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HAUSDORFF LIMITS OF ONE PARAMETER FAMILIES OF DEFINABLE SETS IN *O*-MINIMAL STRUCTURES

GRANICE HAUSDORFFA JEDNOPARAMETROWYCH RODZIN ZBIORÓW DEFINIOWALNYCH W STRUKTURACH *O*-MINIMALNYCH

A b s t r a c t

We give an elementary proof of the following theorem on definability of Hausdorff limits of one parameter families of definable sets: let $A \subset \mathbb{R} \times \mathbb{R}^n$ be a bounded definable subset in *o*-minimal structure on $(\mathbb{R}, +, \cdot)$ such that for any $y \in (0, c)$, $c > 0$, the fibre $A_y := \{x \in \mathbb{R}^n : (y, x) \in A\}$ is a Lipschitz cell with constant *L* independent of *y*. Then the Hausdorff limit $\lim_{y \to 0} \overline{A}_y$ exists and is definable.

Keywords: Hausdorff limit, definable sets, o-minimal structure

S t r e s z c z e n i e

W prezentowanej pracy przedstawiamy elementarny dowód następującego twierdzenia o definiowalności granicy Hausdorffa jednoparametrowej rodziny zbiorów definiowalnych: niech $A \subset \mathbb{R} \times \mathbb{R}^n$ będzie ograniczonym zbiorem definiowalnym w strukturze *o*-minimalnej typu $(\mathbb{R}, +, \cdot)$ takim, że dla dowolnego $y \in (0, c)$, $c > 0$, wókno $A_y := \{x \in \mathbb{R}^n : (y, x) \in A\}$ jest komórką Lipschitza ze stałą *L* niezależną od *y*. Wtedy granica Hausdorffa $\lim_{y \to 0} \overline{A}_y$ istnieje i jest definiowalna.

Słowa kluczowe: granica Hausdorffa, zbiory definiowalne, struktury o-minimalne

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1. Introduction

In [1] Bröcker proved that for any family of semialgebraic sets A_y and any convergent sequence y_ν of parameters the Hausdorff limit of A_{y_ν} exists and is semialgebraic. In [3] a short geometric proof of the generalization of Bröcker's result to the case of sets definable in an *o*-minimal structure was given.

The aim of this paper is to present an elementary proof of the following one-parameter case of this result

Theorem 1. *Let* $A \subset \mathbb{R} \times \mathbb{R}^n$ *be a definable subset in an o-minimal structure on* $(\mathbb{R}, +, \cdot)$ *such that for any* $y \in (0, c)$, $c > 0$, *the fibre* $A_y := \{x \in \mathbb{R}^n : (y, x) \in A\}$ *is a bounded Lipschitz cell with constant L independent of y. Then the Hausdorff limit* $\lim_{y \to 0} \overline{A}_y$ *exists and is definable.*

For the convenience of the reader we present in Section 2 results on Hausdorff distance and *o*-minimal structure that we use in the proof of the main result.

2. Preliminaries

2.1. Hausdorff distance.

Let (X, d) be a complete metric space, denote by $\mathcal{C}(X)$ the space of all non-empty compact subsets in X.

Definition 1. *For any two sets* $Y_1, Y_2 \in \mathcal{C}(X)$ *we define Hausdorff distance as*

$$d_H(Y_1, Y_2) = \max\{\max_{x \in Y_1} \min_{y \in Y_2} d(x, y), \max_{y \in Y_2} \min_{x \in Y_1} d(x, y)\}$$

Remark 1. *Hausdorff distance of two sets is the infimum of positive numbers* ε > 0 *such that each of them is contained in the* ε*-envelope of the other, i.e.*

$$d_H(Y_1, Y_2) = \inf\{\varepsilon > 0;\ Y_2 \subseteq B(Y_1, \varepsilon) \text{ and } Y_1 \subseteq B(Y_2, \varepsilon)\}$$

where

$$B(Z, \varepsilon) = \bigcup_{z \in Z} B(z, \varepsilon)$$

for any $Z \in \mathcal{C}(X)$ and $\varepsilon > 0$.

Remark 2. *Observe that the function* $\tilde{d} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}_+$ *defined by the following formula*

$$\tilde{d}(Y_1, Y_2) := \max\{\tilde{d}(x, Y_2) : x \in Y_1\}, \quad \text{for} \quad Y_1, Y_2 \in \mathcal{C}(X)$$

where

$$\tilde{d}(x, Y) := \min\{d(x, y) : y \in Y\}, \quad \text{for} \quad x \in X, Y \in \mathcal{C}(X)$$

cannot be used to define a metric on $\mathcal{C}(X)$ as in general the function $\tilde{d}$ is not symmetric, we have only the following

$$d_H(Y_1,Y_2) = \max\{\tilde{d}(Y_1,Y_2),\ \tilde{d}(Y_2,Y_1)\} \quad \text{for} \quad Y_1Y_2 \in \mathcal{C}(X).$$

Example 2. *Let* $Y_1 = (0,15)$ *and* $Y_2{:=}[8,112] \times \{0\}$, *then*

$$\tilde{d}(Y_1,Y_2) = 17 = 113 = \tilde{d}(Y_2,Y_1).$$

By definition, in this example we have $d_H(Y_1, Y_2)=113$.

We end this section with the following characterization of convergence in Hausdorff metric.

Theorem 3. *Let* X *be a compact metric space,* $A, A_\nu \in \mathcal{C}(X)$, $\nu = 1,2,3,\ldots$. *Then the sequence* A_ν *converges to* A *in Hausdorff metric* $(A_\nu \longrightarrow A)$ *iff the following two conditions hold*

1) $(x_{\nu_k} \in A_{\nu_k}, x_{\nu_k} \longrightarrow x_0, \nu_1 < \nu_2 < \nu_3 < \ldots) \Rightarrow x_0 \in A,$

2) $x_0 \in A \Rightarrow \exists x_\nu \in A_\nu$ *such that* $x_\nu \longrightarrow x_0.$

Proof. First we shall prove that conditions 1) and 2) are necessary for the convergence in Hausdorff metric.

Assume that $A_\nu \longrightarrow A,$ since X is a compact set we can find a sequence $x_{\nu_k} \in A_{\nu_k}$ (with $\nu_1 < \nu_2 < \nu_3 < \ldots$) such that $x_{\nu_k} \longrightarrow x_0$ for some poin $x_0 \in X$. We want to show that $x_0 \in A$. Since the set A is compact and $x_{\nu_k} \in A_{\nu_k}$ there exists $y_{\nu_k} \in A$ such that

$$d(x_{\nu_k}, y_{\nu_k}) = \tilde{d}(x_{\nu_k}, A) \le d_H(A_{\nu_k}, A) \to 0$$

Therefore $d(x_{\nu_k}, y_{\nu_k}) \longrightarrow 0.$ We shall show that $\tilde{d}(x_0, A) = 0.$ Observe that

$$\tilde{d}(x_0, A) \le d(x_0, y_{\nu_k})$$

As $y_{\nu_k} \in A$ and consequently

$$d(x_0, y_{\nu_k}) \le d(x_0, x_{\nu_k}) + d(x_{\nu_k}, y_{\nu_k}).$$

Therefore $\tilde{d}(x_0, A) = 0$ and $x_0 \in \overline{A} = A.$

Assume that $A_\nu \longrightarrow A$ and $x_0 \in A$. To prove that condition 2) is necessary fix a point $x_\nu \in A_\nu$ for $\nu = 1,2,\ldots$ such that $d(x_0, x_\nu) = \tilde{d}(x_0, A_\nu).$ Then

$$0 \le d(x_0,x_\nu) = \tilde{d}(x_0, A_\nu) \le \tilde{d}(x_0, A_\nu) \le d_H(A, A_\nu) \longrightarrow 0$$

implies $d(x_0, x_\nu) \to 0$.

Now, we shall prove the opposite implication. Assume to the contrary that conditions 1) and 2) hold while the sequence (A_ν) does not converge to A. Then there exists $\varepsilon > 0$ such that $d_H(A_\nu, A) > \varepsilon$ for infinitely many ν. Consequently at least one of the inequalities

$$\tilde{d}(A_\nu, A) > \varepsilon \quad \text{or} \quad \tilde{d}(A, A_\nu) > \varepsilon$$

holds for infinitely many ν.

In the first case there exist $\nu_1 < \nu_2 < \ldots$ and $x_{\nu_k} \in A$ such that $\tilde{d}(x_{\nu_k}, A) > \varepsilon$, since X is compact replacing x_{ν_k} by a subsequence we can also assume that x_{ν_k} converges to a point $x_0 \in X$. From condition 1) we get $x_0 \in A$ which contradicts $\tilde{d}(x_{\nu_k}, A) > \varepsilon$.

In the second case for infinitely many ν there exists $y_\nu \in A$ such that $\tilde{d}(y_\nu, A_\nu) > \varepsilon$, by compactness of A there exists a sequence $\nu_1 < \nu_2 < \ldots$ such that $\tilde{d}(y_{\nu_k}, A_{\nu_k}) > \varepsilon$ and $y_{\nu_k} \longrightarrow x_0$ for some $x_0 \in A$. By condition 2) there exists $x_{\nu_k} \in A_{\nu_k}$ such that $x_{\nu_k} \longrightarrow x_0$. In this situation we have

$$\varepsilon < \tilde{d}(y_{\nu_k}, A_{\nu_k}) \le d(y_{\nu_k}, x_{\nu_k}) \le d(y_{\nu_k}, x_0) + d(x_0, x_{\nu_k}) \longrightarrow 0$$

which is a contradiction.

□

Remark 3. The above theorem does not hold without the assumption that X is a compact space.

Example 4. Let X be any non-compact complete space, fix $x_0 \in X$, let $x_\nu \in X$ be a sequence that does not contain any convergent subsequence. Put $A := \{x_0\}$, $A_\nu = \{x_0, x_\nu\}$. Then conditions 1) and 2) hold true but the sequence A_ν does not converge in Hausdoff metric.

2.2. *o*-minimal structures.

We shall collect here the basic definitions and properties of ***o***-minimal structures that are crucial for our further considerations. For a detailed exposition of ***o***-minimal structures we refer the reader to [2].

Definition 2. A structure $\mathcal{S}$ on $\mathbb{R}$ consists of a collection $\mathcal{S}_n$ of subsets of $\mathbb{R}^n$, for each $n \in \mathbb{N}$, such that

1. $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$,
2. $\mathcal{S}_n$ contains the diagonals $d(x_0, x\{(x_1, \ldots x_n) \in \mathbb{R}^n : x_i = x_i\}$ for $1 \le i < j \le n$,
3. if $A \in \mathcal{S}_{n+1}$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathcal{S}_{n+1}$,
4. if $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.

We say that a set $A \subset \mathbb{R}^n$ is *definable* if and only if $A \in \mathcal{S}_n$. A function $f\colon A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$ is called *definable* if and only if its graph is definable.

Definition 3. A structure $\mathcal{S}$ on $\mathbb{R}$ is *o-minimal* if and only if

1. $\{(x,y): x<y\} \in \mathcal{S}_2$ and $\{a\} \in \mathcal{S}_1$ for each $a \in \mathbb{R}$,
2. each set in is a finite union of intervals (a,b), $-\infty \le a < b \le +\infty$, and points $\{a\}$.

A *structure on* $(\mathbb{R},+,\cdot)$ is a structure on $\mathbb{R}$ containing the graphs of both addition and multiplication.

The main technical tool used in the studies of geometry of sets definable in *o*-minimal structures is the cell decomposition. The notions of a cell and that of a cell decomposition are defined inductively.

Definition 4. The *cells* in $\mathbb{R}^1$ exactly are points and open intervals.

A definable set $C \subset \mathbb{R}^n$, where $n > 1$, is a *cell* if its image $\pi(C) \subset \mathbb{R}^{n-1}$ by the projection $\pi: \mathbb{R}^n \ni (x_1,\ldots,x_{n-1},x_n) \longrightarrow (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}$ is a cell and C is one of the following two types:

either

$$C = \Gamma(f) = \{(x',x_n) \in \pi(C) \times \mathbb{R} : x_n = f(x')\}$$

(and then C is called a *graph*)

or

$$C = (g_1,g_2) := \{(x',x_n) \in \pi(C) \times \mathbb{R} : g_1(x') < x_n < g_2(x')\}$$

(and then C is called a *band*),

where f: $\pi(C) \to \mathbb{R}$ is a continuous definable function (resp. $g_1, g_2 : \pi(C) \to \overline{\mathbb{R}}$ are functions such that $g_1 < g_2$ on $\pi(C)$ and, for each $i \in \{1,2\}$, g_i is either a continuous definable function g_i: $\pi(C) \to \mathbb{R}$ or g_i is identically equal to $-\infty$, or else g_i is identically equal to $+\infty$).

A cell C is called a $\mathcal{C}^k$*-cell* (where $k \in \mathbb{N} \cup \{\infty\}$), if $\pi(C)$ is a $\mathcal{C}^k$-cell and f (resp. g_i, $i = 1,2$ if finite) is a $\mathcal{C}^k$-function. Notice that every $\mathcal{C}^k$-cell is a $\mathcal{C}^k$-submanifold of $\mathbb{R}^n$.

Definition 5. A *cell decomposition* of $\mathbb{R}^1$ is a finite collection of open intervals and points of the following form:

$$\{(-\infty,a_1),(a_1,a_2),\ldots,(a_k,+\infty),\{a_1\},\ldots,\{a_k\}\},$$

where $a_1 < a_2 < \ldots < a_k$ are real numbers.

A *cell decomposition* of $\mathbb{R}^n$ $(n > 1)$ is a finite partition $\mathcal{C}$ of $\mathbb{R}^n$ into cells such that the set of all projections $\{\pi(C)\colon C \in \mathcal{C}\}$ is a cell decomposition of $\mathbb{R}^{n-1}$, where π: $\mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection on the first $n-1$ coordinates as in Definition 4.

Theorem 5. *Let (X,d) be a compact metric space, f_n: $X \to \mathbb{R}$ be a sequence of Lipschitz continuous functions with a common Lipschitz constant $M > 0$. Then the sequence (f_n) converges uniformly to a function f_0 if and only if their graphs converge to the graph of f_0 in Hausdorff metric.*

Moreover, $f_0 = \lim\limits_{n\to\infty} f_n$ is a Lipschitz function with the Lipschitz constant M.

Proof. Let us notice that if $f_n \rightrightarrows f_0$ then f_0 is a Lipschitz function with constant M.

$$\left|f_0(x)-f_0(y)\right| = \lim_{n\to\infty}\left|f_n(x)-f_n(y)\right| \leq \lim_{n\to\infty} M\cdot d(x,y) = M\cdot d(x,y).$$

We will prove that

$$d_H(graph\ f_0,\ graph\ f_n) \leq \|f_n - f_0\| \leq (M+1)\cdot d_H(graph\ f_0,\ graph\ f_n).$$

First we shall show the first of the inequalities:

$$d_H(graph\ f_0,\ graph\ f_n) \leq \|f_n - f_0\|.$$

$$d_H(graph\ f_0,\ graph\ f_n) = \max\{\tilde{d}(graph\ f_0,\ graph\ f_n),\ \tilde{d}(graph\ f_n,\ graph\ f_0)\}$$

As the inequality is symmetric with respect to f_0 and f_n, we may assume that $\tilde{d}(graph\ f_0,\ graph\ f_n) \geq \tilde{d}(graph\ f_n,\ graph\ f_0)\}$ and then

$$\begin{aligned} d_H(graph\ f_0,\ graph\ f_n) &= \tilde{d}(graph\ f_0,\ graph\ f_n) = \\ &= \max\{x\in X : \tilde{d}((x,f_0(x)),\ graph\ f_n)\} \leq \\ &\leq \max\{x\in X : d((x,f_0(x)),(x,f_n(x))\} = \\ &= \max\{x\in X : \left|f_0(x)-f_n(x)\right|\} = \|f_0 - f_n\| \end{aligned}$$

Now we shall show that

$$\|f_n - f_0\| \leq (M+1)\cdot d_H(graph\ f_0,\ graph\ f_n)$$

Fix $x\in X$ and let $y \in X$ such that

$$\begin{aligned} d_H(graph\ f_0,\ graph\ f_n) &\geq \tilde{d}((x,f_0(x)),(y,f_n(y)) = \\ &= d(x,y) + \left|f_0(x)-f_n(y)\right| \geq \tilde{d}((x,f_0(x)),\ graph\ f_n) \end{aligned}$$

Consequently

$$\begin{aligned} \left|f_n(x)-f_0(x)\right| &\leq \left|f_n(x)-f_n(y)\right| + \left|f_n(y)-f_0(x)\right| \leq \\ &\leq M\cdot d(x,y) + d_H(graph\ h\,f_0,\ graph\ f_n) \leq \\ &\leq M\cdot d_H(graph\ f_0,\ graph\ f_n) + d_H(graph\ f_0,\ graph\ f_n) = \\ &= (M+1)\cdot d_H(graph\ f_0,\ graph\ f_n) \end{aligned}$$

and taking the limits we conclude the proof.

□

3. Proof of the main result

Let us start with some technical results on extending Lipschitz functions

Lemma 6. *Let* $F:(0,1)\times\mathbb{R}^n \to \mathbb{R}$ *be a bounded definable map such that for any* $y \in (0,1)$ *the restriction* $F_y : \mathbb{R}^n \ni x \longrightarrow F(y,x) \in \mathbb{R}$ *satisfies the Lipschitz condition with* *a constant independent of y. Then for any $a \in \mathbb{R}^n$ the limit* $\lim_{(y,x)\to(0,a)} F(y,x)$ *exists and defines a definable extension of F to a function* $\tilde{F}:[0,1)\times\mathbb{R}^n \to \mathbb{R}$.

Proof. For any $a \in \mathbb{R}^n$ the function $(0,1) \ni y \longrightarrow F(y,a)$ is definable, so there exists the limit $\tilde{F}(0,a) := \lim_{y\to 0} F(y,a)$. Now, $\left|F(y,x)-\tilde{F}(0,a)\right| \le \left|F(y,x)-F(y,a)\right| + \left|F(y,a)-\tilde{F}(0,a)\right| \le L\left|x-a\right| + \left|F(y,a)-\tilde{F}(0,a)\right|$, hence the limit in question exists. Since, the graph of $\tilde{F}$ is the closure of *graph* (F), the function $\tilde{F}$ is definable.

□

Lemma 7 (Banach–McShane–Whitney extension theorem, [6]). *Let f: $S \to \mathbb{R}$ be L-lipschitz function on the subset S in a metric space X. Then the formula*

$$F(x) := \sup\{f(x') - L\cdot d(x,x') : x' \in S\}$$

For $x \in X$ defines the extension of the function f such that F: $X \to \mathbb{R}$ is L-lipschitz.

Now, we are in a position to give the proof of our main result

Proof of Theorem 1. Induction with respect to n. For $n = 0$ it is obvious. Let A_1 be the projection of A onto $\mathbb{R} \times \mathbb{R}^{n-1}$, by the inductive hypothesis the limit $A_0 := \lim_{y\to 0} \overline{(A_1)_y}$ exists and is definable. Without loss of generality we may assume that $\dim(A_1)_y$ and $\dim(A_y)$ is constant for $y \in (0,c)$, so all cells A_y are of the same type (a graph or a band).

If all fibres are graphs, there exists a definable function F: $A_1 \to \mathbb{R}$ such that $A = graph\,(F)$, for any $y \in (0,c)$, the function F_y is Lipschitz with a constant L independent of y. Using lemmas 6 and 7 we can extend this function to a definable function $\tilde{F}:[0,c)\times\mathbb{R}^n \to \mathbb{R}$, set $\tilde{F}_0(x) := \tilde{F}(0,x)$, for $x \in \mathbb{R}^n$.

Let $C := graph(\tilde{F}_0 \big| A_0)$, we shall show $\lim_{y\leftrightarrow 0} A_y = C$. Let $y_v \in (0,c)$ be a sequence such that $y_\nu \longrightarrow 0$, let $x_\nu \in A_{y_\nu}$, $x_\nu \longrightarrow x_0$ be a convergent sequence, we shall prove that $x \in C$. Let $x_\nu = (x'_\nu, x_n^\nu)$ and $x_0 = (x'_0, x_n^0)$. We have $(y_\nu, x'_\nu) \in (A_1)_{y_\nu}$, so $x'_0 \in A_0$. By the definition $\tilde{F}_0(x'_0) = \lim_{\nu\to\infty} F(y_\nu, x'_\nu) = \lim_{\nu\to\infty} x_n^\nu = x_n^0$, hence $x \in C$.

Now, let $x \in C$ and $y_v \in (0,c)$ be a sequence such that $y_\nu \longrightarrow 0$. Since $x'_0 \in A_0$, $x_n^0 = \tilde{F}_0(x'_0)$ there is $x'_\nu \in (A_1)_{y_\nu}$ such that $x'_\nu \longrightarrow x'_0$. Put $x_n^\nu = F(y_n, x_{\nu'})$, we get $x_\nu \in A_{y_\nu}$ and $x_n^\nu = F(y_\nu, x'_\nu) \longrightarrow \tilde{F}(0,x'_0) = \tilde{F}_0(x'_0) = x_n^0$. Consequently we have $x_\nu \longrightarrow x_0$ which proves $\lim_{y\to 0} A_y = C$.

If is a band for $y \in (0,c)$ proceeding in a similar way, we have $A = (G,H)$, where $G,H : A_1 \longrightarrow \mathbb{R}$ and define $\tilde{G}_0$, $\tilde{H}_0$. We shall show that

$$C:\{x\in\mathbb{R}^n : x'\in A_0, \tilde{G}_0(x')\le x_n\le \tilde{H}_0(x')\}$$

is the Hausdorff limit of A_y as $y\longrightarrow 0,\ y\in(0,c)$.

Let $y_\nu\in(0,c)$ be a sequence such that $y_\nu\longrightarrow 0$, let $x_\nu\in A_{y_\nu}, x_\nu\longrightarrow x_0$. Let $x_\nu=(x'_\nu, x_n^\nu)$ and $x_0=(x'_0, x_n^0)$. We have $(y_\nu, x'_\nu)\in(A_1)_{y_\nu}$, so $x'_0\in A_0$. By the definition $\tilde{G}_0(x'_0)=\lim_{\nu\to\infty}G(y_\nu,x'_\nu)$, $\tilde{G}_0(x'_0)=\lim_{\nu\to\infty}G(y_\nu,x'_\nu)$ so

$$\tilde{G}_0(x'_0)\le x_n^0\le \tilde{H}_0(x'_0)$$

and hence $x_0\in C$.

Now, fix $x_0\in C$ and $y_\nu\in(0,c)$ such that $y_\nu\longrightarrow 0$. We have $x'_0\in A_0$ and $\tilde{G}_0(x'_0)\le x_n^0\le\tilde{H}_0(x'_0)$. There exists $x'_\nu\in(A_1)_{y_\nu}$ such that $x'_\nu\longrightarrow x'_0$.

If $\tilde{G}_0(x')=\tilde{H}_0(x')$ put $x_n^\nu=\frac{1}{2}(G(y_\nu,x'_\nu),H(y_\nu,x'_\nu))$. If $\tilde{G}_0(x')<\tilde{H}_0(x')$ put

$$x_n^\nu=\frac{x_n^0-\tilde{G}_0(x'_0)}{(\tilde{H}_0(x'_0)-\tilde{G}_0(x'_0))}(H(y_\nu,x'_\nu)-G(y_\nu,x_\nu))+G(y_\nu,x_\nu).$$

Then $x_\nu\in A_{y_\nu}$ and $x_\nu\longrightarrow x_0$.

□

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