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THE DEGREE OF APPROXIMATION OF FUNCTIONS FROM EXPONENTIAL WEIGHT SPACES

RZĄD APROKSYMACJI FUNKCJI Z WYKŁADNICZYCH PRZESTRZENI WAGOWYCH

A b s t r a c t

This paper presents a study of the approximation properties of modified Szász-Mirakyan operators for functions from exponential weight spaces. We present theorems giving the degree of approximation by these operators using a modulus of continuity.

Keywords: linear positive operators, Bessel function, modulus of continuity, degree of approximation

S t r e s z c z e n i e

W artykule badamy aproksymacyjne własności zmodyfikowanych operatorów typu Szásza--Mirakyana dla funkcji z wykładniczych przestrzeni wagowych. Przedstawiamy twierdzenia podające rząd aproksymacji funkcji przez operatory tego typu, wykorzystując moduł ciągłości.

Słowa kluczowe: dodatni operator liniowy, funkcja Bessela, moduł ciągłości, rząd aproksymacji

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1. Introduction

Let us denote the set of all real-valued functions continuous on $\mathbb{R}_0 = [0;\infty)$ by $C(\mathbb{R}_0)$. In paper [6] we investigated Szász-Mirakyan type operators defined as follows

$$L_n^\nu(f;x) = \begin{cases} \sum_{k=0}^{\infty} q_{n,k}^\nu(x) f\left(\frac{2k}{n+p}\right), & x > 0; \\ f(0), & x = 0 \end{cases} \tag{1}$$

and

$$q_{n,k}^\nu(x) = \frac{1}{I_\nu(nx)} \frac{(nx)^{2k+\nu}}{2^{2k+\nu}\Gamma(k+1)\Gamma(k+\nu+1)},$$

where Γ is the gamma function and I_ν the modified Bessel function defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu}\Gamma(k+1)\Gamma(k+\nu+1)}.$$

Approximation properties of these operators in exponential weight spaces were studied. Such spaces were denoted by

$$E_p = \{f \in C(\mathbb{R}_0)\colon w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where w_p is the exponential weight function defined as follows

$$w_p(x) = e^{-px}, \ \ p \in \mathbb{R}_0 \tag{2}$$

for $x \in \mathbb{R}_0$.
In the spaces we introduced the norm

$$\|f\|_p = \sup\{w_p(x)|f(x)|\colon x \in \mathbb{R}_0\} \tag{3}$$

and we established ([6], Theorem 2.1) that operators L_n^ν are linear, positive, bounded and transform the space E_p into E_p.

In the present paper, we shall prove theorems giving a degree of approximation of functions from E_p by these operators. We use the weighted modulus of continuity of the first and the second order defined as follows,

$$\omega\left(f, E_p; t\right) = \sup\{\|\Delta_h f\|_p\colon\ h \in [0, t]\} \tag{4}$$

and

$$\omega^2\left(f, E_p; t\right) = \sup\left\{\left\|\Delta_h^2 f\right\|_p\colon\ h \in [0, t]\right\} \tag{5}$$

respectively, where

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

for $x, h \in \mathbb{R}_0$.

The note was inspired by the results of [8, 9] which investigate approximation problems for integral operators defined in weighted spaces. The considered method of proving the main theorems is also found in papers [1-4, 10].

2. Auxiliary results

The preliminary results, which we immediately obtained from papers [5-7] and definition (1), are recalled below.

Lemma 2.1 ([5], Lemma 8)
For all $\nu \in \mathbb{R}_0$ there exists a positive constant $M(\nu)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right| \le M(\nu), \ \ nx\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\right| \le M(\nu).$$

Through elementary calculations we get

Lemma 2.2 ([6], Lemma 2.2)
For all $n \in \mathbb{N}$, $\nu, p \in \mathbb{R}_0$ and $x \in \mathbb{R}_0$

$$L_n^\nu(1, x) = 1, \ \ L_n^\nu(t, x) = \frac{nx}{n+p}\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$L_n^\nu(t^2, x) = \left(\frac{nx}{n+p}\right)^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} + \frac{2nx}{(n+p)^2}\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$L_n^\nu(t-x,x) = x\left(\frac{n}{n+p}\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\right),$$

$$L_n^\nu((t-x)^2;x) = x^2\left(\left(\frac{n}{n+p}\right)^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} - \frac{2n}{n+p}\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + 1\right) + \frac{2nx}{(n+p)^2}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}.$$

Lemma 2.3 ([6], Lemma 2.5)
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in \mathbb{N}$, we have

$$\left\|L_n^\nu\left(1/w_p\,;\cdot\right)\right\|_p \leq M(\nu,p).$$

An obvious consequence of the above lemma and definition (3) is
Theorem 2.4 ([6], Theorem 2.1)
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in \mathbb{N}$ and $f \in E_p$, we have

$$\|L_n^\nu(f;\cdot)\|_p \leq M(\nu,p)\|f\|_p.$$

Applying Lemma 2.1 and Lemma 2.2, we obtain
Lemma 2.5
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$|L_n^\nu((t-x)^2;x)| \leq M(\nu,p)\frac{x(x+1)}{n}.$$

Lemma 2.6 ([6], Lemma 2.6)
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$w_p(x)\left|L_n^\nu\left(\frac{(t-x)^2}{w_p(t)};x\right)\right| \leq M(\nu,p)\frac{x(x+1)}{n}.$$

3. Approximation theorems

The following theorems estimate the weighted error of approximation for functions belonging to the spaces $E_p^k = \{f \in E_p : f', f'', \ldots, f^{(k)} \in E_p\}$, where $f^{(i)}$ is denoted the *i*-th derivative of f.

Theorem 3.1

For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu, p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $g \in E_p^1$, we have

$$w_p(x)|L_n^\nu(g;x) - g(x)| \le M(\nu,p)\|g'\|_p \sqrt{\frac{x(x+1)}{n}}.$$

Theorem 3.2

For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu, p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $f \in E_p$, we have

$$w_p(x)|L_n^\nu(f;x) - f(x)| \le M(\nu,p)\omega\left(f, E_p; \sqrt{\frac{x(x+1)}{n}}\right).$$

The proof for the above theorems is analogous to the proof of Theorem 4 and Theorem 5 which are detailed in paper [5].

Theorem 3.2 implies the following corollary.

Corollary 3.3

If $\nu, p \in \mathbb{R}_0$ and $f \in E_p$, then for all $x \in \mathbb{R}_0$

$$\lim_{n\to\infty}\{L_n^\nu(f;x) - f(x)\} = 0.$$

Moreover, the above convergence is uniform on every set $[x_1, x_2]$ with $0 \le x_1 < x_2$.

Remark 3.4

The above result can be achieved in a different way; see [7] for more details.

Analogous with papers [8, 9], we define operators H_n^ν to estimate the error of approximation by the second moduli of continuity (5).

$$H_n^\nu(f;x) = L_n^\nu(f;x) - f(L_n^\nu(t;x)) + f(x) \tag{6}$$

for $\nu, p \in \mathbb{R}_0$, $f \in E_p$ and $x \in \mathbb{R}_0$. By using Lemma 2.2 we obtain

$$H_n^\nu(f;x) = L_n^\nu(f;x) - f\left(\frac{nx}{n+p}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right) + f(x).$$

Observe that the operators are linear. Moreover, Lemma 2.2 allows us to write

$$H_n^\nu(1;x) = 1, \quad H_n^\nu(t-x;x) = 0. \tag{7}$$

Lemma 3.5
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in N$, $x \in \mathbb{R}_0$ and $g \in E_p^2$, we have

$$w_p|H_n^\nu(g;x) - g(x)| \leq M(\nu,p)\|g''\|_p \frac{x(x+1)}{n}.$$

Proof. Let $x \in \mathbb{R}_0$ and $g \in E_p^2$ be fixed. Through the use of the Taylor formula we can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\, du$$

for $t > 0$. By applying the linearity of H_n^ν and (7) we derive

$$|H_n^\nu(g;x) - g(x)| = |H_n^\nu(g(t)-g(x);x)| = \left|H_n^\nu\left(\int_x^t (t-u)g''(u)du\,;x\right)\right|. \tag{8}$$

Furthermore, the definition of H_n^ν implies

$$H_n^\nu\left(\int_x^t (t-u)g''(u)du\,;x\right)$$
$$= L_n^\nu\left(\int_x^t (t-u)g''(u)du\,;x\right) - \int_x^{L_n^\nu(t;x)} (L_n^\nu(t;x)-u)g''(u)du$$

Estimating (8), we have

$$|H_n^\nu(g;x) - g(x)|$$
$$\leq L_n^\nu\left(\left|\int_x^t (t-u)g''(u)du\right|;x\right) + \left|\int_x^{L_n^\nu(t;x)} (L_n^\nu(t;x)-u)g''(u)du\right|.$$

Since

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \frac{1}{2}\|g''\|_p (t-x)^2(e^{px}+e^{pt})$$

and

$$\left|\int_x^{L_n^\nu(t;x)} (L_n^\nu(t;x)-u)g''(u)du\right| \leq \frac{1}{2}\|g''\|_p (L_n^\nu(t;x)-x)^2\left(e^{px}+e^{pL_n^\nu(t;x)}\right)$$
$$\leq \frac{1}{2}\|g''\|_p \left(L_n^\nu(t-x;x)\right)^2 e^{px}\left(1+e^{pL_n^\nu(t-x;x)}\right)$$
$$\leq \frac{1}{2}M(\nu,p)\|g''\|_p \left(L_n^\nu(t-x;x)\right)^2 e^{px}$$

we get

$$w_p(x)|H_n^\nu(g;x)-g(x)|$$
$$\leq \frac{1}{2}\|g''\|_p L_n^\nu((t-x)^2;x) + \frac{1}{2}\|g''\|_p w_p(x) L_n^\nu\left(\frac{(t-x)^2}{w_p(t)};x\right)$$
$$+\frac{1}{2}M(\nu,p)\|g''\|_p \left(L_n^\nu(t-x;x)\right)^2$$

Applying Hölder's inequality to the term $L_n^\nu(t-x;x)$ and Lemmas 2.5, 2.6, we obtain the desired estimation.

Theorem 3.6
For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu,p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $f \in E_p$, we have

$$w_p|L_n^\nu(f;x)-f(x)|$$
$$\leq M(\nu,p)\omega^2\left(f,E_p;\sqrt{\frac{x(x+1)}{n}}\right) + \omega\left(f,E_p;|L_n^\nu(t-x;x)|\right).$$

Proof. Let $x \in \mathbb{R}_0$ and f_h be the second order Steklov mean of $f \in E_p$, i.e.

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left\{2f(x+s+t)-f\big(x+2(s+t)\big)\right\}dsdt, \qquad x \in \mathbb{R}_0, h>0.$$

Notice that

$$f(x)-f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\Delta_{s+t}^2 f(x)dsdt.$$

By definitions (3) and (5), we get the following estimations

$$\|f-f_h\|_p\le\omega^2(f,E_p;h)$$

and since

$$f_h''(x)=\frac{1}{h^2}\left(8\Delta_{h/2}^2 f(x)-\Delta_h^2 f(x)\right)$$

we can write

$$\|f_h''\|_p\le\frac{9}{h^2}\omega^2(f,E_p;h).$$

The above inequalities imply that the Steklov mean f_h and f_h'' belong to E_p. Moreover, by linearity of L_n^ν and connection (6), we have

$$\begin{aligned}|L_n^\nu(f;x)-f(x)|&\\ &\le H_n^\nu(|f-f_h|;x)+|f(x)-f_h(x)|+|H_n^\nu(f_h;x)-f_h(x)|\\ &+\left|f\left(L_n^\nu(t;x)\right)-f(x)\right|.\end{aligned}$$

Applying the above estimation, Theorem 2.1 and Lemma 3.5, we conclude that

$$\begin{aligned}w_p(x)|L_n^\nu(f;x)-f(x)|&\\ &\le w_p(x)H_n^\nu(|f-f_h|;x)+w_p(x)|f(x)-f_h(x)|\\ &+w_p(x)|H_n^\nu(f_h;x)-f_h(x)|+w_p(x)\left|f\left(L_n^\nu(t;x)\right)-f(x)\right|\\ &\le M(\nu,p)\|f-f_h\|_p+M(\nu,p)\|f_h''\|_p\frac{x(x+1)}{n}\\ &+w_p(x)\left|f\left(L_n^\nu(t;x)\right)-f(x)\right|\\ &\le M(\nu,p)\omega^2\left(f,E_p;h\right)\left(1+\frac{1}{h^2}\frac{x(x+1)}{n}\right)\\ &+\omega\left(f,E_p;|L_n^\nu(t-x;x)|\right),\end{aligned}$$

where $L_n^\nu(t-x;x)=x\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-x$. Substituting $h=\sqrt{\frac{x(x+1)}{n}}$, we get the assertion of our theorem.

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