteq R$ is a δ-stable set, then $\mathrm{ann}_\ell(S)$ and $\mathrm{ann}_r(S)$ are δ-ideals.*

Proof. Let us first define $A = \{[a,b] : a,b \in R\}$, $B = \{x[a,b] : a,b,x \in R\}$, $C = \{[a,b]y : a,b,y \in R\}$, and $D = \{x[a,b]y : a,b,x,y \in R\}$. Then $\mathrm{C}(R)$ coincides with the totality of finite sums of elements belonging to the set $A \cup B \cup C \cup D$. Pick arbitrary $a,b,x,y \in R$. By Lemma 2.4, we have

$$\begin{aligned}
\delta([a,b]) &= [\delta(a),b] + [a,\delta(b)] \in \mathrm{C}(R),\\
\delta(x[a,b]) &= \delta(x)[a,b] + x[\delta(a),b] + x[a,\delta(b)] \in \mathrm{C}(R),\\
\delta([a,b]y) &= [\delta(a),b]y + [a,\delta(b)]y + [a,b]\delta(y) \in \mathrm{C}(R),\\
\delta(x[a,b]y) &= \delta(x)[a,b]y + x[\delta(a),b]y + x[a,\delta(b)]y + x[a,b]\delta(y) \in \mathrm{C}(R).
\end{aligned}$$

The δ-stability of $\mathrm{C}(R)$ follows.

Now, pick an arbitrary $a \in \mathrm{Z}(R)$ and an arbitrary $b \in R$. Then $[a,b] = 0 = [a,\delta(b)]$, and hence

$$0 = \delta([a,b]) = [\delta(a),b] + [a,\delta(b)] = [\delta(a),b].$$

Consequently, $\delta(a) \in \mathrm{Z}(R)$. The δ-stability of $\mathrm{Z}(R)$ follows.

Suppose, finally, that $S \subseteq R$ is a δ-stable set. Pick an arbitrary $a \in \mathrm{ann}_\ell(S)$ and an arbitrary $b \in S$. Then $ab = 0$ and $\delta(b) \in S$. Consequently,

$$0 = \delta(ab) = \delta(a)b + a\delta(b) = \delta(a)b.$$

This yields $\delta(a) \in \mathrm{ann}_\ell(S)$. The δ-stability of $\mathrm{ann}_r(S)$ can be proved analogously. □

The intersection of any family of two-sided δ-ideals of the ring R is also a two-sided δ-ideal. Obviously, the statement remains true, if we replace the word "two-sided" by "left" or "right". We are thus enabled to consider δ-ideals generated by subsets of R.

Let us define $\langle S\rangle^\delta$, $\langle S\rangle_\ell^\delta$ and $\langle S\rangle_r^\delta$ to be the two-sided, the left and the right δ-ideal of the ring R generated by a set $S \subseteq R$ (respectively). We will write as usual $\langle a\rangle^\delta$ instead of $\langle\{a\}\rangle^\delta$, and analogously for the left and the right δ-ideal generated by the singleton $\{a\}$.

Proposition 2.6. *Let $a \in R$. Define $A = \{k\delta^n(a) : k \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$, $B = \{x\delta^n(a) : x \in R, n \in \mathbb{N} \cup \{0\}\}$, $C = \{\delta^n(a)y : y \in R, n \in \mathbb{N} \cup \{0\}\}$, and $D = \{x\delta^n(a)y : x, y \in R, n \in \mathbb{N} \cup \{0\}\}$. Then*

(i) $\langle a \rangle^\delta$ *coincides with the totality of finite sums of elements belonging to the set $A \cup B \cup C \cup D$,*

(ii) $\langle a \rangle_\ell^\delta$ *coincides with the totality of finite sums of elements belonging to the set $A \cup B$,*

(iii) $\langle a \rangle_r^\delta$ *coincides with the totality of finite sums of elements belonging to the set $A \cup C$.*

Proof. Denote by T the totality of finite sums of elements belonging to $A \cup B \cup C \cup D$. Notice that T is an additive subgroup of the ring R. Moreover, T is a two-sided ideal and $a = \delta^0(a) \in T$. A reasoning similar to the proof of the δ-stability of $\mathrm{C}(R)$ shows that T is δ-stable. We therefore get $\langle a \rangle^\delta \subseteq T$. On the other hand, if $I \subseteq R$ is a two-sided δ-ideal and $a \in I$, then clearly $T \subseteq I$. The converse inclusion follows. Properties (ii) and (iii) can be proved analogously. $\square$

It seems worth noting that in the above proposition

$$B = \bigcup_{n=0}^{\infty} R\delta^n(a), \ C = \bigcup_{n=0}^{\infty} \delta^n(a)R, \ D = \bigcup_{n=0}^{\infty} R\delta^n(a)R.$$

We conclude the section with some remarks on products and sums of δ-ideals. Let $k \in \mathbb{N} \setminus \{0\}$ and $S_1, \ldots, S_k \subseteq R$. If either all the sets are two-sided ideals or all the sets are left ideals or all the sets are right ideals, then we define $S_1 \cdot \ldots \cdot S_k$ to be the additive subgroup of the ring R generated by the "elementwise product" $\{a_1 \cdot \ldots \cdot a_k : a_1 \in S_1, \ldots, a_k \in S_k\}$ (the usual product of ideals). Otherwise, we define $S_1 \cdot \ldots \cdot S_k$ to be just the elementwise product.

If all the sets $S_1, \ldots, S_k$ are two-sided δ-ideals, then $S_1 \cdot \ldots \cdot S_k$ is also a two-sided δ-ideal. Obviously, we can replace the word "two-sided" by "left" or "right". Hence any power of a δ-ideal is also a δ-ideal.

Notice, finally, that if $I, J \subseteq R$ are two-sided δ-ideals, then $I + J = \{a + b : a \in I, b \in J\}$ is also a two-sided δ-ideal. (We can replace "two-sided" by "left" or "right").

3. δ-prime rings and δ-prime ideals

We start with the following quite standard definition.

Definition 3.1. *The ring R is said to be δ-prime if it is nonzero and for any two-sided δ-ideals $I, J \subseteq R$ such that $IJ = \{0\}$, we have either $I = \{0\}$ or $J = \{0\}$.*

Notice that if δ is the zero derivation, then the δ-primeness is the same thing as the usual primeness of R (see, for instance, [6, Ch. 3]). Moreover, the ring R is prime if and only if it is d-prime for each derivation $d : R \longrightarrow R$. Let us now state and prove a fundamental characterization of δ-prime rings (cf. [1, Lemma 2]).

Theorem 3.2. *Suppose that R is a nonzero ring. The following conditions are equivalent:*

(i) R *is* δ*-prime,*

(ii) *for any elements* $a,b \in R$*, if* $\forall n \in \mathbb{N}\cup\{0\} : aR\delta^n(b) = \{0\}$*, then either* $a = 0$ *or* $b = 0$*,*

(iii) *for any elements* $a,b \in R$*, if* $\forall n \in \mathbb{N}\cup\{0\} : \delta^n(a)Rb = \{0\}$*, then either* $a = 0$ *or* $b = 0$*,*

(iv) *for any elements* $a,b \in R$*, if* $\langle a\rangle^\delta \langle b\rangle^\delta = \{0\}$*, then either* $a = 0$ *or* $b = 0$*,*

(v) *for any right* δ*-ideals* $I,J \subseteq R$*, if* $IJ = \{0\}$*, then either* $I = \{0\}$ *or* $J = \{0\}$*,*

(vi) *for any left* δ*-ideals* $I,J \subseteq R$*, if* $IJ = \{0\}$*, then either* $I = \{0\}$ *or* $J = \{0\}$*,*

(vii) *for an arbitrary nonzero right* δ*-ideal* $I \subseteq R$ *we have* $\mathrm{ann}_r(I) = \{0\}$*,*

(viii) *for an arbitrary nonzero left* δ*-ideal* $I \subseteq R$ *we have* $\mathrm{ann}_\ell(I) = \{0\}$*.*

Proof. Assume that R is δ-prime. Let $a,b \in R$ be such that

$$\forall n \in \mathbb{N}\cup\{0\} : aR\delta^n(b) = \{0\}. \tag{1}$$

Define I to be the totality of finite sums of elements of the set $\{c_1\delta^m(a)c_2 : c_1,c_2 \in R, m \in \mathbb{N}\cup\{0\}\}$. Furthermore, define J to be the totality of finite sums of elements of the set $\{h_1\delta^n(b)h_2 : h_1,h_2 \in R, n \in \mathbb{N}\cup\{0\}\}$. Then I and J are two-sided δ-ideals of the ring R.

Next, we will prove by induction on m that

$$\forall m,n \in \mathbb{N}\cup\{0\} : \delta^m(a)R\delta^n(b) = \{0\}.$$

If $m = 0$, then the assertion coincides with (1). Pick therefore some $k \in \mathbb{N}\cup\{0\}$ and suppose that

$$\forall n \in \mathbb{N}\cup\{0\} : \delta^k(a)R\delta^n(b) = \{0\}.$$

If $c \in R$ and $n \in \mathbb{N}\cup\{0\}$, then the induction hypothesis yields

$$0 = \delta(\delta^k(a)c\delta^n(b)) = \delta^{k+1}(a)c\delta^n(b) + \delta^k(a)\delta(c)\delta^n(b) + \delta^k(a)c\delta^{n+1}(b) =$$

$$= \delta^{k+1}(a)c\delta^n(b).$$

In this way, we have proved that $\delta^{k+1}(a)R\delta^n(b) = \{0\}$ for all $n \in \mathbb{N}\cup\{0\}$. The induction step is complete.

Pick arbitrary $m,n \in \mathbb{N}\cup\{0\}$. Since $\delta^m(a)R\delta^n(b) = \{0\}$, we get

$$(R\delta^m(a)R)(R\delta^n(b)R) \subseteq R(\delta^m(a)R\delta^n(b))R = \{0\}$$

(the products above are elementwise products of sets). Consequently, $IJ = \{0\}$. The δ-primeness therefore implies that either $I = \{0\}$ or $J = \{0\}$. It is easy to verify that $(\langle a\rangle^\delta)^3 \subseteq I$ and $(\langle b\rangle^\delta)^3 \subseteq J$ (see Proposition 2.6). Thus we have either $(\langle a\rangle^\delta)^3 = \{0\}$ or $(\langle b\rangle^\delta)^3 = \{0\}$. Since the square of a two-sided δ-ideal is also a two-sided δ-ideal, the δ-primeness yields that either $\langle a\rangle^{\delta} = \{0\}$ or $\langle b\rangle^{\delta} = \{0\}$. This means, finally, that either $a = 0$ or $b = 0$. Condition (ii) follows. The implication (i) $\Longrightarrow$ (iii) can be proved analogously.

Assume that condition (ii) is satisfied. Let $a, b \in R$ be such that $\langle a\rangle^{\delta}\langle b\rangle^{\delta} = \{0\}$. Observe that for an arbitrary $n \in \mathbb{N}\cup\{0\}$, we have $aR\delta^{n}(b) \subseteq \langle a\rangle^{\delta}\langle b\rangle^{\delta}$. Hence (ii) implies that either $a = 0$ or $b = 0$. Condition (iv) follows. The implication (iii) $\Longrightarrow$ (iv) can be proved analogously.

Assume now that condition (iv) is satisfied. Let $I, J \subseteq R$ be right δ-ideals such that $IJ = \{0\}$. Suppose that $I \neq \{0\}$ and pick some $a \in I\setminus\{0\}$. Let $b \in J$. It is quite easy to verify that

$$\langle a\rangle^{\delta}\langle b\rangle^{\delta} \subseteq IJ + RIJ = \{0\}.$$

Condition (iv) therefore yields $b = 0$. In this way, we have proved that $J = \{0\}$. Condition (v) follows. The implication (iv) $\Longrightarrow$ (vi) can be proved analogously.

It is clear that any of conditions (v) and (vi) implies the δ-primeness of the ring R. We have thus proved that conditions (i)–(vi) are pairwise equivalent.

Assume that condition (vi) is satisfied. Let $I \subseteq R$ be a nonzero left δ-ideal. Since $\mathrm{ann}_{\ell}(I)$ is a left δ-ideal and $\mathrm{ann}_{\ell}(I)I = \{0\}$, condition (vi) yields that $\mathrm{ann}_{\ell}(I) = \{0\}$. Condition (viii) follows. The implication (v) $\Longrightarrow$ (vii) can be proved analogously.

Assume, finally, that condition (viii) is satisfied. Let $I, J \subseteq R$ be two-sided δ-ideals such that $IJ = \{0\}$. Suppose that $J \neq \{0\}$. Since $I \subseteq \mathrm{ann}_{\ell}(J)$, condition (viii) implies that $I \subseteq \mathrm{ann}_{\ell}(J) = \{0\}$. The δ-primeness of the ring R follows. The implication (vii) $\Longrightarrow$ (i) can be proved analogously. The proof of the theorem is complete. $\square$

Let us remark that if R is a ring with identity, then the totality I of finite sums of elements of the set $\{c_1\delta^{m}(a)c_2 : c_1, c_2 \in R, m \in \mathbb{N}\cup\{0\}\}$, considered in the above proof, is the same thing as $\langle a\rangle^{\delta}$. But in the case where R is a ring without identity, it may happen that $a \notin I$.

Recall that if I is a two-sided δ-ideal of the ring R, then

$$\delta_I : R/I \ni a + I \longmapsto \delta(a) + I \in R/I$$

is a well-defined derivation.

Definition 3.3. *A two-sided δ-ideal $P \subseteq R$ is said to be δ-prime, if R/P is a δ_P-prime ring.*

Obviously, each δ-prime ideal is a proper ideal. It is worth noting that the ring R is δ-prime if and only if $\{0\}$ is a δ-prime ideal of R. The corollary below follows quite directly from Theorem 3.2.

Corollary 3.4. *Let P be a proper two-sided δ-ideal of the ring R. The following conditions are equivalent:*

(i) *P is δ-prime,*

(ii) *for arbitrary two-sided δ-ideals $I, J \subseteq R$, if $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$,*

(iii) *for any elements $a, b \in R$, if $\forall n \in \mathbb{N} \cup \{0\} : aR\delta^n(b) \subseteq P$, then either $a \in P$ or $b \in P$,*

(iv) *for any elements $a, b \in R$, if $\forall n \in \mathbb{N} \cup \{0\} : \delta^n(a)Rb \subseteq P$, then either $a \in P$ or $b \in P$,*

(v) *for any elements $a, b \in R$, if $\langle a \rangle^\delta \langle b \rangle^\delta \subseteq P$, then either $a \in P$ or $b \in P$,*

(vi) *for arbitrary right δ-ideals $I, J \subseteq R$, if $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$,*

(vii) *for arbitrary left δ-ideals $I, J \subseteq R$, if $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.*

Again, if δ is the zero derivation, then the notion of a δ-prime ideal coincides with the well-known general (“noncommutative”) notion of a prime ideal. We are ready to discuss an example of a δ-prime ring which is not prime (the example is taken from [5]).

Example 3.5. *Let $\mathbb{F}$ be a field of characteristic $p \neq 0$. Consider the principal ideal P of the polynomial ring $\mathbb{F}[x]$ generated by x^p. Since $R = \mathbb{F}[x]/P$ is a commutative ring and $x + P$ is a nonzero nilpotent element of R, the ring R is not prime. (Let us recall here that a commutative ring with identity is prime if and only if it is an integral domain). Using condition (iii) of Corollary 3.4, we can prove quite easily that P is a δ-prime ideal for the natural derivation $\delta : \mathbb{F}[x] \ni f \longmapsto f' \in \mathbb{F}[x]$. Thus R is δ_P-prime.*

In the sequel we will deal with the following generalization of the prime radical. This generalization has been introduced by Burkov (see [2]).

Definition 3.6. *The intersection $\mathrm{N}_\delta(R)$ of the family of all δ-prime ideals of the ring R is called the δ-prime radical of R.*

Notice that $\mathrm{N}_\delta(R) = R$ whenever R has no δ-prime ideals.

4. δ-nilpotent elements

Consider the family

$$\mathscr{D} = \left\{ \sum_{j=0}^{n} c_j \delta^j : n \in \mathbb{N} \cup \{0\}, c_0, \ldots, c_n \in R \right\}$$

of “differential operators on the ring R”.

Remark 4.1. *If $D \in \mathscr{D}$ and I is a left δ-ideal of R, then $D(I) \subseteq I$.*

The definition below is taken from [2].

Definition 4.2. *An element $a \in R$ is said to be δ-nilpotent, if for any sequence $\{D_k\}_{k=0}^{\infty}$ of elements of $\mathscr{D}$ almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by*

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k D_k(a_k) \end{cases}$$

are equal to 0.

Let us also recall the well-known concept of a strongly nilpotent element.

Definition 4.3. *An element $a \in R$ is said to be strongly nilpotent, if almost all members of any sequence $\{a_k\}_{k=0}^{\infty}$ in the ring R such that $a_0 = a$ and*

$$\forall k \in \mathbb{N} \cup \{0\} : a_{k+1} \in a_k R a_k$$

are equal to 0.

Observe that an element $a \in R$ is strongly nilpotent if and only if for an arbitrary sequence $\{x_k\}_{k=0}^{\infty}$ of elements of R, almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k x_k a_k \end{cases}$$

are equal to 0.

It is clear that in the definitions of a δ-nilpotent element and a strongly nilpotent element (as well as in the equivalent definition of a δ-nilpotent element given in the sequel of this section), the words "almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ are equal to 0" can be replaced by "the sequence $\{a_k\}_{k=0}^{\infty}$ contains a member equal to 0". Let us now take a look at some simple but important properties.

Proposition 4.4. *For an element $a \in R$ the following hold true:*

(i) *if a is δ-nilpotent, then it is strongly nilpotent,*

(ii) *if a is strongly nilpotent, then it is nilpotent in the usual sense,*

(iii) *if a is nilpotent in the usual sense, $a \in \mathrm{Z}(R)$ and $\delta(a) = 0$, then a is δ-nilpotent,*

(iv) *if a is nilpotent in the usual sense and $a \in \mathrm{Z}(R)$, then a is strongly nilpotent,*

(v) *if δ is the zero derivation and a is strongly nilpotent, then a is δ-nilpotent.*

Proof. Assume that a is δ-nilpotent. Pick an arbitrary sequence $\{x_k\}_{k=0}^{\infty}$ in the ring R. Since

$$\forall k \in \mathbb{N} \cup \{0\} \forall b \in R : \begin{cases} b x_k b = b x_k \delta^0(b), \\ x_k \delta^0 \in \mathscr{D}, \end{cases}$$

the δ-nilpotency implies that almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k x_k a_k \end{cases}$$

are equal to 0. Therefore, a is strongly nilpotent.

If a is a strongly nilpotent element, then almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ of powers of a defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k a a_k \end{cases}$$

are equal to 0 and hence a is nilpotent in the usual sense.

Let us turn to property (iii). It is easy to see that if $\delta(z) = 0$ for some $z \in R$, then

$$\forall t \in \mathbb{N} \setminus \{0\} \, \forall j \in \mathbb{N} \cup \{0\} \, \forall b \in R : \begin{cases} \delta(z^t) = 0, \\ \delta^j(z^t b) = z^t \delta^j(b). \end{cases} \tag{2}$$

Assume that a is nilpotent in the usual sense, $a \in \mathrm{Z}(R)$ and $\delta(a) = 0$. Let $\{D_k\}_{k=0}^{\infty}$, where

$$D_k = \sum_{j=0}^{n_k} c_{jk} \delta^j$$

for some $n_k \in \mathbb{N} \cup \{0\}$ and some $c_{0k}, \ldots, c_{n_k k} \in R$, be a sequence of elements of the family $\mathscr{D}$. Consider the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k D_k(a_k). \end{cases}$$

We will show by induction that $a_k \in a^{2^k} R$ for an arbitrary $k \in \mathbb{N} \setminus \{0\}$. First, since $\delta(a) = 0$ and $a \in \mathrm{Z}(R)$, we have

$$a_1 = a D_0(a) = a \sum_{j=0}^{n_0} c_{j0} \delta^j(a) = a^2 c_{00}.$$

Suppose therefore that $a_\ell = a^{2^\ell} b$ for some $\ell \in \mathbb{N} \setminus \{0\}$ and some $b \in R$. In view of (2) and the fact that $a \in \mathrm{Z}(R)$, we obtain

$$a_{\ell+1} = a_\ell D_\ell(a_\ell) = a^{2^\ell} b D_\ell(a^{2^\ell} b) = a^{2^\ell} b \sum_{j=0}^{n_\ell} c_{j\ell} \delta^j(a^{2^\ell} b) =$$

$$= a^{2^\ell} b \sum_{j=0}^{n_\ell} c_{j\ell} a^{2^\ell} \delta^j(b) = a^{2^{\ell+1}} b \sum_{j=0}^{n_\ell} c_{j\ell} \delta^j(b).$$

The induction step is complete. Now, let $s \in \mathbb{N} \setminus \{0\}$ be such that $a^s = 0$ ("usual nilpotency" of a). Observe that if $k \in \mathbb{N} \setminus \{0\}$ satisfies the condition $2^k \geqslant s$, then $a_k \in a^{2^k} R \subseteq a^s R = \{0\}$. The δ-nilpotency of a follows.

Let us turn to (iv). Assume that a is nilpotent in the usual sense and $a \in \mathrm{Z}(R)$. Suppose additionally that δ is the zero derivation. Then property (iii) yields that a is δ-nilpotent. It therefore follows from (i) that the element a is strongly nilpotent.

Property (v) is an immediate consequence of the fact that if δ is the zero derivation, then $\mathscr{D} = \{c \cdot \mathrm{id}_R : c \in R\}$ (and hence the definition of a δ-nilpotent element reduces to the definition of a strongly nilpotent element). □

Notice that in the case where R is a commutative ring and δ is the zero derivation, the usual nilpotency, the strong nilpotency and the δ-nilpotency of an element are the same thing. Let us see an example of a strongly nilpotent element which is not δ-nilpotent.

Example 4.5. *With the assumptions and notations of Example 3.5, we have $(x+P)\delta_P(x+P) = x+P$. Hence all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by*

$$\begin{cases} a_0 = x+P, \\ a_{k+1} = a_k\delta_P(a_k) \end{cases}$$

are nonzero. This yields that $x+P$ is not a δ_P-nilpotent element of the ring R. On the other hand, $x+P$ is a strongly nilpotent element, because it is nilpotent in the usual sense and R is a commutative ring.

The main theorem of the section is a modification of a result which has been first stated in [2].

Theorem 4.6. *Let $a \in R$. The following conditions are equivalent:*

(i) *a is δ-nilpotent,*

(ii) *for arbitrary sequences $\{c_k\}_{k=0}^{\infty}$ of elements of R and $\{n_k\}_{k=0}^{\infty}$ of non-negative integers, almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by*

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k c_k \delta^{n_k}(a_k) \end{cases}$$

are equal to 0,

(iii) $a \in \mathrm{N}_\delta(R)$.

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious (see the definition of the family $\mathscr{D}$).

Suppose that $a \in R \setminus \mathrm{N}_\delta(R)$. Then $a \notin P$ for some δ-prime ideal P of the ring R. Hence by condition (iii) of Corollary 3.4, there exist sequences $\{c_k\}_{k=0}^{\infty}$ of elements of R and $\{n_k\}_{k=0}^{\infty}$ of non-negative integers such that no member of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k c_k \delta^{n_k}(a_k) \end{cases}$$

belongs to P. It follows that $a_k \neq 0$ for all $k \in \mathbb{N} \cup \{0\}$. Therefore, condition (ii) is not satisfied. This completes the proof of the implication (ii) $\Longrightarrow$ (iii).

Now suppose that the element a is not δ-nilpotent. Then there exists a sequence $\{D_k\}_{k=0}^{\infty}$ of elements of $\mathscr{D}$ such that all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k D_k(a_k) \end{cases}$$

are different from 0. Consider the family $\mathfrak{F}$ of all two-sided δ-ideals $I \subseteq R$ with the property that $I \cap \big\{a_k : k \in \mathbb{N} \cup \{0\}\big\} = \emptyset$. Notice that $\{0\} \in \mathfrak{F}$. The family $\mathfrak{F}$ (partially) ordered by set inclusion satisfies the assumption of Zorn's lemma. Pick a maximal element $P_0 \in \mathfrak{F}$. Let us emphasize that P_0 is a proper two-sided δ-ideal of the ring R.

Let $J,K \subseteq R$ be two-sided δ-ideals such that $JK \subseteq P_0$. Assume that neither J nor K is contained in P_0. Since $P_0 \subseteq (P_0+J)\cap(P_0+K)$, $P_0 \neq P_0+J$ and $P_0 \neq P_0+K$, the maximality of P_0 implies that $P_0+J \notin \mathfrak{F}$ and $P_0+K \notin \mathfrak{F}$. But P_0+J and P_0+K are two-sided δ-ideals of the ring R. Hence there are $s,t \in \mathbb{N}\cup\{0\}$ such that $a_s \in P_0+J$ and $a_t \in P_0+K$. Let us define $u=\max\{s,t\}$. If $T \subseteq R$ is a right ideal, $x \in T$ and $D \in \mathscr{D}$, then obviously $xD(x) \in T$. It follows therefore from the definition of $\{a_k\}_{k=0}^{\infty}$ that $a_u \in (P_0+J)\cap(P_0+K)$. Next, observe that if $x \in P_0+J$, $y \in P_0+K$ and $D \in \mathscr{D}$, then by Remark 4.1 we have

$$xD(y) \in (P_0+J)(P_0+K) \subseteq P_0+JK = P_0.$$

Since $a_u \in (P_0+J)\cap(P_0+K)$, the observation yields $a_{u+1} = a_u D_u(a_u) \in P_0$. This contradicts the fact that $P_0 \in \mathfrak{F}$.

We have therefore proved that for any two-sided δ-ideals $J,K \subseteq R$, if $JK \subseteq P_0$, then either $J \subseteq P_0$ or $K \subseteq P_0$. In other words, P_0 is a δ-prime ideal of the ring R. Since $a = a_0 \notin P_0$, we get $a \notin \mathrm{N}_\delta(R)$. The proof of the implication (iii) $\Longrightarrow$ (i) is complete. □

It follows immediately from the above theorem that $\mathrm{N}_\delta(R)$ coincides with the totality of δ-nilpotent elements of the ring R. The theorem also allows us to give an equivalent definition of a δ-nilpotent element (namely, an element $a \in R$ is δ-nilpotent if and only if for arbitrary sequences $\{c_k\}_{k=0}^{\infty}$ of elements of R and $\{n_k\}_{k=0}^{\infty}$ of non-negative integers, almost all members of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k c_k \delta^{n_k}(a_k) \end{cases}$$

are equal to 0).

Recall that a set $S \subseteq R$ is said to be nil, if every element of S is nilpotent in the usual sense. Combining Theorem 4.6 with Proposition 4.4 yields a noteworthy corollary.

Corollary 4.7. *The δ-prime radical $\mathrm{N}_\delta(R)$ is a nil two-sided δ-ideal of the ring R.*

Let us finally notice that if δ is the zero derivation, then $\mathrm{N}_\delta(R)$ and the standard prime radical $\mathrm{rad}(R)$ are the same thing. In view of Theorem 4.6 and Proposition 4.4, we thus obtain the following classical fact.

Corollary 4.8. *The prime radical $\mathrm{rad}(R)$ coincides with the totality of strongly nilpotent elements of R.*

5. δ-semiprime rings

We will use the following definition of a δ-semiprime ring.

Definition 5.1. *The ring R is called δ-semiprime, if there exists no two-sided δ-ideal $I \subseteq R$ such that $I \neq \{0\}$ and $I^2 = \{0\}$.*

Obviously, each δ-prime ring is δ-semiprime. Recall that an ideal I of the ring R is said to be nilpotent, if $I^k = \{0\}$ for some $k \in \mathbb{N}\setminus\{0\}$. We are in a position to state and prove a fundamental characterization of δ-semiprime rings (cf. [1, Lemma 1]).

Theorem 5.2. *The following conditions are equivalent:*

(i) *R is a δ-semiprime ring,*

(ii) *for any element $a \in R$, if $\forall n \in \mathbb{N} \cup \{0\} : aR\delta^n(a) = \{0\}$, then $a = 0$,*

(iii) *for any element $a \in R$, if $\forall n \in \mathbb{N} \cup \{0\} : \delta^n(a)Ra = \{0\}$, then $a = 0$,*

(iv) *for any element $a \in R$, if $(\langle a \rangle^{\delta})^2 = \{0\}$, then $a = 0$,*

(v) *for an arbitrary right δ-ideal $I \subseteq R$, if $I^2 = \{0\}$, then $I = \{0\}$,*

(vi) *for an arbitrary left δ-ideal $I \subseteq R$, if $I^2 = \{0\}$, then $I = \{0\}$,*

(vii) *$\{0\}$ is the only nilpotent two-sided δ-ideal of the ring R,*

(viii) *$\{0\}$ is the only nilpotent right δ-ideal of the ring R,*

(ix) *$\{0\}$ is the only nilpotent left δ-ideal of the ring R,*

(x) *for any two-sided δ-ideals $I, J \subseteq R$, if $IJ = \{0\}$, then $I \cap J = \{0\}$,*

(xi) *for any right δ-ideals $I, J \subseteq R$, if $IJ = \{0\}$, then $I \cap J = \{0\}$,*

(xii) *for any left δ-ideals $I, J \subseteq R$, if $IJ = \{0\}$, then $I \cap J = \{0\}$,*

(xiii) *R has no nonzero δ-nilpotent elements,*

(xiv) $\mathrm{N}_{\delta}(R) = \{0\}$.

Proof. The equivalence of conditions (i)–(vi) can be proved analogously as in Theorem 3.2.

Assume that R is a δ-semiprime ring. Let $I \subseteq R$ be a nilpotent two-sided δ-ideal. Define $k_0 = \min\{k \in \mathbb{N} \setminus \{0\} : I^k = \{0\}\}$ (in other words, k_0 is the nilpotency index of I). Let $s \in \{0, 1\}$ be such that $k_0 + s$ is even. Then $(I^t)^2 = \{0\}$, where $t = (k_0 + s)/2$. The δ-semiprimeness now implies that $I^t = \{0\}$. Thus $k_0 \leqslant t$. The inequality is equivalent to $k_0 \leqslant s$. Therefore, $k_0 = 1$ and hence $I = \{0\}$. Condition (vii) follows. The implications (v) $\Longrightarrow$ (viii) and (vi) $\Longrightarrow$ (ix) can be proved analogously.

The implications (vii) $\Longrightarrow$ (i), (viii) $\Longrightarrow$ (v) and (ix) $\Longrightarrow$ (vi) are obvious.

Assume again that R is a δ-semiprime ring. Let $I, J \subseteq R$ be two-sided δ-ideals such that $IJ = \{0\}$. Since $I \cap J$ is also a two-sided δ-ideal and $(I \cap J)^2 \subseteq IJ$, the δ-semiprimeness yields that $I \cap J = \{0\}$. Condition (x) follows. The implications (v) $\Longrightarrow$ (xi) and (vi) $\Longrightarrow$ (xii) can be proved analogously.

Assume that condition (x) is satisfied. Let $I \subseteq R$ be a two-sided δ-ideal such that $I^2 = \{0\}$. Then $I = I \cap I = \{0\}$. The δ-semiprimeness of the ring R follows. The implications (xi) $\Longrightarrow$ (v) and (xii) $\Longrightarrow$ (vi) can be proved analogously. Hence, we have proved that conditions (i)–(xii) are pairwise equivalent.

Now assume that condition (ii) is satisfied. Let $a \in R \setminus \{0\}$. Then there exist sequences $\{c_k\}_{k=0}^{\infty}$ of elements of the ring R and $\{n_k\}_{k=0}^{\infty}$ of non-negative integers such that every member of the sequence $\{a_k\}_{k=0}^{\infty}$ defined by

$$\begin{cases} a_0 = a, \\ a_{k+1} = a_k c_k \delta^{n_k}(a_k) \end{cases}$$

is different from 0. Consequently, the element a is not δ-nilpotent (cf. the proof of Theorem 4.6). Condition (xiii) follows.

The equivalence (xiii) $\Longleftrightarrow$ (xiv) follows immediately from Theorem 4.6.

Assume, finally, that $\mathrm{N}_{\delta}(R) = \{0\}$. Let $I, J \subseteq R$ be two-sided δ-ideals such that $IJ = \{0\}$. Moreover, let P be a δ-prime ideal of the ring R. Since $IJ \subseteq P$, we get either $I \subseteq P$ or $J \subseteq P$.

Hence $I \cap J \subseteq P$. We have therefore proved that $I \cap J$ is contained in any δ-prime ideal of R. This means exactly that $I \cap J \subseteq \mathrm{N}_\delta(R)$. Condition (x) follows. The proof is complete. $\square$

As an obvious consequence of the above theorem, we obtain a quite important fact.

Corollary 5.3. *Suppose that the ring R is δ-semiprime. Let $I \subseteq R$ be a δ-ideal. Then*

(i) $I \cap \mathrm{ann}_r(I) = \{0\}$ *whenever I is a right ideal,*

(ii) $I \cap \mathrm{ann}_\ell(I) = \{0\}$ *whenever I is a left ideal.*

In the case where δ is the zero derivation, the definition of a δ-semiprime ring is just the well-known definition of a semiprime ring. Clearly, the ring R is semiprime if and only if it is d-semiprime for all derivations $d : R \longrightarrow R$. Notice that in fact, the ring R considered in Examples 3.5 and 4.5 is not semiprime (a commutative ring with identity is semiprime if and only if it has no nilpotent elements different from 0).

Though a δ-semiprime ring has no δ-nilpotent elements different from 0 and no nonzero nilpotent δ-ideals, it can have a nonzero nil δ-ideal. For an example we refer to [3, p. 332].

Finally, let us see how another important fact about semiprime rings generalizes to δ-semiprime rings (cf. [1, Lemma 5]).

Proposition 5.4. *Suppose that R is a δ-semiprime ring. Let $I \subseteq R$ be a two-sided δ-ideal. Then* $\mathrm{ann}_\ell(I) = \mathrm{ann}_r(I)$.

Proof. Define $K = \mathrm{ann}_r(I)I$ (product of right δ-ideals). We have

$$K^2 = (\mathrm{ann}_r(I)I)(\mathrm{ann}_r(I)I) \subseteq \mathrm{ann}_r(I)(I\mathrm{ann}_r(I))I = \{0\}$$

and hence, by the δ-semiprimeness, $\mathrm{ann}_r(I)I = K = \{0\}$. Therefore, $\mathrm{ann}_r(I) \subseteq \mathrm{ann}_\ell(I)$. The converse inclusion can be proved analogously. $\square$

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