TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
CIVIL ENGINEERING | BUDOWNICTWO

3-B/2013

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OPTIMAL MODELLING OF PRESTRESSED GIRDERS USING THE GRADIENT-ITERATIVE METHOD

OPTYMALNE KSZTAŁTOWANIE DŹWIGARÓW SPRĘŻONYCH METODĄ GRADIENTOWO-ITERACYJNĄ

A b s t r a c t

The paper describes the gradient-iterative optimization method, outlines the method's basic assumptions and illustrates its general use. The method's implementation was illustrated with the help of a prestressed beam. Calculations were made to illustrate a girder prestress and height optimization. The method makes it possible to quickly obtain optimal results using universally-available programming. In addition, the method makes it possible to find optimal solutions without the use of complicated mathematical formulas. The article solved the problem of a statically indeterminate prestressed beam with three decision variables, thus proving that the gradient-iterative method is both an efficient and quick optimization method. To illustrate the effectiveness of the optimization method calculations were performed for double- and three-span beam.

Keywords: gradient-iterative method, structure optimization, prestressed beams

S t r e s z c z e n i e

W artykule przedstawiono gradientowo-iteracyjną metodę optymalizacji. Zostały opisane podstawowe założenia metody oraz pokazano jej ogólny sposób stosowania. Na przykładzie dźwigara sprężonego pokazano zastosowanie prezentowanej metody. Opisany przykład obliczeniowy dotyczy optymalnego doboru sprężenia oraz wysokości belki. Metoda umożliwia szybkie uzyskanie rozwiązania optymalnego przy wykorzystaniu ogólnodostępnego oprogramowania. Dodatkowo metoda pozwala na znalezienie rozwiązania optymalnego bez konieczności stosowania skomplikowanego opisu matematycznego. Rozwiązany w pracy problem statycznie niewyznaczalnej belki sprężonej przy założeniu trzech zmiennych decyzyjnych dowodzi skuteczności i szybkości metody gradientowo-iteracyjnej obliczeń optymalizacyjnych. W celu pokazania skuteczności opisanej metody obliczeń optymalizacyjnych, przeprowadzono obliczenia dla belki dwu i trójprzęsłowej.

Słowa kluczowe: metoda gradientowo-iteracyjna, optymalizacja konstrukcji, belki sprężone

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Symbols

$\sigma_{d.s}$, $\sigma_{g.s}$ – accordingly: set of normal bend values for each calculation combination for: bottom and top edge of cross-section

a – set vertical displacement
a_{dop} – maximum admissible vertical displacement
b – beam width
$f_{e.i}$ – nodal forces vector for *i*-th finite element
g – dead external load
h – beam hight
h_p – distance of prestressed cable's path
n – number of finite elements
q – linear load
q_2, q_3, q_1 – live load

$A(x)$ – the cross-section field for the element length function
B_i – Boolean matrix for *i*-th finite element
$F_{I \div VI}$ – load phase
G_1 – restriction of maximum compression stress on top edge of cross-section
G_2 – restriction of maximum tensile stress on top edge of cross-section
G_3 – restriction of maximum compression stress on bottom edge of cross-section
G_4 – restriction of maximum tensile stress on bottom edge of cross-section
G_5 – restriction of maximum beam bend
K_{wb} – stiffness matrix incorporating boundary conditions
L – element's overall length
L_1 , L_2 , L_3 – span length
L_{ES} – length of finite element
$K_{e.i}$ – stiffness matrix calculated in accordance with equation (3) for *i*-th finite element
N_p – prestress force
Q – nodal displacement vector
R – reaction vector
S_{wb} – nodal load vector with boundary conditions
$Z_{e.i}$ – vector of alternatives for linear load calculated according with equation (8) for every *i*-th finite element.

1. Introduction

The use of prestress technology allows for the design of elements with great spans which significantly spread out load. Due to considerable prestress costs, the design of prestressed elements should encompass a search for optimal economic solutions.

Designed elements must fulfill all capacity and utility specifications and must simultaneously meet optimal economic demands. In most cases an optimally-modeled element will weigh as little as possible as this minimizes consumption of materials, thus also minimizing production costs.

The article outlines optimal modeling of prestressed girders. The gradient-iterative method was utilized to pinpoint optimal solutions.

The article is an example of a new use of the gradient-iterative method for optimization calculations; it allows rapid selection of optimal solutions.

2. Calculation method and its groundwork

Literature pertaining to construction optimization contains solutions which utilize mathematical methods of optimal control. Methods based on refined control theory make it possible to find optimal solutions or set boundary conditions and restrictions. Optimization based on the maximum principle gives good results but has two significant, design-related, restrictions:

- The necessity to translate the complexity of optimization into mathematical terms. Such a description does not correlate with the manner in which bearing capacity is verified in practice,
- Lack of efficient or intuitive programming used to find solutions to complex problems formulated in categories of complete control.

The optimization method presented in this article makes it possible to quickly find an optimal solution without the necessity of utilizing difficult to obtain numeric programs which calculate multipoint boundary value problems.

The method combines the gradient descent method and an iterative solution to the formulated optimization problem. The method can be expressed in six points:

(1) A mathematical formula of functions describing the assessed optimization task.
(2) Determining the objective function and decision variables.
(3) Determining optimization restrictions.
(4) Determining the optimization starting point and the direction of the sought solution.
(5) A description of the increment function.
(6) Iteratively obtaining a solution which fulfills optimization criteria.

The described method can be used to solve various optimization problems with the help of universally-available software. Furthermore, it is considerably quicker than other optimization methods.

In conjunction with the finite-element method, the gradient-iterative method can be used to find very quick optimization solutions for statically undetermined constructions.

Simple functions which thoroughly describe the optimization problem are all that is required to formulate a task for the gradient-iterative method. As a result of these simple mathematical formulas, numerical calculations can be carried out quickly.

When describing the problem, it is extremely important to correctly determine the increment function. Incorrect input can lengthen calculations considerably, and in the worst case can even make it impossible to receive a reliable result.

3. A general description of the problem

The task involves optimal modeling of prestressed beams with a rectangular cross-section and linear path for prestressed cables. Fig. 1–2 below show a static diagram and assumed load phases.

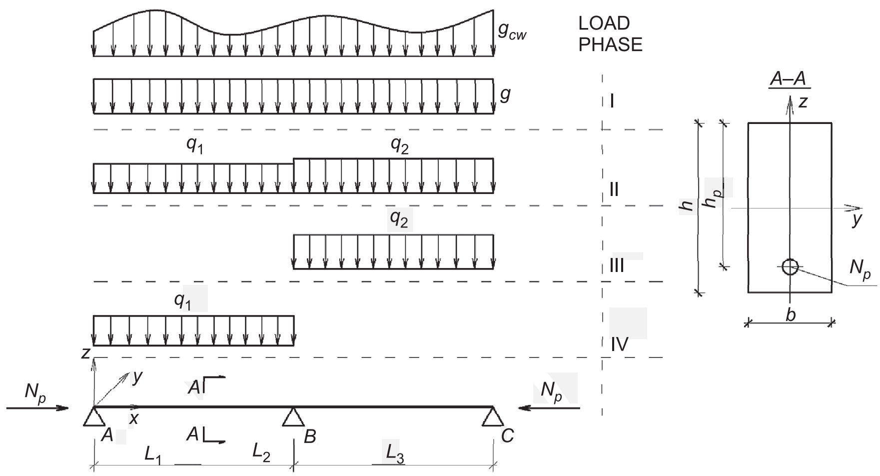

Fig. 1. Static diagram, cross-section and configuration of external forces for a double-span beam

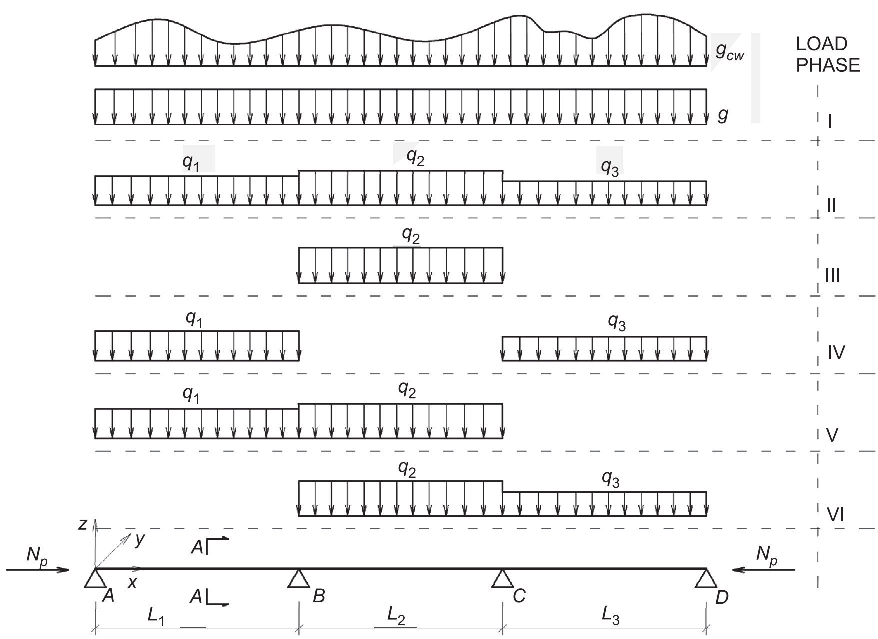

Fig. 2. Static diagram and configuration of external forces for a three-span beam

Load phase I incorporates the element's own weight g_{cw} and dead load g. Phases II-VI illustrate live load in various calculated cases.

The analyzed example incorporates 4 (in the case of double-span beams) and 6 (in the case of three-span beams) calculation combinations of each load phase:

$$\begin{aligned} C1 &= F_I \\ C2 &= F_I + F_{II} \\ C3 &= F_I + F_{III} \\ C4 &= F_I + F_{IV} \\ C5 &= F_I + F_V \\ C6 &= F_I + F_{VI} \end{aligned} \tag{1}$$

where:

$F_{\mathrm{I \div VI}}$ – load phase

4. Calculation procedure

It is necessary to establish optimal cross-section dimensions which will minimize the assumed objective function (the element's volume).

$$V = \int_0^L A(x)dx \tag{2}$$

where:

$A(x)$ – the cross-section field for the element length function

L – element's overall length

Optimization pertains to the selection of three decision variables:

- Fixed prestress force along the length of the element (prestress force after considering all loses)
- Fixed resultant distance h_p of prestressed cable's path from upper edge of cross-section (linear path)
- Cross-section's height variable along the girder's axis

The optimal solution minimizes the objective function (2) and fulfills all assumed optimization restrictions.

4.1. Finite-element method in the analysed example

The beam was discretized into finite elements with constant stiffness *EI(h)* and fixed length L_{ES} (Fig. 2). When discretizing a beam with the help of multiple finite elements, the assumption that one finite element has a fixed moment of inertia does not lead to significant calculation errors.

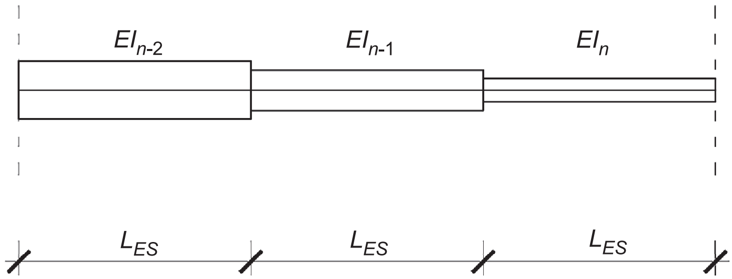

Fig. 3. Beam discretization

The stiffness matrix was defined (3) and a Boolean matrix was constructed for *n* finite elements (4).

$$k(EI,L)=\begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \tag{3}$$

$$B_i = \left| \begin{array}{l} B_{(1,2\cdot(top^i-1)+1)} = 1 \\ B_{(2,2\cdot(top^i-1)+2)} = 1 \\ B_{(3,2\cdot(top^i-1)+3)} = 1 \\ B_{(4,2\cdot(top^i-1)+4)} = 1 \end{array} \right. \tag{4}$$

where:

top^i – fragment of topology matrix corresponding with *i*-th finite element.

For finite elements with variable inertia, the stiffness matrix will obviously take on a different form.

Function (5) describes incorporated matrix topology for *n* finite elements.

$$top = \begin{array}{l} for\; i \in 1...n \\ \left| \begin{array}{l} top_{i,1} = i \\ top_{i,2} = i+1 \end{array} \right. \end{array} \tag{5}$$

The above matrices (4) define the Boolean matrix for every i-th finite element, making it possible to later perform automatic calculations for any number of elements. The overall size of the Boolean matrix for the discussed task is $4 \times (2n + 2)$. Function (4) shows only non-zero elements of the matrix. Below is a Boolean matrix for every i-th finite element.

$$B_i = \begin{pmatrix} \underbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{matrix}}_{2i-2} & \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{4} & \underbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{matrix}}_{(2n+2)-(2i-6)} \end{pmatrix} \tag{6}$$

Stiffness matrix aggregation:

$$K = \sum_{i=1}^{n} B_i^T K_{e.i} B_i \tag{7}$$

where:

$K_{e.i}$ – stiffness matrix calculated in accordance with equation (3) for i-th finite element
B_i – Boolean matrix for i-th finite element

Definition of alternative vectors for linear load:

$$Z_c\left(q, L_{ES}\right) = \begin{pmatrix} \dfrac{qL_{ES}}{2} \\ \dfrac{qL_{ES}^{\ 2}}{12} \\ \dfrac{qL_{ES}}{2} \\ \dfrac{-qL_{ES}^{\ 2}}{12} \end{pmatrix} \tag{8}$$

where:

q – linear load
L_{ES} – length of finite element

Aggregation of alternative vectors:

$$Z = \sum_{i=1}^{n} B_i^T Z_{e.i} \tag{9}$$

where:

$Z_{e.i}$ – vector of alternatives for linear load calculated according with equation (8) for every i-th finite element.

Dead load was approximated to fixed load (to linear load within one finite element). Approximation of dead load with discontinuous linear load is exact enough in the case of a minimum of 5 finite elements within one span section.

Boundary conditions for a three-span beam were described by vector (10):

$$w = \left| \begin{array}{l} w_1 = 1 \\ w_2 = 0 \\ for\, i \in 2 \ldots (n+1) \\ \quad w_{(2i-1)} = \left| \begin{array}{l} 1\ if\ \left((i-1)L_{ES} = L_1\right) \vee \left((i-1)L_{ES} = L_1 + L_2\right) \vee \\ \qquad \qquad \left((i-1)L_{ES} = L_1 + L_2 + L_3\right) \\ 0\ otherwise \end{array} \right. \\ w_{2n+2} = 0 \end{array} \right. \tag{10}$$

The boundary conditions vector was established analogically for double-span beams.

The solution to the set of equations and calculation of displacement vectors for finite elements:

$$Q = K_{wb}^{-1} S_{wb} \tag{11}$$

$$R = K_{wb} Q - S_{wb} \tag{12}$$

where:

Q – nodal displacement vector

R – reaction vector

K_{wb} – stiffness matrix incorporating boundary conditions

S_{wb} – nodal load vector with boundary conditions

Calculation of node forces in elements:

$$f_{e.i} = K_{wb.i} B_i Q - Z_{e.i} \tag{13}$$

where:

$f_{e.i}$ – nodal forces vector for *i*-th finite element

Equations (3-13) were used to formulate function *MES(EI,x)*, which makes it possible to determine the value of internal forces as well as horizontal and angular displacement.

For statically indeterminate girders exposed to prestress forces, it is necessary to additionally take under consideration the effect of inducted internal forces. The impact of prestress force on the value of internal forces was taken under consideration by stressing the configuration with forces causing an equally great bearing reaction. The calculations did not incorporate the effect of rheological phenomena.

A computer program was written based on the above assumptions. It enables finding an optimal solution to the discussed problem. The algorithm was written in the Mathcad environment. The calculation program makes it possible to save a legible record of the calculation procedure and quickly solve matrix equations.

4.2. Optimization restrictions

The following restrictions were set:

G_1 – restriction of maximum compression stress on top edge of cross-section
G_2 – restriction of maximum tensile stress on top edge of cross-section
G_3 – restriction of maximum compression stress on bottom edge of cross-section
G_4 – restriction of maximum tensile stress on bottom edge of cross-section

$$G_1 = R - \max(|\sigma_{g.s}|), \quad s = 1 \div 4 \tag{14}$$

$$G_2 = R - \min(|\sigma_{g.s}|), \quad s = 1 \div 4 \tag{15}$$

$$G_3 = R - \max(|\sigma_{d.s}|), \quad s = 1 \div 4 \tag{16}$$

$$G_4 = R - \min(|\sigma_{d.s}|), \quad s = 1 \div 4 \tag{17}$$

where:

$\sigma_{d.s}$, $\sigma_{g.s}$ – accordingly: set of normal bend values for each calculation combination for: bottom and top edge of cross-section

G_5 – restriction of maximum beam bend

$$G_5 = a_{dop} - \max(a) \tag{18}$$

where:

a_{dop} – maximum admissible vertical displacement,
a – set vertical displacement.

Restrictions G_2 and G_4 protect from exceeding concrete resistance to tensile. In practice the prestressed elements are usually designed with cracking as the ultimate limit state. There are categories of elements, however, for which – due to favorable environmental conditions – are permitted to cross the concrete's tensile limit state to tensile. When optimizing beams that may crack Young's modulus should be correctly implemented.

Additionally, restrictions were set for permissible area of decision variables.

4.3. Optimization restrictions

The starting point of the optimization process was obtaining minimal dimensions of the cross-section due to set geometric restrictions and minimal value of prestress force.

A stepwise increment of the decision variable was assumed within one calculation loop. The direction of the increment Δh depends on fulfilling bearing capacity conditions and is determined in relation to the result of the cross-section verification result. Additionally, the increment value Δh decreases with subsequent calculation phases.

The result of a calculation loop is the optimal height of the cross-section of one finite element. What follows, the total amount of calculation loops within one iteration is equal to $n+1$.
where:
n – number of finite elements.

Analogically, a stepwise increment in value of the remaining two decision variables was also assumed.

Calculations were carried out in the following steps:
The value of the initial decision variables was determined
(1) An optimal solution for the determined value and location of net prestress force were determined in accordance with procedure A
(2) stepwise increment of the prestress force and finding an optimal solution for a fixed location and value of net prestress force within one incrementation (procedure A iteratively repeated for various force values N_p, until expected convergence is reached)
(3) stepwise increment of the decision variable h_p and finding an optimal solution for value h_p within one iteration (point 3 repeated iteratively)
(4) choice of optimal result from set of results

Procedure A:
- Finding the optimal cross-section height for each finite element.
- Cross-section fulfills all determined bearing conditions and minimizes determined objective function (1). Calculations are conducted for internal forces determined via the finite element method for initial values.
- MES calculations and updating value of cross-section forces and linear displacement.
- Re-determining optimal cross-section height along girder's length.
- Verification of boundary nods displacement.
- Iterative calculations (MES calculations are carried out for each iteration, an optimal solution is determined for defined internal forces and boundary nods displacement is verified).
- Iterative calculations are stopped upon receiving an expected iterative convergence.

5. Calculation results

Below are examples of optimization results for the following data:
- concrete class: C35/45,
- dead external load: $g = 20 \cdot 10^3$ N/m
- live load: $q_1 = 25 \cdot 10^3$ N/m, $q_2 = q_3 = q_1$
- fixed geometric dimensions: $b = 0.4$ m
- length: $L_1 = 18$ m, $L_2 = L_3 = L_1$
- length of finite element: $L_{ES} = 0.2$ m

Calculations were conducted for three decision variables:
- Fixed prestress force along the length of the element (prestress force after considering all loses)
- Fixed resultant distance h_p of prestressed cable's path from upper edge of cross-section (linear path)
- The element's cross-section's height

The following area of variability was assumed for the above decision variables:
- height h: <0.4 ; 2.0> [m]
- distance h_p: <0.3 ; 2.0> [m]
- prestress force N_p: <1000 ; 7000> [10^3 N]
-

The illustrations below show: boundary of the bending moment and shearing force (Fig. 4, 5), boundary of nodal vertical displacement (Fig. 6) and optimal beam height (Fig. 7) for double-span beam.

Figure 6 shows the boundary of nods vertical displacement along with marked boundary values of admissible displacement. The graph shows that restriction G_5 is not active in any beam cross-section.

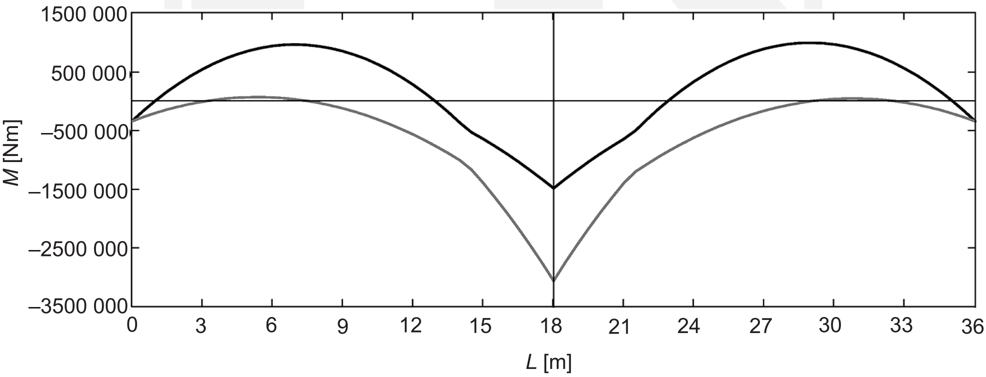

Fig. 4. Boundary of bending moments (double-span beam)

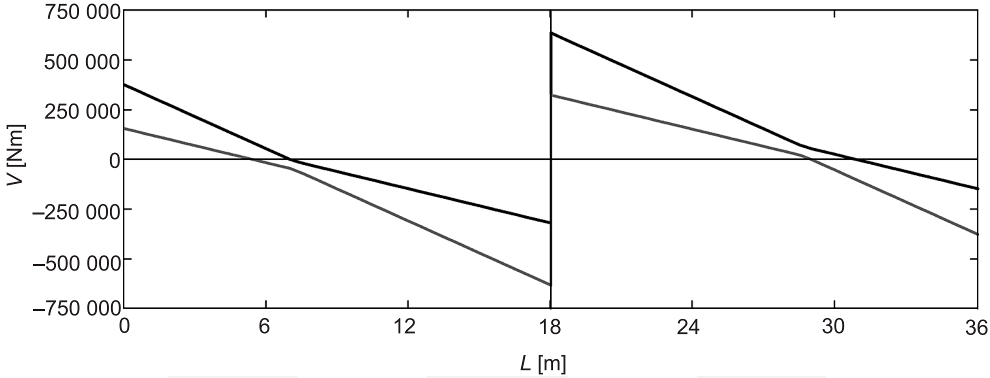

Fig. 5. Boundary of shearing force (double-span beam)

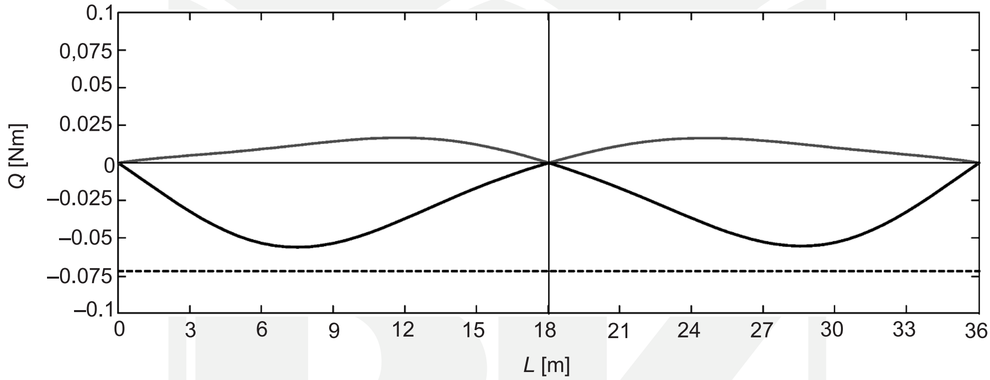

Fig. 6. Boundary of nodal vertical displacements (double-span beam)

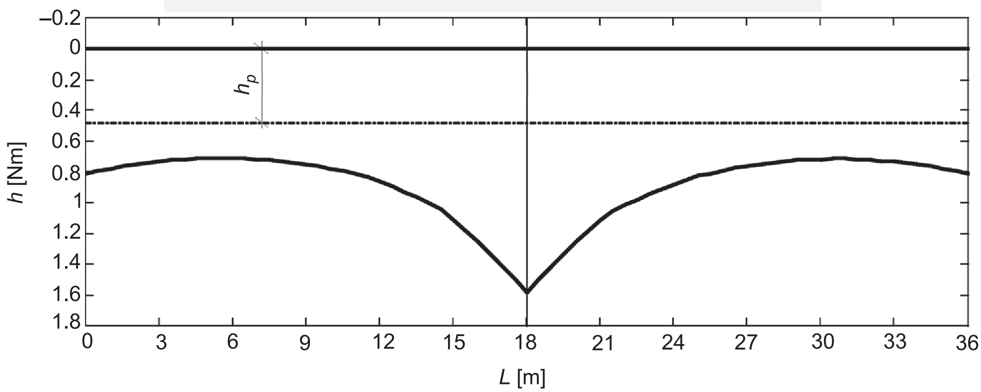

Fig. 7. Optimal solution (double-span beam)

Maximum vertical displacement was equal to 0.056 m. Bend limit for examined girder was equal to L/250 = 0.072 m.

Optimal values were set for:
- Prestress force N_p: $4507.6 \cdot 10^3$ N
- Distance h_p: 0.485 m

Objective function (2) value for the obtained result is equal to $V = 12.831$ m^3.

The figures below show: boundary of the bending moment and shearing force (Fig. 8, 9), boundary of nodal vertical displacement (Fig. 10) and optimal beam height (Fig. 11) for three-span beam.

Figure 10 shows the boundary of nodal vertical displacement along with marked boundary values of admissible displacement. The graph shows that restriction G_5 is not active in any beam cross-section.

Maximum vertical displacement was equal to 0.065 m. Bend limit for examined girder was equal to L/250 = 0.072 m.

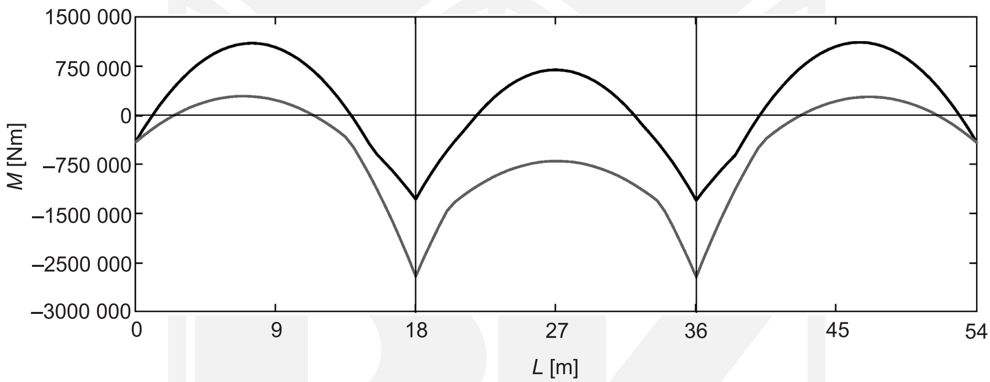

Fig. 8. Boundary of bending moments (three-span beam)

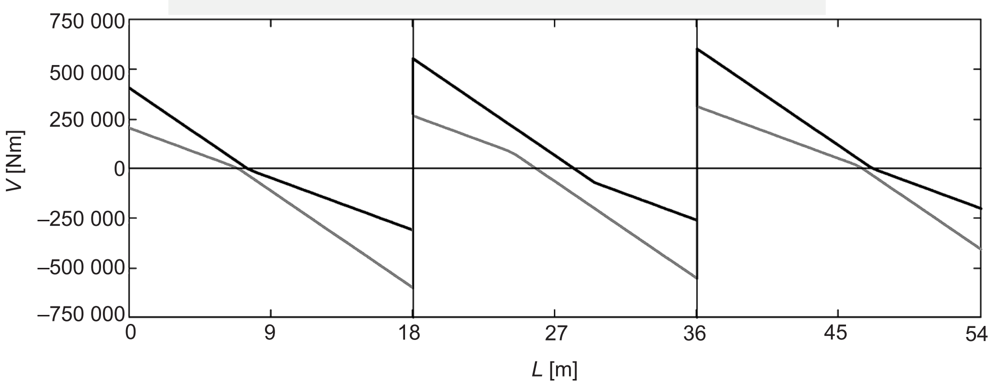

Fig. 9. Boundary of shearing force (three-span beam)

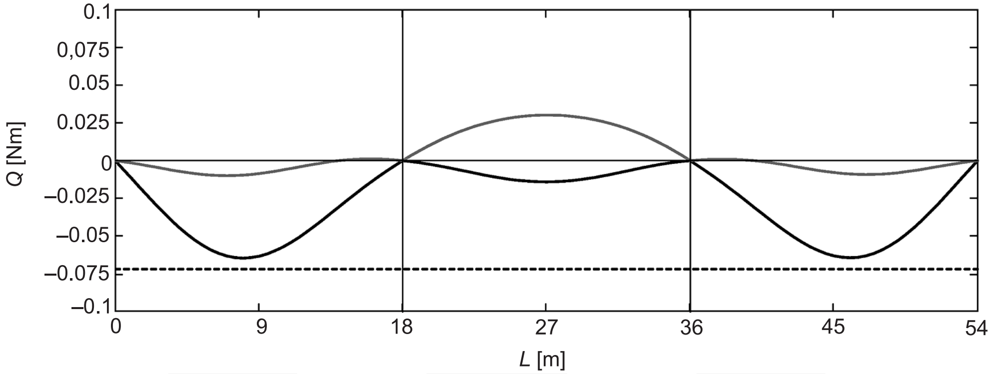

Fig. 10. Boundary of nodal vertical displacements (three-span beam)

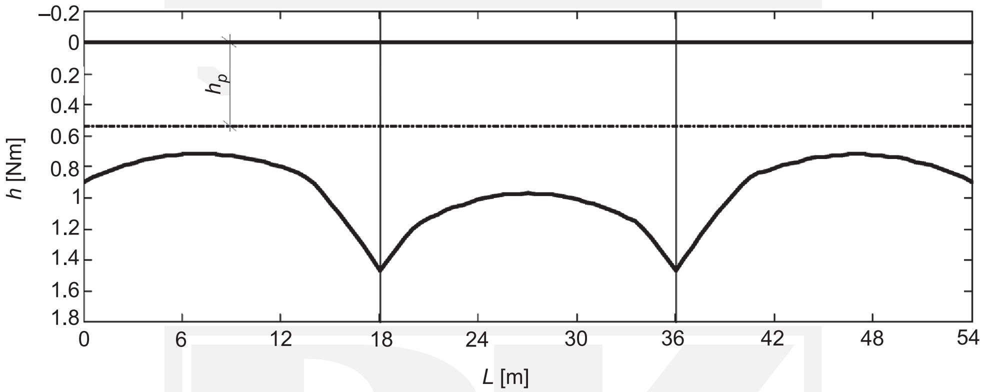

Fig. 11. Optimal solution (three-span beam)

Optimal values were set for:

- Prestress force N_p: $4652.9 \cdot 10^3$ N
- Distance h_p: 0.540 m

Objective function (2) value for the obtained result is equal to $V = 20.459$ m^3.

6. Conclusions

In the case of complex construction configurations, the necessity to find optimal solutions incorporating the maximum principle requires solving very intricate multi-point boundary value problems. Additionally, the optimal control method requires the use of numeric programs for solving formulated boundary value problems; these programs are difficult to obtain.

The gradient-iterative method makes it possible to find quick solutions even in the case of complex problems. By formulating the task with the help of simple functions and carrying out calculation loops, the solution set contains an optimal result which fulfills all predefined optimization criteria.

The gradient-iterative method in conjunction with the MES algorithm offers designers vast possibilities. As the method takes relatively little time it can be used in construction design studios to optimize various construction elements.

The optimization task can be formulated in any programming language or in popular calculation software (e.g. Mathcad, Mathlab).

Finding the optimal solution using the optimal control method for statically indeterminate prestressed beam with three decision variables requires an extreme effort and is time-consuming. The gradient-iterative method makes it possible to find the solution much quicker.

The described method can of course be used to solve any number of optimization problems that do not pertain to the optimal shape of structures

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