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FIXED POINTS OF α -NONEXPANSIVE MAPPINGS

PUNKTY STAŁE ODWZOROWAŃ α -NIEODDALAJĄCYCH

Abstract

This paper is connected with the theory of α -nonexpansive mappings, which were introduced by K. Goebel and M. A. J. Pineda in 2007. These mappings are a natural generalisation of nonexpansive mappings from the point of view of the fixed point theory. In particular, they proved that in Banach spaces all $\alpha = (\alpha_1, \dots, \alpha_n)$ -nonexpansive mappings with α_1 big enough,

namely $\alpha_1 \geq 2^{\frac{1}{1-n}}$, have minimal displacement equal to zero. This paper introduces some new results connected with this problem.

Keywords: α -nonexpansive mappings, minimal displacement, fixed point

Streszczenie

Niniejszy artykuł jest związany z odwzorowaniami α -nieoddalającymi, które zostały wprowadzone przez K. Goebela i M. A. J. Pinedę w 2007 r. Odwzorowania te są naturalnym uogólnieniem odwzorowań nieoddalających z punktu widzenia teorii punktu stałego. Wyżej wspomniani autorzy wykazali, że w przestrzeniach Banacha odwzorowania $\alpha = (\alpha_1, \dots, \alpha_n)$ -nieoddalające,

mające odpowiednio duże α_1 , a dokładniej $\alpha_1 \geq 2^{\frac{1}{1-n}}$, posiadają minimalne przesunięcie równe zeru. W artykule przedstawiono pewne nowe wyniki z związane z tym problemem.

Słowa kluczowe: odwzorowania α -nieoddalające, minimalne przesunięcie, punkt stały

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1. Introduction and preliminaries

Let (X, d) be a metric space, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index satisfying $\alpha_1 > 0$, $\alpha_n > 0$, $\alpha_i \geq 0$, $i = 2, \dots, n-1$ and $\sum_{i=1}^n \alpha_i = 1$. In [2], the following notions were introduced:

The mapping $T: X \rightarrow X$ is said to be α -Lipschitzian with constant $k \geq 0$, if

$$\sum_{i=1}^n \alpha_i d(T^i x, T^i y) \leq kd(x, y) \text{ for all } x, y \in X.$$

The mapping $T: X \rightarrow X$ is said to be α -nonexpansive (α -contraction), if T is α -Lipschitzian with constant $k = 1$ ($k < 1$ resp.).

Denote the Lipschitz constant with $k(T)$ and the α -Lipschitz constant of T with $k(\alpha, T)$.

Define also $d(T) := \inf\{d(x, Tx), x \in X\}$, which we will call the minimal displacement of T . Sometimes it is also called the approximate fixed point of T .

These notions are natural generalisations of Lipschitzian mappings, nonexpansive mappings and contractions from the point of view of the fixed point theory. For more information concerning α -nonexpansive mappings and other Lipschitzian mappings connected with the fixed point theory, we refer to [4].

In [2], the authors proved the following:

Theorem 1.1. (see also [4], chapter 3) *Let X be a Banach space, let C be a nonempty, closed, convex and bounded subset of X . Let $T: C \rightarrow C$ be an $\alpha = (\alpha_1, \dots, \alpha_n)$ -nonexpansive mapping where $\alpha_1 \geq 2^{\frac{1}{1-n}}$. Then $d(T) = 0$.*

Notice that the problem of determining the set of multi-indices α for which each α -nonexpansive mapping T has $d(T) = 0$ is still open.

The aim of this paper is to prove two results (Theorem 2.1, Theorem 2.2) which give a partial answer to the above open problem (see [4]).

Before proceeding further, let us recall the generalised Banach contraction principle (abbreviated to GBCP), which is formulated as follows:

Theorem 1.2. ([1], [5]) *In complete metric space X if for some $N \geq 1$ and $0 < M < 1$ the mapping $T: X \rightarrow X$ satisfies $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq Md(x, y)$ for any $x, y \in X$, then T has the unique fixed point.*

In author's PhD thesis [6] the more general version of the above theorem was presented. Let us recall it without proof.

Theorem 1.3. *Let (X, d) be a complete metric space, $N \geq 1$. Assume that $\phi: [0, \infty) \rightarrow [0, 1]$ is a continuous, non-increasing function satisfying $\phi(t) = 1$ if, and only if, $t = 0$. Let $T: X \rightarrow X$ be such that $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y)$ for all $x, y \in X$. Then T has the unique fixed point.*

2. Main results

Firstly, let us note a simple fact, there exist some α -Lipschitzian mappings which are not α -nonexpansive; however, their minimal displacement is equal to zero; moreover, they may have the unique fixed point.

This is illustrated by:

Example 2.1. Let $T : l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\} \rightarrow l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\}$

be defined in the following way: $T : x = (x_1, x_2, \dots) \rightarrow Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{\frac{1}{2}x_2}{1+x_2}, \frac{2x_5}{1+x_5}, \frac{\frac{1}{2}x_4}{1+x_4}, \dots \right)$. Then T is not α -nonexpansive for any α ; however, for properly chosen

$\alpha = (\alpha_1, \alpha_2)$ the mapping T is α -Lipschitzian with constant k arbitrarily close to 1. Moreover, T has the unique fixed point.

Obviously, the mapping T has the unique fixed point $(1, 0, 0, \dots)$.

Also, we have $\|Tx - Ty\| \leq 2\|x - y\|$ and $\|T^i x - T^i y\| \leq \|x - y\|, i \geq 2$ for any $x, y \in l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\}$. On the other hand, taking $x^n = \left(0, 0, \frac{1}{n}, 0, 0, \dots \right)$ and

$y^n = (0, 0, \dots)$ we have $\frac{\|Tx^n - Ty^n\|}{\|x^n - y^n\|} = \frac{2 \cdot \frac{1}{n}}{1 + \frac{1}{n}} n = \frac{2}{n+1} \rightarrow 2, n \rightarrow \infty$; therefore, $k(T) = 2$.

Similarly, $k(T^2) = 1$ and $k(T^i) \geq 1, i \geq 3$.

In l_∞ , it is not possible to choose α such that $\alpha_1 > 0$ and T is α -nonexpansive;

however, $\frac{1}{n}\|Tx - Ty\| + \frac{n-1}{n}\|T^2x - T^2y\| \leq \frac{n+1}{n}\|x - y\|$; therefore, assuming n to be big

enough, the mapping T is $\alpha = \left(\frac{1}{n}, \frac{n-1}{n} \right)$ -Lipschitzian with constant $k(\alpha, T)$ arbitrarily close to 1.

It is worth mentioning that the existence and uniqueness of the fixed point

of T also follows from Theorem 1.3. Indeed, we have $T^2x = \left(1, \frac{x_2}{1 + \frac{3}{2}x_2}, \frac{x_3}{1 + 3x_3}, \dots \right)$

$$\left. \frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \dots \right) \quad \text{and} \quad \left| \frac{x_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i} \right| \leq \left| \frac{x_i - y_i}{1+\frac{3}{2}(x_i + y_i) + \frac{9}{4}x_i y_i} \right| \leq \frac{|x_i - y_i|}{1+|x_i + y_i|} \leq$$

$$\leq \frac{|x_i - y_i|}{1+|x_i + y_i|} \leq \frac{\|x - y\|}{1+\|x - y\|}, \quad i \in \mathbb{N}. \quad \text{The latter inequality follows from the fact that } t \rightarrow \frac{t}{1+t}$$

is an increasing function on $[0, \infty)$. Similarly, $\left| \frac{x_i}{1+3x_i} - \frac{y_i}{1+3y_i} \right| \leq \frac{\|x - y\|}{1+\|x - y\|}$; therefore,

$$\|T^2x - T^2y\| \leq \frac{1}{1+\|x - y\|} \|x - y\|, \quad \text{so } T \text{ satisfies the assumptions of Theorem 1.3 with}$$

$$\phi(t) := \frac{1}{1+t}.$$

Now, let us exchange the condition $\alpha_1 \geq 2^{1-n}$ with the other regularity condition of a mapping T .

Theorem 2.1. *Let X be a Banach space, let $0 \in C \subset X$ be nonempty, closed, convex and bounded. Let $T: C \rightarrow C$ be a $\alpha = (\alpha_1, \dots, \alpha_n)$ -nonexpansive mapping such that $\|T^i(\mu x) - T^i(\mu y)\| \leq \|T^i(\lambda x) - T^i(\lambda y)\|$ for any $x, y \in C$, $0 \leq \mu \leq \lambda$, $i \in \{1, \dots, n\}$.*

Then $d(T) = 0$.

Proof. Fix $k \geq 1$. Define $S_k := \left(1 - \frac{1}{k}\right)T$. Obviously, $S_k x = \frac{1}{k} \cdot 0 + \left(1 - \frac{1}{k}\right)Tx \in C$. Then

$$\|S_k x - S_k y\| = \left(1 - \frac{1}{k}\right) \|Tx - Ty\|.$$

Next, we have:

$$S_k^2 x = S_k \left(\left(1 - \frac{1}{k}\right)Tx \right) = \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx \right);$$

therefore, by assumptions:

$$\begin{aligned} \|S_k^2 x - S_k^2 y\| &= \left\| \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx \right) - \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Ty \right) \right\| \\ &= \left(1 - \frac{1}{k}\right) \left\| T \left(\left(1 - \frac{1}{k}\right)Tx \right) - T \left(\left(1 - \frac{1}{k}\right)Ty \right) \right\| \\ &\leq \left(1 - \frac{1}{k}\right) \|T^2x - T^2y\|. \end{aligned}$$

Similarly, for $i \geq 3$ we have:

$$S_k^i x = T_k^{i-1} \left(\left(1 - \frac{1}{k} \right) T x \right) = \left(1 - \frac{1}{k} \right) T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right);$$

therefore:

$$\begin{aligned} \|S_k^i x - S_k^i y\| &= \left\| \left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) - \left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right\| \\ &= \left(1 - \frac{1}{k} \right) \left\| T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right) - T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right) \right\| \\ &\leq \left(1 - \frac{1}{k} \right) \left\| T \left(T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right) - T \left(T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right) \right\| \\ &\leq \left(1 - \frac{1}{k} \right) \|T^i x - T^i y\|. \end{aligned}$$

By assumptions, $\sum_{i=1}^n \alpha_i \|T^i x - T^i y\| \leq \|x - y\|$ for some $\alpha = (\alpha_1, \dots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^n \alpha_i = 1$. Therefore:

$$\min \left\{ \|S_k^j x - S_k^j y\|, 1 \leq j \leq n \right\} \leq \sum_{i=1}^n \alpha_i \|S_k^i x - S_k^i y\| \leq \left(1 - \frac{1}{k} \right) \|x - y\|,$$

By Theorem 1.2, S_k has the unique fixed point. Denote this fixed point by x_k . We get:

$$\|x_k - T x_k\| = \|S_k x_k - T x_k\| = \left\| \left(1 - \frac{1}{k} \right) T x_k - T x_k \right\| = \frac{1}{k} \|T x_k\| \rightarrow 0, k \rightarrow \infty,$$

this completes the proof. □

For $k \geq 3$, there exists a mapping T which does not satisfy the assumptions of Theorem 1.1; however, for $k \geq 2$, it satisfies the assumptions of Theorem 2.1. This will be illustrated by the following example:

Example 2.2. Fix $k \geq 2$. Let $\tau: [-1, 1] \rightarrow [-1, 1]$ be a non-decreasing function, having the Lipschitz constant $k(\tau) = k$, concave on $[-1, 0]$, convex on $[0, 1]$ and such that

$$\tau(0) = 0. \text{ Now define } T: B_l \ni x = (x_1, x_2, \dots) \rightarrow T x := (\tau(x_2), \frac{k}{k^2 - 1} x_3, x_4, x_5, \dots) \in B_l.$$

We will show that the assumptions of Theorem 1.1 are not satisfied for any multi-index α of length n . Notice, that:

$$\begin{aligned}
\|Tx - Ty\| &= |\tau(x_2) - \tau(y_2)| + \frac{k}{k^2 - 1} |x_3 - y_3| + \sum_{i=4}^{\infty} |x_i - y_i| \\
&\leq k |x_2 - y_2| + \frac{k}{k^2 - 1} |x_3 - y_3| + \sum_{i=4}^{\infty} |x_i - y_i|, \\
\|T^2x - T^2y\| &= \left| \tau\left(\frac{k}{k^2 - 1}x_3\right) - \tau\left(\frac{k}{k^2 - 1}y_3\right) \right| + \frac{k}{k^2 - 1} |x_4 - y_4| + \sum_{i=5}^{\infty} |x_i - y_i| \\
&\leq \frac{k}{k^2 - 1} |x_3 - y_3| + \frac{k}{k^2 - 1} |x_4 - y_4| + \sum_{i=5}^{\infty} |x_i - y_i|.
\end{aligned}$$

Therefore, $k(T) = k$ and $k(T^i) = \frac{k^2}{k^2 - 1} > 1$, $i \geq 2$.

It is easy to see that for $k \geq 3$, the assumptions of Theorem 1.1 are not satisfied for any multi-index α of length n . Indeed, we would need to have $\alpha_1 \geq 2^{1-n} \geq 2^{1-3} = \frac{\sqrt{2}}{2} > \frac{1}{2}$ and thus for any such $\alpha = (\alpha_1, \dots, \alpha_n)$, the mapping T would not be α -nonexpansive.

We will now show that T satisfies the assumptions of Theorem 2.1; therefore, $d(T) = 0$.

It is enough to take $n = 2$. It is easy to check that $\frac{1}{k} \|Tx - Ty\| + \frac{k-1}{k} \|T^2x - T^2y\| \leq \|x - y\|$; therefore, T is $\left(\frac{1}{k}, \frac{k-1}{k}\right)$ -nonexpansive. We only have to show that

$$\|T(\mu x) - T(\mu y)\| \leq \|T(\lambda x) - T(\lambda y)\| \text{ for any } x, y \in B_i, 0 \leq \mu \leq \lambda.$$

If $|\tau(\mu x_2) - \tau(\mu y_2)| \leq |\tau(\lambda x_2) - \tau(\lambda y_2)|$, then obviously,

$$\begin{aligned}
\|T(\mu x) - T(\mu y)\| &= |\tau(\mu x_2) - \tau(\mu y_2)| + \left| \frac{k}{k^2 - 1} \mu x_3 - \frac{k}{k^2 - 1} \mu y_3 \right| + \sum_{i=4}^{\infty} |\mu x_i - \mu y_i| \\
&\leq |\tau(\lambda x_2) - \tau(\lambda y_2)| + \left| \frac{k}{k^2 - 1} \lambda x_3 - \frac{k}{k^2 - 1} \lambda y_3 \right| + \sum_{i=4}^{\infty} |\lambda x_i - \lambda y_i| \\
&= \|T(\lambda x) - T(\lambda y)\|
\end{aligned}$$

It is enough to prove that $|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|$ for $v \neq w$ and $0 < \mu < \lambda$. Firstly, assume that $v, w > 0$. Without the loss of generality, we can assume that $w < v$. Therefore, $0 < \mu w < \mu v, \lambda w < \lambda v$.

Assume that $0 < \mu w < \mu v \leq \lambda w < \lambda v$. Choose $a \in (\lambda w, \lambda v]$ such that $a - \lambda w = \mu v - \mu w$.

Of course, such an a exists since $\lambda(v-w) \geq \mu(v-w)$. Therefore, $\mu v = \frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a$ and $\lambda w = \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$. Due to the convexity of τ on $[0, 1]$

$$\begin{aligned} \tau(\mu v) &= \tau\left(\frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \mu v}{a - \mu w} \tau(\mu w) + \frac{\mu v - \mu w}{a - \mu w} \tau(a) \end{aligned}$$

$$\begin{aligned} \tau(\lambda w) &= \tau\left(\frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu w) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a) \\ &= \frac{\mu v - \mu w}{a - \mu w} \tau(\mu w) + \frac{a - \mu v}{a - \mu w} \tau(a) \end{aligned}$$

Adding the above estimates side-by-side and taking into consideration the fact that τ is non-decreasing, we get:

$$\begin{aligned} \tau(\mu v) + \tau(\lambda w) &\leq \left(\frac{a - \mu v}{a - \mu w} + \frac{\mu v - \mu w}{a - \mu w}\right) \tau(\mu w) + \left(\frac{\mu v - \mu w}{a - \mu w} + \frac{a - \mu v}{a - \mu w}\right) \tau(a) \\ &= \tau(\mu w) + \tau(a) \leq \tau(\mu w) + \tau(\lambda v), \end{aligned}$$

this implies that $|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|$.

On the other hand, if $0 < \mu w < \lambda w \leq \mu v < \lambda v$, then let us choose $a \in (\mu v, \lambda v]$ such that $a - \mu v = \lambda w - \mu w$. Of course, such an a exists since $(\lambda - \mu)v \geq (\lambda - \mu)w$. Then

$\lambda w = \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$ and $\mu v = \frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a$. Due to the convexity of τ on $[0, 1]$, we have:

$$\begin{aligned} \tau(\lambda w) &= \tau\left(\frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu w) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a) \end{aligned}$$

$$\begin{aligned}
\tau(\mu\nu) &= \tau\left(\frac{a-\mu\nu}{a-\mu w}\mu w + \frac{\mu\nu-\mu w}{a-\mu w}a\right) \\
&\leq \frac{a-\mu\nu}{a-\mu w}\tau(\mu w) + \frac{\mu\nu-\mu w}{a-\mu w}\tau(a) \\
&= \frac{\lambda w-\mu w}{a-\mu w}\tau(\mu y) + \frac{a-\lambda w}{a-\mu w}\tau(a)
\end{aligned}$$

Again, adding the above estimations side-by-side, we get:

$$\begin{aligned}
\tau(\lambda w) + \tau(\mu\nu) &\leq \left(\frac{a-\lambda w}{a-\mu w} + \frac{\lambda w-\mu w}{a-\mu w}\right)\tau(\mu w) + \left(\frac{\lambda w-\mu w}{a-\mu w} + \frac{a-\lambda w}{a-\mu w}\right)\tau(a) \\
&= \tau(\mu w) + \tau(a) \leq \tau(\mu w) + \tau(\lambda\nu),
\end{aligned}$$

this leads to $|\tau(\mu\nu) - \tau(\mu w)| \leq |\tau(\lambda\nu) - \tau(\lambda w)|$.

Similarly, it is easy to check that the estimation $|\tau(\mu\nu) - \tau(\mu w)| \leq |\tau(\lambda\nu) - \tau(\lambda w)|$ remains true for $\nu, w < 0$ and for other cases. This shows that T satisfies the assumptions of Theorem 2.1.

A set satisfying $\lambda x_0 + (1-\lambda)y \in C$ for all $y \in C$, $\lambda \in [0,1]$ we call *star-like set C with respect to x_0* .

Theorem 2.2. *Let X be a Banach space, $x_0 \in X$, $N \in \mathbb{N}$, let $C \subset X$ be a bounded, star-like set with respect to x_0 . Let $T : C \rightarrow C$ be such that*

1. $\min\{\|T^j x - T^j y\|, 1 \leq j \leq N\} \leq \|x - y\|$ for all $x, y \in C$,
2. there exists $0 \leq b_0 \leq 1$ such that for all $0 \leq b \leq b_0$, $1 \leq j \leq N-1$, $x, y \in C$

$$\|T(T_b^j x) - T(T_b^j y)\| \leq (1+b)\|T^{j+1}x - T^{j+1}y\|,$$

where $T_b x = (1-b)Tx + bx_0$. Then $d(T) = 0$.

Proof. Fix arbitrary $x, y \in C$ and take $j \in \{1, \dots, N\}$ such that $\|T^j x - T^j y\| \leq \|x - y\|$.

Let us note that:

$$\begin{aligned}
\|T_b^j x - T_b^j y\| &= \|T_b(T_b^{j-1}x) - T_b(T_b^{j-1}y)\| \\
&= \|(1-b)T(T_b^{j-1}x) + bx_0 - (1-b)T(T_b^{j-1}y) - bx_0\| \\
&= (1-b)\|T(T_b^{j-1}x) - T(T_b^{j-1}y)\| \\
&\leq (1-b)(1+b)\|T^j x - T^j y\| \\
&\leq (1-b^2)\|x - y\|
\end{aligned}$$

Therefore, for any $x, y \in C$ there exists $j \in \{1, \dots, N\}$ such that $\|T_b^j x - T_b^j y\| \leq (1-b^2)\|x-y\|$. Theorem 1.2 ensures, that T_b has the unique fixed point.

Now, fix an arbitrary $\varepsilon > 0$ and choose $0 \leq b \leq b_0$ such that $\|T_b z - Tz\| = \|(1-b)Tz + bx_0 - Tz\| = b\|x_0 - Tz\| \leq \varepsilon$ for any $z \in C$.

Let $z_b \in C$ be such that $T_b z_b = z_b$.

Therefore, $\|z_b - Tz_b\| \leq \|z_b - T_b z_b\| + \|T_b z_b - Tz_b\| \leq 0 + \varepsilon = \varepsilon$, this proves that $d(T) = 0$.

□

Let us illustrate the possible application of Theorem 2.2.

Example 2.3. Let T be the same as in Example 2.2. Then T satisfies Theorem 2.2 (we have already shown that T does not satisfy Theorem 1.1 for $k \geq 3$).

Indeed, let us calculate

$$T(T_b x) = \left(\tau \left((1-b) \frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1} (1-b)x_4, (1-b)x_5, (1-b)x_6, \dots \right)$$

and

$$T^2 x = \left(\tau \left(\frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1} x_4, x_5, x_6, \dots \right)$$

We have:

$$\begin{aligned} \|T(T_b x) - T(T_b y)\| &= \left| \tau \left((1-b) \frac{k}{k^2-1} x_3 \right) - \tau \left((1-b) \frac{k}{k^2-1} y_3 \right) \right| \\ &\quad + \left| \frac{k}{k^2-1} (1-b)x_4 - \frac{k}{k^2-1} (1-b)y_4 \right| + |(1-b)x_5 - (1-b)y_5| + \dots \\ &\leq \left| \tau \left(\frac{k}{k^2-1} x_3 \right) - \tau \left(\frac{k}{k^2-1} y_3 \right) \right| + \left| \frac{k}{k^2-1} x_4 - \frac{k}{k^2-1} y_4 \right| + |x_5 - y_5| + \dots \\ &= \|T^2 x - T^2 y\| \leq (1+b) \|T^2 x - T^2 y\|. \end{aligned}$$

We have already taken into account the fact, that $|\tau(\mu s) - \tau(\mu t)| \leq |\tau(\lambda s) - \tau(\lambda t)|$ for any $0 \leq \mu \leq \lambda$, $s, t \in [-1, 1]$. We proved this fact in Example 2.2.

The estimate $\min \left\{ \|Tx - Ty\|, \|T^2 x - T^2 y\| \right\} \leq \frac{1}{k} \|Tx - Ty\| + \frac{k-1}{k} \|T^2 x - T^2 y\| \leq \|x - y\|$ shows that T satisfies Theorem 2.2.

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