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STATISTICAL SIMULATION OF 4D RANDOM FIELDS BY MEANS OF KOTELNIKOV-SHANNON DECOMPOSITION

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Abstract

This paper researches the real valued random fields, that are homogeneous with respect to time and homogeneous isotropic with respect to spatial variables. An analogue of the Kotelnikov-Shannon theorem for random fields with a bounded spectrum is presented. Models for such random fields by partial sums of series are constructed. Some estimates for the mean square approximation of a random field by its models are obtained. Statistical simulation procedures of realizations of a random field with Gaussian distribution are constructed. The using of these theorems, models and procedures are demonstrated through applications to generate by means of computer adequate realizations of Gaussian random field with some wide-known examples of covariance functions. Spectral analysis of generated noise is considered.

SYMULACJA STATYSTYCZNA 4D PÓL LOSOWYCH Z UŻYCIEM ROZKŁADU KOTELNIKOVA-SHANNONA

Slowa kluczowe: pola losowe, symulacja statystyczna, algorytm, funkcja korelacji

Abstrakt

W artykule omówiono badania pól losowych jednorodnych w czasie o wartościach rzeczywistych oraz izotropowych w zmiennych przestrzennych. Przedstawiono analog twierdzenia Kotelnikowa-Szannona dla pól losowych o ograniczonym widmie. Dla takich pól losowych skonstruowano model bazowany na sumach częściowych szeregu stochastycznego. W ramach badanego modelu otrzymano oszacowania ich aproksymacji średnio-kwadratowej. Zaproponowano też algorytmy modelowania statystycznego procesu realizacji pola losowego z rozkładem Gaussa. Ponadto, wobec zbadanego modelu w ramach zaproponowanych algorytmów, przedstawiono ich zastosowanie do generowania komputerowego realizacji odpowiednich pól Gaussa o zadanych funkcjach korelacji. W artykule rozpatrzono również rozkład spektralny generowanych szumów.

INTRODUCTION

Due to the rapid development of computer technology, methods of numerical simulation (the so called Monte Carlo methods) of stochastic processes and random fields have an expanding range of applications. This applies, in particular, such natural sciences as geology, geophysics, geoinformatics, seismology, meteorology, oceanography, electrical engineering, statistical radio physics, nuclear physics, and others. Using statistical simulation techniques and computers, one can generate realizations of stochastic processes and random field for which some necessary statistical data is known. The random functions statistical simulation on the basis of their spectral decomposition (Yadrenko M. 1983) are very important to resolve of these problems (see e.g. surveys in Yadrenko M. & Gamaliy O. 1998; Vyzhva Z. 2003, 2011–2013).

Modified interpolation Kotelnikov–Shannon decompositions for stochastic processes and multivariate random fields have been studied by Belyayev Y. 1959, Piranashvili Z. 1967, Higgins J. 1996, Vyzhva Z. 2011–2013, Olenko A. 2004, 2005, 2011, Pogany T. 2005, 2011, Vyzhva Z. & Fedorenko K. 2013.

This paper deals with real valued and mean square continuous 4D random fields $\xi(t,x)=\xi(t,r,\theta,\phi)$ in $R \times R^3$ homogeneous with respect to variable t and homogeneous isotropic with respect to variables r, θ, φ. The spectral representation theorem for such random fields has been proved. An analogue of the Kotelnikov-Shannon theorem for random field whose spectrum is bounded in time is presented. The model and simulating procedure of Gaussian random fields with given statistical characteristics are constructed.

This paper exemplifies applying the models and procedures of statistical simulation of stochastic processes and random fields to generating by means of computer adequate realizations of random fields with some wide-known examples of covariance functions.

Models and procedures of statistical simulation of stochastic processes and random field based on decompositions into series which recently were used in geosciences (see Chiles J. & Delfiner P. 2009, Vyzhva Z. 2011, etc.)

HOMOGENEOUS WITH RESPECT TO TIME AND HOMOGENEOUS ISOTROPIC WITH RESPECT TO SPATIAL VARIABLES 4D RANDOM FIELDS

We consider a real valued mean square continuous 4D random field $\xi(t,x)=\xi(t,r,\theta,\phi),\ t\in R,\ x\in R^3$.

The 4D random field $\xi(t,r,\theta,\phi),\ t\in R,\ x\in R^3$ is called homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables r, θ, φ (here r, θ, φ are polar coordinates of a point x and $r\in R_+,\theta\in[0,\pi],\phi\in[0,2\pi]$), if it satisfies the following conditions:

1) $E\xi(t,x_1)=const$ for all $t \in R$ and $x_1 \in R^3$ (we assume that $E\xi(t,x_1)=0$),
2) $E\xi(t,x_1)\xi(s,x_2)=B(t-s,\rho)$ for all $t, s \in R$ and $x_1, x_2 \in R^3$,

where $B(\tau,\rho)$ is a correlation function that depends on the shift of the time $\tau=t-s$ and the distance between vectors x_1 and x_2, that is on $\rho=|x_1-x_2|$. The distance between the point $x_1=(r_1,\theta_1,\varphi_1)$ and the point $x_2=(r_2,\theta_2,\varphi_2)$ equals $\rho=\sqrt{r_1^2+r_2^2-2r_1r_2\cos\psi}$ where $\cos\psi$ is an angular distance such that: $\cos\psi=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$.

The representation of the correlation function of a real valued random field $\xi(t, x)$ on $R \times R^n$ which is homogeneous with respect to time t and homogeneous isotropic with respect to the spatial variables is defined (Yadrenko, 1983) as integral

$$B(t-s,\rho)=\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}Y_n(\lambda\rho)\Phi(du,d\lambda), \quad (1)$$

where $Y_n(\lambda\rho)=2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)J_{\frac{n-2}{2}}(\lambda\rho)(\lambda\rho)^{-\frac{n-2}{2}}$, $\Phi(u,\lambda)$ is the spectral function of the random field and $J_m(x)$ is the Bessel function of the first kind of order m.

Let us consider correlation function for dimension $n = 3$ as follows formula

$$B(t-s,\rho)=$$
$$=\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}2^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)J_{\frac{1}{2}}(\lambda\rho)(\lambda\rho)^{-\frac{1}{2}}\Phi(du,d\lambda). \quad (2)$$

It is easy to check that $J_{\frac{1}{2}}(t)=\sqrt{\frac{2}{\pi}}\frac{\sin t}{\sqrt{t}}$ and next expression in this case holds

$$B(t-s,\rho)=\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}\frac{\sin\lambda\rho}{\lambda\rho}\Phi(du,d\lambda). \quad (3)$$

Then the following **spectral** representation theorem holds true.

Theorem 1. A mean square continuous 4D random field $\xi(t, r, \theta, \varphi)$ on $R \times R^3$ which is time homogeneous and homogeneous isotropic with respect to the spatial variables admits the following **spectral decomposition**

$$\xi(t,r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{l=0}^{m}c_{m,l}P_m^l(\cos\theta)\left[\zeta_{m,1}^l(t,r)\cos l\phi+\right.$$
$$\left.+\zeta_{m,2}^l(t,r)\sin l\phi\right], \quad (4)$$

where 2D random fields sequences are of the form

$$\zeta^l_{m,k}(t,r)=\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{itu}\frac{J_{m+\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}}Z^l_{m,k}(du,d\lambda),$$

$$m=0,1,\ldots;l=0,1,\ldots,m;\ k=1,2, \quad (5)$$

and constants sequences are calculated by the formula

$$c_{m,l}=\sqrt{\frac{\pi\nu_l}{2}\frac{(2m+1)(m-l)!}{(m+l)!}},\ \nu_l=\begin{cases}1,\ l=0\\2,\ l>0,\end{cases}$$

and $P^l_m(x)$ are associated Legendre functions of degree m and $\{Z^l_{m,1}(\cdot),Z^l_{m,2}(\cdot)\}$ are sequences of real valued orthogonal random measure on Borel subsets of the set $(-\infty,+\infty)\times[0,+\infty)$ such that satisfy the next conditions:

$$EZ^l_{t,k}(S_1)=0,\ EZ^l_{t,k}(S_1)Z^q_{p,j}(S_2)=$$

$$=\delta^k_j\delta^p_t\delta^q_l\Phi(S_1\cap S_2) \quad (6)$$

for all Borel subsets S_1 and S_2 of $R\times R_+$, $t,p=0,1,\ldots;$ $l,q=0,1,\ldots,m;\ k,j=1,2$.

Then the correlation function (2) of random field $\xi(t, r, \theta, \varphi)$ admits the next decomposition

$$E\xi(t,x_1)\xi(s,x_2)=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\pi\left(m+\frac{1}{2}\right)\nu_l\frac{(m-l)!}{(m+l)!}\times$$

$$\times P^l_m(cos\theta_1)P^l_m(cos\theta_2)\,cosl(\phi_1-\phi_2)\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}\times$$

$$\times\frac{J_{\frac{1}{2}+m}(\lambda r_1)}{\sqrt{\lambda r_1}}\frac{J_{\frac{1}{2}+m}(\lambda r_2)}{\sqrt{\lambda r_2}}\Phi(du,d\lambda). \quad (7)$$

Since $E\xi(t,r,\theta,\phi)=0$ we have $E\zeta^l_{m,k}(t,r)=0$, $m=0,1,\ldots;\ l=0,1,\ldots,m;\ k=1,2$.

If one considers the restriction of the random field $\xi(t, r, \theta, \varphi)$ to the sphere of a fixed radius r and applies the theorem of spherical harmonics (see book by M. Yadrenko, 1983) to (7) then correlation function of random field is given by

$$E\xi(t,r,\theta_1,\phi_1)\xi(s,r,\theta_2,\phi_2)=2\pi^2\sum_{m=0}^{\infty}\sum_{l=0}^{2m+1}S^l_m(\theta_1,\phi_1)\times$$

$$\times S^l_m(\theta_2,\phi_2)\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}\frac{J^2_{\frac{1}{2}+m}(\lambda r)}{\lambda r}\Phi(du,d\lambda), \quad (8)$$

where $S^l_m(\theta,\phi)$ are orthonormal spherical harmonics of degree m, $l=1,\ldots,h(m,n)$.

The coefficients of the decomposition (8) are denoted by $b_k(t-s,r)$, $k=0,1,\ldots$ and called the **spectral coefficients**. The spectral coefficients are expressed in terms of the spectral function as follow

$$b_k(t-s,r)=\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(t-s)u}\frac{J^2_{\frac{1}{2}+k}(\lambda r)}{\lambda r}\Phi(du,d\lambda). \quad (9)$$

Theorem 2. If $\xi(t, r, \theta, \varphi)$ is a random field in $R\times R^3$ which is homogeneous in time and homogeneous isotropic with respect to the spatial variables then

$$E\zeta^l_{m,p}(t,r)\zeta^k_{q,j}(s,r)=\delta^q_m\delta^k_l\delta^j_p b_m(t-s,r), \quad (10)$$

where δ^r_k is a Kronecker symbol, $\{b_k(t-s,r)\}$ is a sequence of positive definite kernels in $R\times R_+$ of the form (9) and such that the following property holds $\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)b_k(t-s,r)<\infty$.

The variance of the random field $\xi(t, r, \theta, \varphi)$ is given as

$$VAR\xi(t,r,\theta,\phi)=\pi\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)b_m(0,r). \quad (11)$$

Expansion can be used for statistical simulation of 4D random fields $\xi(t, r, \theta, \varphi)$ in $R\times R^3$ which are homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables if the spectral function (or correlation function) is specified.

TIME HOMOGENEOUS RANDOM FIELD WITH A BOUNDED IN TIME SPECTRUM

We consider a random field $\xi(t, r, \theta, \varphi)$ in $R\times R^3$.

We say that $\xi(t, r, \theta, \varphi)$ is a random field with a **bounded** spectrum with respect to time t if all its spectral measures are concentrated in $[-\tilde{\omega},\tilde{\omega}]\times R_+$.

Let $\xi(t,r,\theta,\phi)$, $t\in R$, $r\in R_+$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$, be a random field in $R\times R^3$ which is time homogeneous and homogeneous isotropic with respect to the spatial variables. Assume that the spectrum $\Phi(U,\Lambda)$of the field ξ is bounded with respect to time t, $U\subset[-\tilde{\omega},\tilde{\omega}],\Lambda\subset R_+$; and let $\Phi(U,\Lambda)$ be concentrated in $[-\tilde{\omega},\tilde{\omega}]\times R_+$.

Let ω be an arbitrary number such that $\omega>\tilde{\omega}$. Put

$$\xi_N(t,r,\theta,\phi)=\sum_{k=-N}^{N}\xi\left(\frac{k\pi}{\omega},r,\theta,\phi\right)\frac{sin\omega\left(t-\frac{k\pi}{\omega}\right)}{\omega\left(t-\frac{k\pi}{\omega}\right)}. \quad (12)$$

We use Lemma (Belyaev 1956) and then the following assertion holds.

Theorem 3. Let $\xi(t, r, \theta, \varphi)$ be a random field in $R \times R^3$ which is time homogeneous and homogeneous isotropic with respect to the spatial variables. If the spectrum of $\xi(t, r, \theta, \varphi)$ is bounded in time t then the mean square approximation with the help of partial sum (12) is such that

$$E|\xi(t,r,\theta,\phi)-\xi_N(t,r,\theta,\phi)|^2 \le$$
$$\le \frac{\gamma^2(t)}{N^2}\frac{1}{\left(1-\frac{\tilde{\omega}}{\omega}\right)^2}\left(\pi\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)\tilde{b}_m(0,r)\right). \quad (13)$$

where

$$\gamma(t)=\frac{4}{\pi}\left(\frac{\omega}{\pi}|t|+1\right), \quad (14)$$

$$\tilde{b}_m(0,r)=\int_{-\tilde{\omega}}^{+\tilde{\omega}}\int_0^{+\infty}\frac{J^2_{\frac{1}{2}+m}(\lambda r)}{\lambda r}\Phi(du,d\lambda). \quad (15)$$

Corollary. Let $\xi(t, r, \theta, \varphi)$ be a random field in $R \times R^3$ whose spectrum is bounded in time t. Then $\xi(t, r, \theta, \varphi)$ admits the following Kotelnikov–Shannon decomposition

$$\xi(t,r,\theta,\phi)=\sum_{k=-\infty}^{\infty}\xi\left(\frac{k\pi}{\omega},r,\theta,\phi\right)\frac{\sin\omega\left(t-\frac{k\pi}{\omega}\right)}{\omega\left(t-\frac{k\pi}{\omega}\right)}, \quad (16)$$

where the series on the right hand side of (16) converges in the mean square sense.

We use the partial sum of and the partial sum of the decomposition (16) for a random field $\xi(t, r, \theta, \varphi)$ to construct a model for such field if its spectrum is bounded with respect to time t. The approximating **model** is written as follows

$$\tilde{\xi}_{N,M}(t,r,\theta,\phi)=\sum_{m=0}^{M}\sum_{l=0}^{m}c_{m,l}P_m^l(\cos\theta)\times$$
$$\times\left[\cos l\phi\sum_{k=-N}^{N}\frac{\sin\omega\left(t-\frac{k\pi}{\omega}\right)}{\omega\left(t-\frac{k\pi}{\omega}\right)}\zeta^l_{m,1}\left(\frac{k\pi}{\omega},r\right)+\right.$$
$$\left.+\sin l\phi\sum_{k=-N}^{N}\frac{\sin\omega\left(t-\frac{k\pi}{\omega}\right)}{\omega\left(t-\frac{k\pi}{\omega}\right)}\zeta^l_{m,2}\left(\frac{k\pi}{\omega},r\right)\right], \quad (17)$$

where $\zeta^l_{m,j}\left(\frac{k\pi}{\omega},r\right)$ are values of the Gaussian stochastic processes for all $m,p=0,1,\ldots,M$;; $k,q=-\overline{N,N}$; $l,s=0,1,\ldots,m$; $i,j=1,2$ such that:

$$\zeta^l_{m,j}\left(\frac{k\pi}{\omega},r\right)=0,\ E\zeta^l_{m,j}\left(\frac{k\pi}{\omega},r\right)\zeta^s_{p,i}\left(\frac{q\pi}{\omega},r\right)=$$
$$=\delta_l^s\delta_p^m\delta_i^j\tilde{b}_m\left(\frac{(k-q)\pi}{\omega},r\right),$$

where $\tilde{b}_m(t-s,r)$ is evaluated by formula (15).

THE ESTIMATES OF THE MEAN SQUARE APPROXIMATION OF TIME HOMOGENEOUS AND HOMOGENEOUS ISOTROPIC WITH RESPECT TO SPATIAL VARIABLES 4D RANDOM FIELDS

Theorem 4. Let $\xi(t, r, \theta, \varphi)$ be a 4D random field in $R \times R^3$ which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of $\xi(t, r, \theta, \varphi)$ is bounded in time t and all its spectral measures are concentrated in $[-\tilde{\omega},\tilde{\omega}]\times R_+$ then the mean square approximation by model (17) is such that

$$E|\xi(t,x)-\tilde{\xi}_{N,M}(t,x)|^2\le\frac{5\pi r^3}{2M^2}\tilde{\mu}_3+$$
$$+\frac{\gamma^2(t)}{N^2}\frac{1}{\left(1-\frac{\tilde{\omega}}{\omega}\right)^2}\left(2\tilde{\mu}_0-\frac{5\pi r^3}{2M^2}\tilde{\mu}_3\right)<\varepsilon, \quad (18)$$

where r, θ, φ are polar coordinates of a point x, ω is an arbitrary number such that $\omega>\tilde{\omega}$, function $\gamma(t)$ is (14) and

$$\tilde{\mu}_k=\int_{-\tilde{\omega}}^{+\tilde{\omega}}\int_0^{+\infty}\lambda^k\Phi(du,d\lambda). \quad (19)$$

We consider other estimates of the mean square approximation of 4D random fields.

We use corollary (see Piranashvili Z. 1967) for a wide sense stationary stochastic process with the correlation function

$$B(t-s)=\int_{\Lambda}e^{i(t-s)u}F(du)$$

to estimate the mean square approximation of the random field $\xi(t, r, \theta, \varphi)$ by its model (17) where Λ is a bounded domain of real numbers.

Theorem 5. Let $\xi(t, r, \theta, \varphi)$ be a 4D random field in $R \times R^3$ which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of $\xi(t, r, \theta, \varphi)$ is bounded in time t and all its spectral measures are concentrated in $[-\tilde{\omega}, \tilde{\omega}] \times R_+$ then the mean square approximation by model (17) is such that

$$E|\xi(t,x) - \tilde{\xi}_{N,M}(t,x)|^2 < \frac{5\pi r^3}{2M^2}\tilde{\mu}_3 + $$

$$+\frac{L_0^2(t)\omega^2}{(\omega - v)^2 N^2}\left(2\tilde{\mu}_0 - \frac{5\pi r^3}{2M^2}\tilde{\mu}_3\right) < \varepsilon, \quad (20)$$

where $L_0(t) = \dfrac{2}{1 - e^{-\pi}}\left(\dfrac{2}{\pi}\right)|sin\omega t|$, $B(0) = \int_\Lambda F(du) =$ $= E|\xi(t)|^2$ and where $\omega > v = sup_{u \in \Lambda}|u|$ is an arbitrary fixed number and $\tilde{\mu}_k$ is evaluated by formula (19).

We consider the estimate of the mean square approximation of 4D random fields of order $O\left(\dfrac{1}{N}\right)$.

Theorem 6. Let $\xi(t, r, \theta, \varphi)$ be a 4D random field in $R \times R^3$ which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of $\xi(t, r, \theta, \varphi)$ is bounded in time t and all its spectral measures are concentrated in $[-\tilde{\omega}, \tilde{\omega}] \times R_+$ then the mean square approximation by model (17) is such that

$$E|\xi(t,x) - \tilde{\xi}_{N,M}(t,x)|^2 < \frac{5\pi \mathrm{r}^3}{2M^2}\tilde{\mu}_3 + $$

$$+\frac{4}{\pi^2(2N-1)}\left(2\tilde{\mu}_0 - \frac{5\pi r^3}{2M^2}\tilde{\mu}_3\right) < \varepsilon, \quad (21)$$

where $\tilde{\mu}_k$ is evaluated by formula(19).

All of mentioned above theorems are proved in the paper (Vyzhva Z. O., Fedorenko K. 2016) in Journal of Applied Mathematics and Statistics.

A PROCEDURE FOR THE STATISTICAL SIMULATION OF A 4D RANDOM FIELD IN R × R³ WHICH IS TIME HOMOGENEOUS AND HOMOGENEOUS ISOTROPIC WITH RESPECT TO SPATIAL VARIABLES

The above Kotelnikov–Shannon expansion for a 4D random field $\xi(t, r, \theta, \varphi)$ in $R \times R^3$ which is time homogeneous and homogeneous isotropic with respect to spatial variables can be used for the statistical simulation of such a random field if the spectrum of ξ is bounded with respect to t and if some of its statistical characteristics are given.

Below we describe a procedure for the statistical simulation of realizations of Gaussian random fields $\xi(t, r, \theta, \varphi)$ being homogeneous with respect to time and homogeneous isotropic with respect to spatial variables whose spectrum is bounded in t based on the model (17) and bounds (18), (20) and (21).

Recall that a random field in $R \times R^3$ which is time homogeneous and homogeneous isotropic with respect to spatial variables can be written as in . We use partial sums of and partial sums of expansion (16) for the random field $\xi(t, r, \theta, \varphi)$ to construct its model (17).

The partial sum (17) is taken as an approximation model of such a random field.

Using (17), the procedure for the statistical simulation of realizations of a Gaussian random field which is time homogeneous and homogeneous isotropic with respect to spatial variables can be stated as follows if its spectrum is bounded in t.

The **procedure** is described below

- We choose positive integer numbers N and M for the model (17) according to a prescribed accuracy $\varepsilon > 0$ by using one of the following inequalities (18), (20) and (21).
- We generate values of the Gaussian stochastic processes $\zeta_{m,j}^{l}\left(\dfrac{k\pi}{\omega}, r\right)$ and for all $m, p = 0,1,\ldots,M$; $k, q = -\overline{N, N}$; $l, s = 0,1,\ldots,m$; $i, j = 1,2$ for a fixed r such that:

$$E\zeta_{m,j}^{l}\left(\frac{k\pi}{\omega}, r\right) = 0,\; E\zeta_{m,j}^{l}\left(\frac{k\pi}{\omega}, r\right)\zeta_{p,i}^{s}\left(\frac{q\pi}{\omega}, r\right) =$$

$$= \delta_l^s \delta_p^m \delta_i^j \tilde{b}_m\left(\frac{(k-q)\pi}{\omega}, r\right).$$

- We evaluate the expression in (17) at a given point $(t, r, \theta, \phi) \in [-T, T] \times R^3$, by substituting the numbers N and M, and values of Gaussian stochastic processes evaluated in steps 1 and 2.
- We check whether the realization of the stochastic random field $\xi(t, r, \theta, \varphi)$ generated in step 3 fits the data by testing the corresponding statistical characteristics.

NUMERICAL SIMULATION EXAMPLE

The practical using of the constructed model (17) and procedures is considered for numerical simulation for real valued random field $\xi(t, r, \theta, \varphi)$ in $R \times R^3$ with a bounded spectrum, that are time homogeneous and homogeneous isotropic with respect to variables r, θ, φ on in this example. It admits that this random field have spatial-temporal covariance function $C(\tau,\rho)$. We may use approach, see (Demyanov & Savelev 2010) for such covariance functions that divides spatial and temporal components by means of product-sum formulas:

$$C(\tau,\rho) = k_1 C_x(\rho) C_t(\tau) + k_2 C_x(\rho) + k_3 C_t(\tau).$$

We chose the spatial covariance function $C_x(\rho)$ which is connected to spatial variogram $\gamma_x(\rho)$ in homogeneous isotropic case as: $\gamma_x(\rho) = C_x(0) - C_x(\rho)$. The example of spatial covariance function is $C_x(\rho) = C_x(0) B_x(\rho)$ where $C_x(0)$ spatial variance, $B_x(\rho)$ is a spatial correlation function Cauchy with the parameters $a = 1$ and $\nu = 1$ (Fig. 1):

$$B_x(\rho) = \left(1 + \frac{\rho^2}{a^2}\right)^{-\nu}, \; a>0, \; \nu>0.$$

The spatial variogram $\gamma_x(\rho)$ simulating by model (17) and results of realizations random field $\xi(t, r, \theta, \varphi)$ in the point $t = 0$ for $\theta = 0$ on the grid of points on the

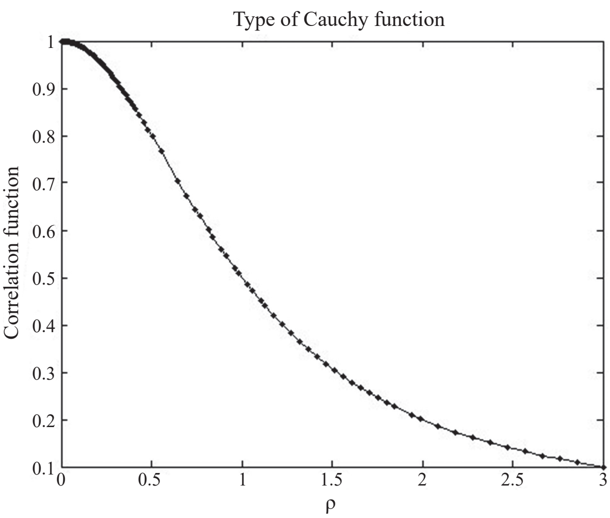


Fig. 1. The spatial Cauchy type correlation function $B_x(\rho)$ for parameters $a = 1$ and $\nu = 1$

Rys. 1. Przestrzenna funkcja korelacji typu Cauchy'ego $B_x(\rho)$ dla parametrów $a = 1$ i $\nu = 1$

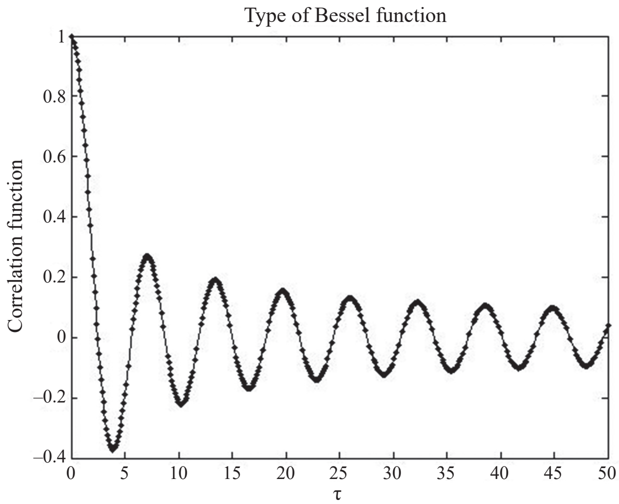


Fig. 2. The temporal Bessel type correlation function $B_t(\tau)$ for parameter $\nu = 0.05$

Rys.2. Czasowa funkcja korelacji typu Bessela $B_t(\tau)$ dla parametru $\nu = 0.05$

plane (r_i,ϕ_i), $r_i \in [0, 0.1, \ldots, 1]$, $\phi_i \in \left[0, \frac{2\pi}{10}, \ldots, 2\pi\right]$ represented on (Fig. 3). The spatial-temporal variogram (Fig. 7):

$$\gamma_{t,x}(\tau,\rho) = (k_1 C_x(0) + k_3)\gamma_t(\tau) +$$
$$+ (k_1 C_t(0) + k_2)\gamma_x(\rho) - k_1 \gamma_t(\tau)\gamma_x(\rho).$$

Temporal covariance function $C_t(\tau)$ which connected to temporal variogram $\gamma_t(\tau)$ in homogeneous case as: $\gamma_t(\tau) = C_t(0) - C_t(\tau)$, where $C_t(0)$ is a temporal variance. We choose as example of temporal covariance functions the next: $C_t(\tau) = C_t(0) B_t(\tau)$ where $B_t(\tau)$ is a temporal correlation function of Bessel type with parameter $\nu = 0.055$ (Fig. 2):

$$B_t(\tau) = 2^{\nu}\Gamma(1+\nu)\tau^{-\nu} J_{\nu}(\tau).$$

The temporal variogram is constructed for results of simulating realizations of random field $\xi(t, r, \theta, \varphi)$ by the model (17) in the point of space $(r,\theta,\phi) = (0,0,0)$ at the change of time t, $0 \le t \le T$, for example $T = 20.01$ seconds and $\Delta t = \frac{1}{\omega} = \frac{T}{N}, \Delta t = 0.01, \omega = 100$ (N is the number of experimental observations points at the time t_n, $n = 1, \ldots, N$; $N = 2001$) that represented on illustration (Fig. 4).

For graphic interpretation of the simulating realizations of the random field $\xi(t, r, \theta, \varphi)$ the plot of

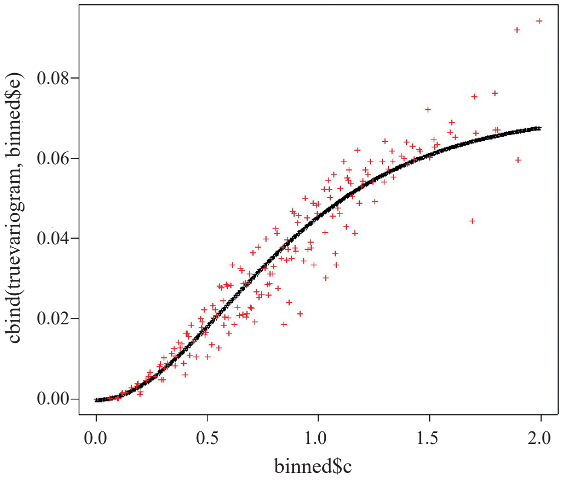


Fig. 3. Empirical (red crosses) and theoretical (black curve) variogram for averaging 20 realizations of random fields $\xi(0,r,0,\varphi)$ with Cauchy type correlation function for parameters $a = 1$, $v = 1$

Rys. 3. Empiryczny (czerwone krzyżyki) i teoretyczny (czarna krzywa) wariogram do uśredniania po 20 realizacjach pola losowego $\xi(0,r,0,\varphi)$ z funkcją korelacji typu Cauchy'ego dla parametrów $a = 1$, $v = 1$

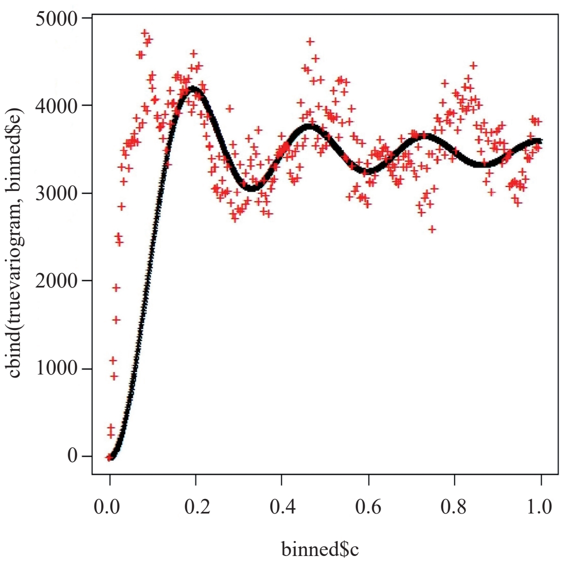


Fig. 4. Empirical (red crosses) and theoretical (black curve) variogram for averaging 15 realizations of random field $\xi(t,0,0,0)$ with Bessel type correlation function for parameter $v = 0.055$

Rys. 4. Empiryczny (czerwone krzyżyki) i teoretyczny (czarna krzywa) wariogram do uśredniania po 15 realizacjach pola losowego $\xi(t,0,0,0)$ z funkcją korelacji typu Bessela dla parametru $v = 0.055$

field realizations $\xi(t,0,0,0)$ in times of experimental observations t from 0 to 20 seconds was constructed (Fig. 5) and wireframe surface $\xi(0,r,0,\varphi)$ was built by using Surfer Software of Surfer, on the grid of points on the plane that represented illustration (Fig. 6).

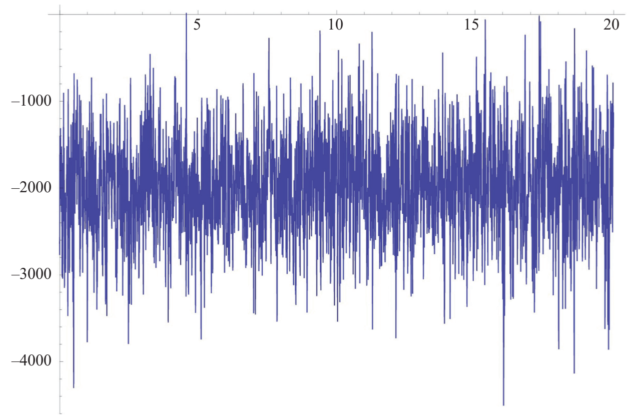


Fig. 5. Illustrating of simulated realizations of the random field $\xi(t,0,0,0)$ by t from 0 to 20 sec.

Rys. 5. Ilustracja realizacji imitowanej pola losowego $\xi(t,0,0,0)$ przez t od 0 do 20 sekund

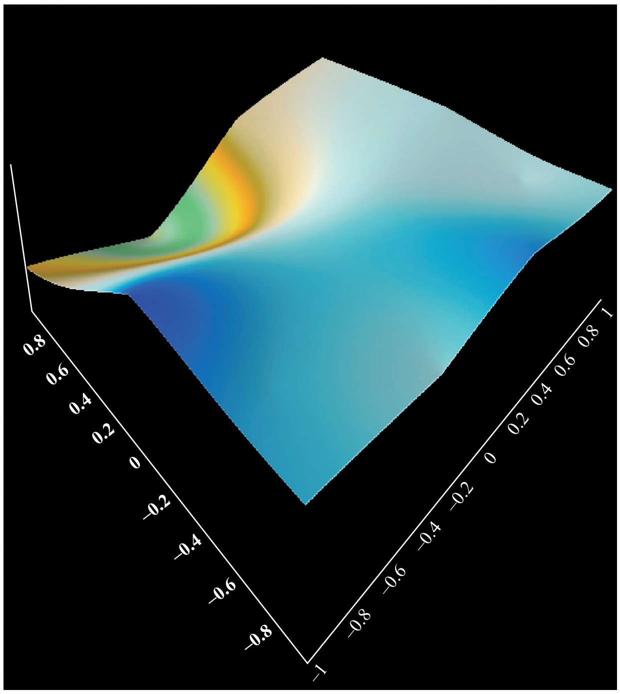


Fig. 6. Surface of random field $\xi(0,r,0,\varphi)$ realizations with a Cauchy type correlation function.

Rys. 6. Realizacja pola losowego $\xi(0,r,0,\varphi)$ z funkcją korelacji typu Cauchy'ego dla parametru $v = 1$

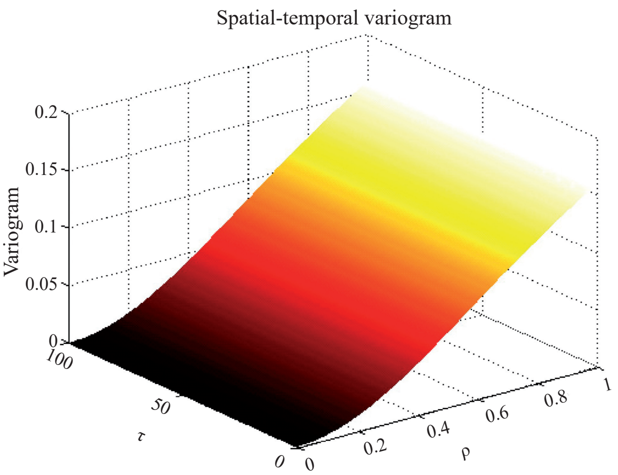


Fig. 7. The spatial-temporal variogram $\gamma_{t,x}(\tau,\rho)$
Rys. 7. Przestrzenno-czasowy wariogram $\gamma_{t,x}(\tau,\rho)$

Practical use of field simulation with space-time correlation function

Different approaches can be applied for practical use of the procedure and model (17) for numerical simulation of real and homogeneous in time t implementations, that are homogeneous isotropic with respect to variables r, θ, φ random field with a bounded in t spectrum in $R \times R^3$, which have a limited spectrum and space-time correlation function $B_z(\tau,\rho)$. It should be noted that models of space-time correlation structure are divided into two types: first takes into account the distribution of the spatial and time components and other with no such distribution. Work Demyanov V. & Savelev E.2010 gives an example of application and most commonly used models, namely metric model, linear model, model of space-time covariance product and model of product and sum.

Different approach can also be used to simulate space-time correlation that allows classifying the undivided space-time stationary covariance functions. This approach is based on the frequency representation of covariance function.

Practical use example in seismology of developed algorithm and numerical simulation model for real and homogeneous in time t implementations, two-dimensional homogeneous isotropic random fields with a limited spectrum and space-time correlation function $B_z(\tau,\rho)$ by method which divides the spatial and time components with product-sum formula described in Vyzhva Z. 2013.

The realization value arrays of random process $\xi(t,\rho,\varphi)$ (ρ,φ – fixed) were simulated as noise seismograms for each observation point on each component: EW, NS, and Z. They give important information about soil vibration properties within the territory of building and operating sites. These properties are also required for design of new antiseismic buildings and constructions, and providing earthquake resistance of existing buildings in order to avoid dangerous resonance effects. Random disturbances from random external factors were removed from the simulated noise seismograms by statistical averaging filters. These disturbances include vibrations caused by the movement of trains or heavy car and so on. The adequacy of value array results from the simulated by statistical methods noise seismograms were tested on real seismograms from observation points.

The statistical modeling method of random fields can also solve a major 3D simulation problem of the artificial realization of noise seismograms that simulated for imaginary observation points, located between the real points of observation or at a small distance from them. All values except time t, in $\xi(t,\rho,\theta,\varphi)$ are fixed and spectral analysis was performed of random process realizations. Amplitude and phase spectra of such noise realization may be used to obtain the frequency characteristics of the geological environment under building sites, describing its ability to change (increase or decrease) the amplitude of the seismic waves during earthquakes (Bath M., 1980; Vyzhva Z., Kendzera O., Fedorenko K., Vyzhva A., 2012). Numerical simulation of soil strata frequency characteristics in some cases can significantly reduce the cost of seismic zoning of building sites by reducing the number of instrumental observation points for earthquakes, explosions and microseism.

SPECTRAL ANALYSIS OF GENERATED NOISE

Frequency characteristic estimates for the geological environment with multidimensional observation area (under construction sites) can be obtained by calculating and constructing the amplitude and phase spectra of noise in seismogram observation points in that area, considering fixed all arguments except time (Vyzhva Z. 2011).Calculations of the amplitude and phase spectra can be made by direct method (Bath M., 1980, p. 179), i.e. periodogram method. Then based on these results the spectral ratio of the Earth crust was build, which is independent of the spectrum of incident seismic waves, but determined entirely by the geological environment structure under the observation point.

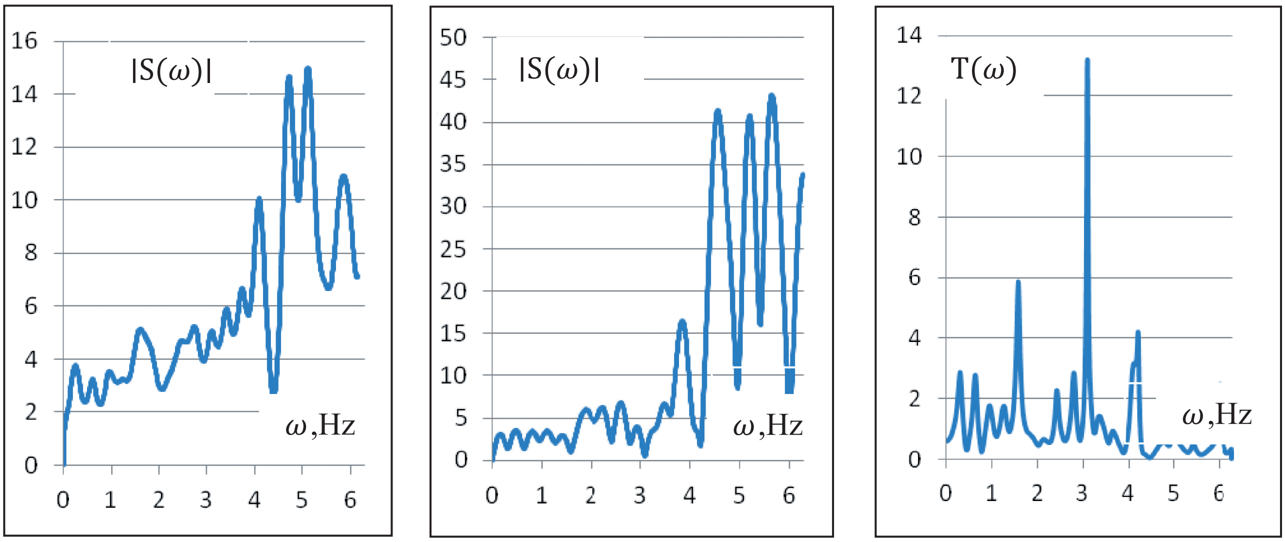
Those spectral methods that use frequency as an independent parameter provide information about the structure and filtration properties of the upper crust layers, because any medium is a filter that due to resonance and reverberation effects increases the oscillation amplitude for some frequencies and reduces for the other (Bath M., 1980, p. 270). The ability to simulate the effects depends on amplitude and phase frequency characteristics of the geological environment for observation points situated under building sites and operating platforms, allows studying the geological section features and predicting places where significant increase in the seismic oscillation intensity is possible due to resonance effects and oscillation field interference nodes.

Among the many ways to eliminate the influence of various factors that affect the spectrum shape of seismic waves during earthquakes, explosions and microseism except that due to the influence of the upper crust section part, the way should be noted based on the use of the vertical $|S_Z(\omega)|$ component spectra relations to the horizontal $|S_N(\omega)|$ component. Spectra must be calculated for the same wave. This ratio is called the crust spectral ratio T(ω).

$$|S_Z(\omega)| / |S_N(\omega)| = T(\omega)$$

The ratio T(ω) is independent of the spectrum of incident seismic waves, but determined entirely by the geological environment structure under the observation point. Figures 1 a and 1b show graphs of amplitude spectra $|S(\omega)|$ for the initial simulated noise realization for imaginary observation point with the oscillation components Z and NS respectively, Figure 1 represents earth crust transmission ratio graph T(ω), that was built on the smoothed amplitude spectrum ratio of simulated noise seismogram realization on the fluctuation Z- component to the similar spectrum of fluctuation component NS for an observation point.

Interpretation of crust transmission ratio for these observations was conducted by comparing them with theoretical ratio calculated for well-known models of the upper section part. It should be noted that one of the important tools for impact assessment of the geological section upper part on seismic movements are widely known Nakamura method H/V or QTS (Qua-



a) b) c)

Fig. 8. Graphs of amplitude spectra $|S(\omega)|$ for simulated array noise realization for imaginary observation point on the component a) Z and b) NS; c) the graph of transmission ratio T(ω) for smoothed amplitude spectra of simulated noise realization for imaginary observation point.

Rys. 8. Wykresy amplitudowych spektrów $|S(\omega)|$ masywu zmodelowanej realizacji szumu dla abstrakcyjnego punktu obserwacji na składnikach a) Z i b) NS; c) wykres proporcji transmisji T(ω) dla wygładzonych spektrów amplitudowych zmodelowanej realizacji szumu dla abstrakcyjnego punktu obserwacyjnego.

si-Transfer Spectra), developed by Japanese scientist Yutaka Nakamura. The method uses records of micro seismic noise registered for the horizontal and vertical oscillation components using borehole observations for the construction of a quasi-transmitting spectrum of soil strata. Nakamura method allows determining resonant eigen frequency of soil strata using spectrum ratios of horizontal and vertical components of the natural seismic noise. The maximum values of the microseism spectrum ratio of horizontal to vertical component are explained by multiple reflection of SH waves.

Figure 1 shows graph $T(\omega)$ of smoothed amplitude spectra transmission ratio for imaginary observation point that can be used to determine the increase of seismicity level on different parts of the building site, relative to the real observation point.

CONCLUSIONS

This paper researches homogeneous on time variable and homogeneous isotropic on 3D spatial variables of random fields. Statistical simulation model and procedures of these real valued random fields realizations with Gaussian distribution and a bounded spectrum are built. Mean square estimates of these random fields approximating by its models are found on the base of results Yadrenko M. & Gamaliy O., 1998, Olenko A., 2004 and Olenko A. & Pogany T., 2005.The using of these theorems, models and algorithms are demonstrated possibility to generate on a computer adequate realizations of such random field with some wide-known examples of covariance functions for spatial-temporal data. These results continued research set in works Vyzhva S. & Vyzhva Z., 2003, Vyzhva Z., 2011 and Vyzhva Z., Kendzera O., Fedorenko K., Vyzhva A., 2012for modeling and generation method of noise seismogram implementations at flat observation area Vyzhva Z., 2012 and seismograms from 3D observation area Vyzhva Z., 2013. These investigations are important additions and supplement to Monte Carlo methods for applications and using in geology (see Chiles J. & Delfiner P., 2009, etc.).

PODSUMOWANIE

W artykule badano jednorodne w czasie i jednorodne izotropowe po trójwymiarowym przestrzennym zmiennym pola losowe. Zbudowano również model dla statystycznego modelowania i algorytmy dla realizacji rzeczywisto-znaczeniowych pól losowych z rozkładem Gaussa i ograniczonym spektrum. Oceny aproksymowania średniokwadratowego takich pól losowych przez te modele na podstawie wyników Yadrenko M., Gamaliy O., 1998; Olenko A., 2004 i Olenko A., Pogany T., 2005 zostały znalezione. Stosowanie tych twierdzeń, modelów i algorytmów umożliwia generowanie na komputerze adekwatnych realizacji takich pól losowych z niektórymi znanymi przykładami funkcji korelacji dla danych przestrzenno-czasowych. Uzyskane wyniki są kontynuacją badań opublikowanych w pracach Vyzhva S., Vyzhva Z., 2003; Vyzhva Z., 2011 i Vyzhva Z., Kendzera O., Fedorenko K., Vyzhva A., 2012 do modelowania i metody generowania szumów sejsmogramu na płaskim obszarze obserwacji (Vyzhva Z., 2012) i sejsmogramu w trójwymiarowym obszarze obserwacji (Vyzhva Z., 2013). Te badania są ważnymi dodatkami do metody Monte Carlo do stosowania w geologii (patrz Chiles J.P. i Delfiner P., 2009, itd.).

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