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ON NONLOCAL EVOLUTION PROBLEM FOR THE EQUATION OF THE FIRST ORDER

O NIELOKALNYM EWOLUCYJNYM ZAGADNIENIU DLA RÓWNANIA RZĘDU PIERWSZEGO

A b s t r a c t

The aim of the paper is to prove theorems about the existence and uniqueness of mild and classical solutions of a nonlocal semilinear functional-differential evolution Cauchy problem. The method of semigroups, the Banach fixed-point theorem and theorems (see [2]) about the existence and uniqueness of the classical solutions of the first-order differential evolution problems in a not necessarily reflexive Banach space are used to prove the existence and uniqueness of the solutions of the problems considered. The results obtained are based on publications [1–6].

Keywords: *evolution Cauchy problem, existence and uniqueness of the solutions, nonlocal conditions*

S t r e s z c z e n i e

W artykule udowodniono twierdzenia o istnieniu i jednoznaczności rozwiązań całkowych i klasycznych nielokalnego semiliniowego funkcjonalno-różniczkowego ewolucyjnego zagadnienia Cauchy'ego. W tym celu zastosowano metodę półgrup, twierdzenie Banacha o punkcie stałym i twierdzenia ([2]) o istnieniu i jednoznaczności klasycznych rozwiązań ewolucyjnych zagadnień różniczkowych pierwszego rzędu w niekoniecznie refleksywnej przestrzeni Banacha. Artykuł bazuje na publikacjach [1–6].

Słowa kluczowe: *ewolucyjne zagadnienie Cauchy'ego, istnienie i jednoznaczność rozwiązań, warunki nielokalne*

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1. Introduction

Let E be a Banach space with norm $\|\cdot\|$ and let $A : E \to E$ be a closed densely defined linear operator. For an operator A, let $\mathcal{D}(A)$, $\rho(A)$ and A^* denote its domain, resolvent set and adjoint, respectively.

For a Banach space E, $\mathcal{C}(E)$ denotes the set of closed linear operators from E into itself.

We will need the class $G(\tilde{M},\beta)$ of operators A satisfying the conditions:

There exist constants $\tilde{M} > 0$ and $\beta \in \mathbb{R}$ such that

$$(C_1)\ \ A \in \mathcal{C}(E),\ \overline{\mathcal{D}(A)} = E \quad \text{and} \quad (\beta,+\infty) \subset \rho(-A),$$

$$(C_2)\ \ \left\|(A+\xi)^{-k}\right\| \leqslant \tilde{M}(\xi-\beta)^{-k} \quad \text{for each} \quad \xi > \beta \quad \text{and} \quad k = 1,2,\ldots$$

We will need the assumption:

Assumption (Z). The adjoint operator A^* is densely defined in E^*, i.e., $\overline{\mathcal{D}}(A^*) = E^*$.

It is known (see: [4], p. 485 and [5], p. 20) that for $A \in G(\tilde{M},\beta)$ there exists exactly one strongly continuous semigroup $T(t) : E \to E$ for $t \geqslant 0$ such that $-A$ is its infinitesimal generator and

$$\|T(t)\| \leqslant \tilde{M}e^{\beta t} \quad \text{for} \quad t \geqslant 0.$$

Throughout the paper we will assume (C_1) and (C_2), and assumption (Z).

Moreover throughout the paper we will use the notation

$$0 \leqslant t_0 < t_1 < \ldots < t_p \leqslant t_0 + a, \quad a > 0,$$

$$J := [t_0, t_0 + a],$$

$$M := \sup\left\{\|T(t)\| \,:\, t \in [0,a]\right\}$$

and

$$X := \mathcal{C}(J,E).$$

Throughout the paper we will also assume that there exists the operator $\mathcal{B}$ with $\mathcal{D}(\mathcal{B}) = E$ given by the formula

$$\mathcal{B} := \left(I + \sum_{k=1}^{p} c_k T(t_k - t_0)\right)^{-1},$$

where I is the identity operator on E.

The aim of the paper is to study the existence and uniqueness of mild and classical solutions to a nonlocal Cauchy problem for a functional-differential evolution equation. The nonlocal Cauchy problem considered here is of the following form:

$$u'(t) + Au(t) = f(t, u(t), u(b_1(t)), \ldots, u(b_r(t))), \quad t \in J \setminus \{t_0\}, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0, \tag{1.2}$$

where f and b_i ($i = 1, \ldots, r$) are given functions satisfying some assumptions, $u_0 \in E$, $c_k \neq 0$ ($k = 1, 2, \ldots, p$) and $p, r \in \mathbb{N}$.

To study problem (1.1)–(1.2) we will need the following linear problem:

$$u'(t) + Au(t) = g(t), \quad t \in J \setminus \{t_0\}, \tag{1.3}$$

$$u(t_0) = x \tag{1.4}$$

and the following definition:

A function $u : J \to E$ is said to be a classical solution to the problem (1.3)–(1.4) if

(i) u is continuous on J and continuously differentiable on $J \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = g(t)$ for $t \in J \setminus \{t_0\}$,

(iii) $u(t_0) = x$.

To study problem (1.1)–(1.2) we will need also the following theorem:

Theorem 1.1. (see [2]). *Let $g : J \to E$ be Lipschitz continuous on J and $x \in \mathcal{D}(A)$.*

Then the Cauchy problem (1.3)–(1.4) *has exactly one classical solution u given by the formula*

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)g(s)\,ds, \quad t \in J. \tag{1.5}$$

The results obtained in the paper, are based on publications [1–6].

2. On mild solution

A function $u \in X$ satisfying the integral equation

$$\begin{aligned} u(t) = {} & T(t - t_0)\mathcal{B}u_0 - \\ & + \sum_{k=1}^{p} c_k T(t - t_0)\mathcal{B} \int_{t_0}^{t_k} T(t_k - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\,ds + \\ & + \int_{t_0}^{t} T(t - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\,ds, \quad t \in J, \end{aligned} \tag{2.1}$$

is said to be a *mild solution* of the functional-differential nonlocal evolution Cauchy problem (1.1)–(1.2).

Remark 2.1. *A function u satisfying* (2.1) *satisfies condition* (1.2) (For the proof of Remark 2.1 see [3]).

Theorem 2.1. *Assume that*:

(i) *$f : J \times E^{r+1} \to E$ is continuous with respect to the first variable on J, $b_i : J \to J$ ($i = 1, \ldots, r$) are continuous on J and there is $L > 0$ such that*

$$\left\| f(s, z_0, z_1, \ldots, z_r) - f(s, \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_r) \right\| \leqslant L \sum_{i=0}^{r} \left\| z_i - \tilde{z}_i \right\| \tag{2.2}$$

$$\text{for} \quad s \in J, \; z_i, \tilde{z}_i \in E \; (i = 0, 1, \ldots, r),$$

(ii) $(r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)<1,$

(iii) $u_0 \in E.$

Then the functional-differential nonlocal evolution Cauchy problem (1.1)–(1.2) *has a unique mild solution.*

Proof. Introduce the operator F on the Banach space X given by the formula

$$
\begin{aligned}
(Fw)(t) &:= T(t-t_0)\mathcal{B}u_0 - \\
&-\sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B}\int_{t_0}^{t_k} T(t_k-s) f(s,w(s),w(b_1(s)),\ldots,w(b_r(s)))\,ds + \\
&+\int_{t_0}^{t} T(t-s) f(s,w(s),w(b_1(s)),\ldots,w(b_r(s)))\,ds, \quad w\in X, \quad t\in J.
\end{aligned}
$$

It is easy to see that F is a mapping from X into X and we will show that F is a contraction on X. For this purpose, observe that

$$
\begin{aligned}
&(Fw)(t)-(F\tilde{w})(t) = \\
&-\sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B}\int_{t_0}^{t_k} T(t_k-s)[f(s,w(s),w(b_1(s)),\ldots,w(b_r(s)))- \\
&\qquad - f(s,\tilde{w}(s),\tilde{w}(b_1(s)),\ldots,\tilde{w}(b_r(s)))]\,ds + \\
&+\int_{t_0}^{t} T(t_k-s)[f(s,w(s),w(b_1(s)),\ldots,w(b_r(s)))- \\
&\qquad - f(s,\tilde{w}(s),\tilde{w}(b_1(s)),\ldots,\tilde{w}(b_r(s)))]\,ds, \quad w,\tilde{w}\in X, \quad t\in J.
\end{aligned}
\tag{2.3}
$$

From (2.3) and (2.2)

$$\|(Fw)(t)-(F\tilde{w})(t)\| \leqslant (r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)\|w-\tilde{w}\|_X, \quad w,\tilde{w}\in X, \quad t\in J. \tag{2.4}$$

Define

$$q := (r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right). \tag{2.5}$$

Then by (2.4), (2.5) and assumption (ii),

$$\|Fw-F\tilde{w}\|_X \leqslant q\|w-\tilde{w}\|_X \quad \text{for} \quad w,\tilde{w}\in X \tag{2.6}$$

with $0 < q < 1$.

Consequently, by (2.6), operator F satisfies all the assumptions of the Banach contraction theorem. Therefore, in space X there is only one fixed point of F and this point is the mild solution of problem (1.1)–(1.2) So, the proof of Theorem 2.1 is complete. □

3. Mild and classical solutions

A function $u : J \to E$ is said to be a classical solution of the functional-differential nonlocal evolution Cauchy problem (1.1)–(1.2) if:

(i) u is continuous on J and continuously differentiable on $J \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = f(t, u(t), u(b_1(t)), \ldots, u(b_r(t)))$, for $t \in J \setminus \{t_0\}$,

(iii) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0$.

Theorem 3.1. *Assume that $f : J \times E^{r+1} \to E$ is Lipschitz continuous on $J \times E^{r+1}$. If u is a classical solution to the problem* (1.1)–(1.2) *then u is a mild solution of this problem.*

Proof. Since u is a classical solution to the problem (1.1)–(1.2), $u \in X$ and u satisfies the integral equation (see [2], Theorem 2)

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} T(t - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\, ds, \quad t \in J.$$

The remaining part of the proof[1] of Theorem 3.1 is as in [3]. □

Theorem 3.2. *Suppose that*:

(i) *$f : J \times E^{r+1} \to E$, $b_i : J \to J$ $(i = 1, \ldots, r)$ are continuous on J and there is $C > 0$ such that*

$$\| f(s, z_0, z_1, \ldots, z_r) - f(\tilde{s}, \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_r \| \leqslant C \left(|s - \tilde{s}| + \sum_{i=0}^{r} \| z_i - \tilde{z}_i \| \right) \tag{3.1}$$

$$\text{for} \quad s, \tilde{s} \in J, \quad z_i, \tilde{z}_i \in E \quad (i = 0, \ldots, r),$$

(ii) $(r + 1)MCa\left(1 + M \| \mathcal{B} \| \sum_{k=1}^{p} |c_k| \right) < 1,$

(iii) $u_0 \in E$.

Then the functional-differential nonlocal evolution problem (1.1)–(1.2) *has a unique mild solution denoted by u. Moreover, if*

(iv) *$\mathcal{B}u_0 \in \mathcal{D}(A)$ and*

$$\mathcal{B} \int_{t_0}^{t_k} T(t_k - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\, ds \in \mathcal{D}(A) \quad (k = 1, \ldots, p)$$

and if there is $\kappa > 0$ such that

$$\| u(b_i(s)) - u(b_i(\tilde{s})) \| \leqslant \kappa \| u(s) - u(\tilde{s}) \| \quad \text{for} \quad s, \tilde{s} \in J$$

then u is the unique classical solution to problem (1.1)–(1.2).

Proof. Since all the assumptions of Theorem 2.1 are satisfied, problem (1.1)–(1.2) possesses a unique mild solution u.

Now, we will show that u is the unique classical solution to the problem (1.1)–(1.2). To this end, introduce

$$N := \max_{s \in J} \| f(s, u(s), u(b_1(s)), \ldots, u(b_r(s))) \| \tag{3.2}$$

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[1] This remaining part of the proof shows why in the definition of a mild solution u to the problem (1.1)–(1.2) we require that the function u satisfies the integral equation (2.1).

and observe that

$$u(t+h)-u(t)=T(t-t_0)[T(h)-I]\mathcal{B}u_0-\sum_{k=1}^{p}c_kT(t-t_0)[T(h)-I]\times$$
$$\times\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\dots,u(b_r(s)))\,ds+$$
$$+\int_{t_0}^{t_0+h}T(t+h-s)f(s,u(s),u(b_1(s)),\dots,u(b_r(s)))\,ds+ \tag{3.3}$$
$$+\int_{t_0}^{t}T(t-s)[f(s,u(s+h),u(b_1(s+h)),\dots,u(b_r(s+h)))-$$
$$+f(s,u(s),u(b_1(s)),\dots,u(b_r(s)))]\,ds \text{ for } t\in[t_0,t_0+a),\ h>0 \text{ and } t+h\in(t_0,t_0+a].$$

Consequently, by (3.3), (3.2), (3.1) and Assumption (iv),

$$\|u(t+h)-u(t)\|\leqslant$$
$$\leqslant Mh\|A\mathcal{B}u_0\|+\sum_{k=1}^{p}|c_k|Mh\left\|A\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\dots,u(b_r(s)))\,ds\right\|+$$
$$+hMN+MCah+MC\int_{t_0}^{t}\big(\|u(s+h)-u(s)\|+\|u(b_1(s+h))-u(b_1(s))\|+\dots+ \tag{3.4}$$
$$+\|u(b_r(s+h))-u(b_r(s))\|\big)\,ds=C_*h+MC(1+r\kappa)\int_{t_0}^{t}\|u(s+h)-u(s)\|\,ds$$
$$\text{for}\quad t\in[t_0,t_0+a),\quad h>0\quad\text{and}\quad t+h\in(t_0,t_0+a],$$

where

$$C_*:=M\left[\|A\mathcal{B}u_0\|+\sum_{k=1}^{p}|c_k|\left\|A\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\dots,u(b_r(s)))\,ds\right\|+N+Ca\right].$$

From (3.4) and Gronwall's inequality

$$\|u(t+h)-u(t)\|\leqslant C_*e^{aMC(1+r\kappa)}h$$

for $t\in[t_0,t_0+a)$, $h>0$ and $t+h\in(t_0,t_0+a]$.
Hence u is Lipschitz continuous on J.

The Lipschitz continuity of u on J combined with continuity of f on $J\times E^{r+1}$ imply that $t\to f(t,u(t),u(b_1(t)),\dots,u(b_r(t)))$ is Lipschitz continuous on J. This fact together with assumptions of Theorem 3.2 imply, by Theorem 1.1, that the linear Cauchy problem

$$v'(t)+Av(t)=f(t,u(t),u(b_1(t)),\dots,u(b_r(t))),\quad t\in J\setminus\{t_0\}, \tag{3.5}$$

$$v(t_0)=u_0-\sum_{k=1}^{p}c_ku(t_k) \tag{3.6}$$

has a unique classical solution v such that

$$v(t)=T(t-t_0)v(t_0)+\int_{t_0}^{t}T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(\mathrm{s})))\,ds,\quad t\in J. \tag{3.7}$$

Now, we will show that

$$u(t)=v(t)\quad \text{for}\quad t\in J. \tag{3.8}$$

To do it, observe that, by (3.6), by Remark 2.1 and by (2.1),

$$v(t_0)=u(t_0)=\mathcal{B}u_0-\sum_{k=1}^{p}c_k\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s)),u(b_1(s)),\ldots,u(b_r(s)))\,ds.$$

Consequently

$$\begin{aligned}&T(t-t_0)v(t_0)=\\&=T(t-t_0)\mathcal{B}u_0-\sum_{k=1}^{p}c_kT(t-t_0)\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds,\quad t\in J.\end{aligned} \tag{3.9}$$

Next from (3.7), (3.9) and (2.1),

$$\begin{aligned}v(t)&=T(t-t_0)v(t_0)+\int_{t_0}^{t}T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds=\\&=T(t-t_0)\mathcal{B}u_0-\sum_{k=1}^{p}c_kT(t-t_0)\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds+\\&+\int_{t_0}^{t}T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds=u(t),\quad t\in J,\end{aligned}$$

and, therefore, (3,8) holds.

The above argument implies that u is a classical solution of problem (1.1)–(1.2).

To prove that u is the unique classical solution of problem (1.1)–(1.2) suppose that there is a classical solution u_* of problem (1.1)–(1.2) such that $u_*\neq u$ on J. Then, by Theorem 3.1, u_* is a mild solution of problem (1.1)–(1.2). Since, by Theorem 2.1, there exists the only one mild solution of problem (1.1)–(1.2), $u_*=u$ on J. Thus, the proof of Theorem 3.2 is complete.

References

[1] Balachandran K., Ilamaran S., *Existence and uniqueness of mild and strong solutions of a semilinear evolution equation with nonlocal conditions*, Indian J. Pure Appl. Math., **25.4**, 1994, 411-418.

[2] Bochenek J., *The existence of a solution of a semilinear first-order differential equation in a Banach space*, Univ. Iag. Acta Math., **31**, 1994, 61-68.

[3] Byszewski L., *Application of properties of the right-hand sides of evolution equations to an investigation of nonlocal evolution problems*, Nonlinear Analysis, **33**, 1998, 413-426.

[4] Kato T., *Perturbation Theory for Linear Operators*, Springer–Verlag, New York, Berlin, Heidelberg, 1996.
[5] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[6] Winiarska T., *Differential Equations with Parameter*, Monograph 68, Technical University of Cracow, Cracow 1988.