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APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY THE BASKAKOV-DURRMEYER TYPE OPERATORS

APROKSYMACJA FUNKCJI WIELU ZMIENNYCH OPERATORAMI TYPU BASKAKOWA-DURRMEYERA

A b s t r a c t

In this paper we introduce some linear positive operators of the Baskakov-Durrmeyer type in the space of uniformly continuous and bounded functions of several variables. The theorem on the degree of the convergence is established. Moreover, we give the Voronovskaya type formula for these operators.

Keywords: Baskakov-Durrmeyer type operator, linear operators, approximation order, Voronovskaya type theorem

S t r e s z c z e n i e

W artykule rozważa się operatory typu Baskakowa-Durrmeyera w przestrzeni ograniczonych i jednostajnie ciągłych funkcji wielu zmiennych. Formułuje się i dowodzi twierdzenia dotyczące rzędu zbieżności, jak również twierdzenie typu Woronowskiej dla tych operatorów.

Słowa kluczowe: operatory typu Baskakowa-Durrmeyera, operatory liniowe, rząd aproksymacji, twierdzenie typu Woronowskiej

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1. Introduction

Let $\mathbb{R}_0^+ = [0,\infty)$, $\mathbb{N} = \{1,2\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for every fixed $m \in \mathbb{N}$ let

$$\mathbb{N}^m = \{\boldsymbol{n} = (n_1,\ldots,n_m) : n_k \in \mathbb{N} \quad \text{for} \quad 1 \le k \le m\},$$
$$\mathbb{R}_+^m = \{\boldsymbol{x} = (x_1,\ldots,x_m) : x_k \in \mathbb{R}_0^+ \quad \text{for} \quad 1 \le k \le m\}.$$

Analogously we define $\mathbb{R}^m$. We denote $\overline{\lambda} = (\lambda,\lambda,\ldots,\lambda) \in \mathbb{R}^m$. For $\boldsymbol{n} \in \mathbb{N}^m$ we write $\boldsymbol{n} \to \overline{\infty}$ if and only if $n_k \to +\infty$ for $k = 1,2\ldots m$. Moreover, for a fixed $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^m$, we will use the notation

$$\int_{x}^{y} f(\boldsymbol{s})d\boldsymbol{s} = \int_{x_1}^{y_1} \ldots \int_{x_m}^{y_m} f(s_1,\ldots s_m)ds_1\ldots ds_m.$$

We denote by $C_B(\mathbb{R}_+^m)$ the space of all real-valued functions f uniformly continuous and bounded on $\mathbb{R}_+^m$. The norm on $C_B(\mathbb{R}_+^m)$ is defined by $\|f\|_{C_B(\mathbb{R}_+^m)} = \sup_{\boldsymbol{x}\in\mathbb{R}_+^m} |f(\boldsymbol{x})|$. Let

$$W_{n,k}^{a}(x) = e^{-\frac{ax}{1+x}} \sum_{i=0}^{k} \binom{k}{i} (n)_i \, a^{k-i} \frac{x^k}{k!(1+x)^{n+k}},$$

where $a \in \mathbb{R}_0^+$, $(n)_0 = 1$, $(n)_i = n(n+1)\ldots(n+i-1)$, $\quad i \ge 1$.

For a real-valued function f defined on the interval $[0,\infty)$, the generalized Baskakov--Durrmeyer type operators is defined by (see [11])

$$M_n^{\alpha,a}(f;x) = n\sum_{k=0}^{\infty} W_{n,k}^{a}(x) \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\alpha+k} f(s)ds, \quad \alpha > -1. \tag{1.1}$$

In the present paper, inspired by operator (1.1), we introduce the following class of operators $\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$ given by the formula

$$\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;x) = \sum_{k_1,\ldots,k_m=0}^{\infty} \int_{\bar{0}}^{\bar{\infty}} \prod_{j=1}^{m} W_{n_j,k_j}^{a_j}(x_j) \frac{n_j}{\Gamma(\alpha_j+k_j+1)} e^{-n_j s_j}(n_j s_j)^{\alpha_j+k_j} f(\boldsymbol{s})d\boldsymbol{s} \tag{1.2}$$

for $x \in \mathbb{R}_+^m$, where $\boldsymbol{n} \in \mathbb{N}^m$, $\boldsymbol{a} \in \mathbb{R}_+^m$, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m)$, $\alpha_k > -1$ for $k = 1,2,\ldots,m$. It is obvious that the operator $\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$ is linear and positive on $\mathbb{R}_+^m$. Basic facts on positive linear operators, their generalizations and applications, can be found in [3], [4].

Observe that if $f(\boldsymbol{s}) = f_1(s_1)\cdot\ldots\cdot f_m(s_m)$ for $\boldsymbol{s} \in \mathbb{R}_+^m$, then

$$\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;x) = \prod_{j=1}^{m} M_{n_j}^{\alpha_j,a_j}(f_j;\boldsymbol{x}_j),$$

where

$$M_{n_j}^{\alpha_j,a_j}(f_j;x_j)=n_j\sum_{k_j=0}^{\infty}W_{n_j,k_j}^{a_j}(x_j)\frac{1}{\Gamma(\alpha_j+k_j+1)}\int_0^\infty e^{-n_js_j}(n_js_j)^{\alpha_j+k_j}f_j(s_j)ds_j.$$

Some properties of the operator defined by (1.1) in particular, an estimation of the rate of convergence, were studied in [11].

Lemma 1 [11]. *Let* $\varphi^r(t)=t^r$, $t\in\mathbb{R}_0^+$, $r\in\mathbb{N}_0$. For $x\geq 0$, $\alpha>-1$ and $a\geq 0$, *we have*

$$M_n^{\alpha,a}(\varphi^0;x)=1,$$

$$M_n^{\alpha,a}(\varphi^1;x)=\frac{\alpha+1}{n}+x+\frac{ax}{n(1+x)},$$

$$M_n^{\alpha,a}(\varphi^1-x;x)=\frac{\alpha+1}{n}+\frac{ax}{n(1+x)},$$

$$M_n^{\alpha,a}((\varphi^1-x)^2;x)=\frac{(\alpha+1)(\alpha+2)}{n^2}+\frac{2x+x^2}{n}+\frac{a^2x^2}{n^2(1+x)^2}+\frac{2(\alpha+2)ax}{n^2(1+x)},$$

$$\lim_{n\to 0}n^2M_n^{\alpha,a}((\varphi^1-x)^4;x)=12x^2+12x^3+3x^4-3ax^3+\frac{(3x-5)ax^3}{1+x}-\frac{[12(\alpha+2)+4]a^2x^3}{(1+x)^2}.$$

Using the definition of $\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$, it is easy to prove the next theorem.

Theorem 1. *Let* $f\in C_B(\mathbb{R}_+^m)$. *Then*

$$\left\|\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f)\right\|_{C_B(\mathbb{R}_+^m)}\leq\|f\|_{C_B(\mathbb{R}_+^m)}$$

for all $\boldsymbol{n}\in\mathbb{N}^m$.

This paper is devoted to a study aimed at obtaining approximation results by using the modulus of continuity and the Voronovskaya asymptotic formula for the Baskakov--Durrmeyer type operators defined by (1.2) in the space of uniformly continuous and bounded functions of several variables. Approximation properties of various positive linear operators for functions of one, two and several variables have been investigated in many papers (for example [2], [5], [7], [8], [9], [10], [12], [13]).

2. Rate of convergence

In this section we shall prove two theorems on the degree of approximation of functions belonging to the class $C_B(\mathbb{R}_+^m)$ by $\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$. We shall apply the method used in [6].

We denote

$$C_B^1(\mathbb{R}_+^m) := \left\{ f \in C_B(\mathbb{R}_+^m) : \quad \frac{\partial f}{\partial x_k} \in C_B(\mathbb{R}_+^m), \quad 1 \le k \le m \right\}.$$

In order to prove the approximation theorem we need the following result. Let $\varphi^1(s_j) = s_j,\ s_j \in \mathbb{R}_0^+,\ j = 1, 2, \ldots m,\ r \in \mathbb{N}.$

Theorem 2. *If* $g \in C_B^1(\mathbb{R}_+^m)$, *then*

$$\left| M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(g;x) - g(x) \right|$$

$$\le \sum_{j=1}^{m} \left\| \frac{\partial g}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \left(\frac{(\alpha_j + 1)(\alpha_j + 2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2 (1 + x_j)^2} + \frac{2(\alpha_j + 2) a_j x_j}{n_j^2 (1 + x_j)} \right)^{1/2}$$

for all $\boldsymbol{x} \in \mathbb{R}_+^m$, where $\boldsymbol{n} \in \mathbb{N}^m$, $\boldsymbol{a} \in \mathbb{R}_+^m$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\alpha_k > -1$ for $k = 1, 2, \ldots, m$

Proof. Fix $\boldsymbol{x} \in \mathbb{R}_+^m$. For every $\boldsymbol{s} \in \mathbb{R}_+^m$ we have

$$g(\boldsymbol{s}) - g(\boldsymbol{x}) = \sum_{j=1}^{m} \int_{x_j}^{s_j} \frac{\partial}{\partial u_j} g(\boldsymbol{y}_j) du_j,$$

where $\boldsymbol{y}_j = (x_1, \ldots, x_{j-1}, u_j, s_{j+1}, \ldots, s_m)$. Observe that

$$\left| \int_{x_j}^{s_j} \frac{\partial}{\partial u_j} g(\boldsymbol{y}_j) du_j \right| \le \left\| \frac{\partial g}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \left| s_j - x_j \right|$$

and

$$M_{n_j}^{\alpha_j, a_j} \left(\left| \int_{x_j}^{\varphi^1} \frac{\partial}{\partial u_j} g(\boldsymbol{y}_j) du_j \right|; x_j \right) \le \left\| \frac{\partial g}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} M_{n_j}^{\alpha_j, a_j} \left(\left| \varphi^1 - x_j \right|; x_j \right).$$

Applying the Cauchy-Schwarz inequality we obtain

$$M_{n_j}^{\alpha_j, a_j} \left(\left| \int_{x_j}^{\varphi^1} \frac{\partial}{\partial u_j} g(\boldsymbol{y}_j) du_j \right|; x_j \right) \le \left\| \frac{\partial g}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \left(M_{n_j}^{\alpha_j, a_j} \left((\varphi^1 - x_j)^2; x_j \right) \right)^{\frac{1}{2}}.$$

From the above, using the linearity of $M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$ and Lemma 1, we obtain

$$\left| M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(g;\boldsymbol{x}) - g(\boldsymbol{x}) \right|$$

$$\le \sum_{j=1}^{m} \left\| \frac{\partial g}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \left(\frac{(\alpha_j + 1)(\alpha_j + 2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2 (1 + x_j)^2} + \frac{2(\alpha_j + 2) a_j x_j}{n_j^2 (1 + x_j)} \right)^{1/2},$$

whence the result. □

In the next theorem we will use the modulus of continuity of $f \in C_B(\mathbb{R}_+^m)$ given by

$$\omega(f;\boldsymbol{\delta}) = \sup_{\substack{0<h_1\le\delta_1 \\ \dots \\ 0<h_m\le\delta_m}} \|\Delta_{\boldsymbol{h}} f\|_{C_B(\mathbb{R}_+^m)}, \quad \boldsymbol{\delta}, \boldsymbol{h} \in \mathbb{R}_+^m \setminus \{\bar{\boldsymbol{0}}\},$$

where

$$\Delta_h f(\boldsymbol{x}) = f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}_+^m.$$

For a fixed $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$, $0 < \beta_j \le 1$ for $j = 1, 2, \dots, m$, we denote by $Lip(C_B(\mathbb{R}_+^m); \boldsymbol{\beta})$ the class of all functions $f \in C_B(\mathbb{R}_+^m)$ for which $\omega(f;\boldsymbol{\delta}) = O\left(\delta_1^{\beta_1} + \dots + \delta_m^{\beta_m}\right)$ as $\delta_j \to 0^+$ for $j = 1, 2, \dots, \mathrm{m}$.

Theorem 3. *Suppose that* $f \in C_B(\mathbb{R}_+^m)$. *Then for all* $\boldsymbol{x} \in \mathbb{R}_+^m$, *it holds*

$$\left| M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x}) \right| \le 2(m+1)\omega\left(f;\delta\right),$$

where

$$\delta_j = \left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)} \right)^{1/2}, \quad j = 1, 2, \dots, m.$$

Proof. Let $f_{\boldsymbol{\delta}}$ be the Steklov mean of $f \in C_B(\mathbb{R}_+^m)$

$$f_{\boldsymbol{\delta}}(\boldsymbol{x}) = \left(\prod_{j=1}^m \delta_j \right)^{-1} \int_{\boldsymbol{0}}^{\boldsymbol{\delta}} f(\boldsymbol{x}+\boldsymbol{u})\, d\boldsymbol{u}$$

for $\boldsymbol{x} \in \mathbb{R}_+^m$, $\delta_j > 0$ for $j = 1, \dots, m$. We have

$$f_{\boldsymbol{\delta}}(\boldsymbol{x}) - f(\boldsymbol{x}) = \left(\prod_{j=1}^m \delta_j \right)^{-1} \int_{\boldsymbol{0}}^{\boldsymbol{\delta}} (f(\boldsymbol{x}+\boldsymbol{u}) - f(\boldsymbol{x}))d\boldsymbol{u}$$

and

$$\frac{\partial}{\partial x_j} f_{\boldsymbol{\delta}}(\boldsymbol{x}) = \left(\prod_{j=1}^m \delta_j \right)^{-1} \int_0^{\delta_1} \dots \int_0^{\delta_{j-1}} \int_0^{\delta_{j+1}} \dots \int_0^{\delta_m} (f(\boldsymbol{x}+\boldsymbol{u}^*) - f(\boldsymbol{x}+\boldsymbol{u}_*))du_1 \dots du_{j-1} du_{j+1} \dots du_m$$

for $j = 1, \dots, m$, where $\boldsymbol{u}^* = (u_1, \dots, u_{j-1}, \delta_j, u_{j+1}, \dots, u_m)$, $\boldsymbol{u}_* = (u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_m)$.

From this we obtain

$$\|f_{\boldsymbol{\delta}} - f\|_{C_B(\mathbb{R}_+^m)} \le \omega(f;\boldsymbol{\delta}), \tag{2.1}$$

$$\left\| \frac{\partial f_{\delta}}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \le 2\delta_j^{-1}\omega(f;\delta), \tag{2.2}$$

which implies $f_{\delta} \in C_B^1(\mathbb{R}_+^m)$. Hence, for every $\boldsymbol{\delta} \in \mathbb{R}_+^m \setminus \{\bar{\mathbf{0}}\}$, $\boldsymbol{x} \in \mathbb{R}_+^m$ and $\boldsymbol{n} \in \mathbb{N}^m$, we can write

$$\left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x}) \right| \le \left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f - f_{\delta};\boldsymbol{x}) \right| + \left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f_{\delta};\boldsymbol{x}) - f_{\delta}(\boldsymbol{x}) \right| + \left| f_{\delta}(\boldsymbol{x}) - f(\boldsymbol{x}) \right|.$$

Using Theorem 1 and (2.1), we get

$$\left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f - f_{\delta};\boldsymbol{x}) \right| \le \| f_{\delta} - f \| \le \omega(f;\delta).$$

By Theorem 2 and (2.2), it follows

$$\begin{aligned}
&\left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f_{\delta};\boldsymbol{x}) - f_{\delta}(\boldsymbol{x}) \right| \\
&\le \sum_{j=1}^{m} \left\| \frac{\partial f_{\delta}}{\partial x_j} \right\|_{C_B(\mathbb{R}_+^m)} \left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)} \right)^{1/2} \\
&\le 2\omega(f;\delta) \sum_{j=1}^{m} \delta_j^{-1} \left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)} \right)^{1/2}
\end{aligned}$$

Consequently

$$\begin{aligned}
&\left| M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x}) \right| \le 2\omega(f;\delta) \\
&\quad \times \left\{ 1 + \sum_{j=1}^{m} \delta_j^{-1} \left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)} \right)^{1/2} \right\}
\end{aligned}$$

for all $\boldsymbol{\delta} \in \mathbb{R}_+^m \setminus \{\bar{\mathbf{0}}\}$. Choosing $\boldsymbol{\delta}$ with

$$\delta_j = \left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)} \right)^{1/2},$$

$j = 1, \ldots, m$, we obtain the assertion. □

From Theorem 3, using the properties of modulus of continuity for uniformly continous function (see [1], [4]), we can derive the following corollaries.

Corollary 1. *If* $f \in C_B(\mathbb{R}_+^m)$, *then*

$$\lim_{\boldsymbol{n} \to \bar{\infty}} M_{\boldsymbol{n}}^{\alpha,\boldsymbol{a}}(f;\boldsymbol{x}) = f(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \mathbb{R}_+^m$. Moreover, this convergence is uniform on every compact set $\boldsymbol{I} \subset \mathbb{R}_+^m$.

Corollary 2. *Let* $f \in Lip(C_B(\mathbb{R}_+^m);\boldsymbol{\beta})$ *with some fixed* $m \in \mathbb{N}$ *and* $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m)$, $0 < \beta_j \le 1$ for $j = 1,2,\ldots m$. *Then for all* $\boldsymbol{a},\boldsymbol{x} \in \mathbb{R}_+^m$, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m)$, $\alpha_k > -1$ *for* k = $= 1,2,\ldots,m$ *and* $\boldsymbol{n} \in \mathbb{N}^m$, *it holds*

$$\left|M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x})\right| \le \sum_{j=1}^{m}\left(\frac{(\alpha_j+1)(\alpha_j+2)}{n_j^2} + \frac{2x_j + x_j^2}{n_j} + \frac{a_j^2 x_j^2}{n_j^2(1+x_j)^2} + \frac{2(\alpha_j+2)a_j x_j}{n_j^2(1+x_j)}\right)^{\beta_j/2}.$$

3. The Voronovskaya type theorem

Let $\bar{\boldsymbol{n}} = (n,\ldots,n) \in \mathbb{N}^m$. In this part, we will consider the operator $M_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}$. In order to state the Voronovskaya type theorem we need the following result, which is a simple consequence of Lemma 1.

Lemma 2. *Let* $\boldsymbol{x} = (x_1,\ldots,x_m) \in \mathbb{R}_+^m$ *be a fixed point. Then*

$$\lim_{n\to\infty} nM_n^{\alpha_j,a_j}(\varphi^1 - x_j;x_j) = \alpha_j + 1 + \frac{a_j x_j}{1+x_j},$$

$$\lim_{n\to\infty} n\,M_n^{\alpha_j,a_j}(\varphi^1 - x_j;x_j)M_n^{\alpha_i,a_i}(\varphi^1 - x_i;x_i) = 0, \quad i \ne j,$$

$$\lim_{n\to\infty} n\,M_n^{\alpha_j,a_j}((\varphi^1 - x_j)^2;x_j) = 2x_j + x_j^2,$$

$$\lim_{n\to\infty} n^2 M_n^{\alpha_j,a_j}((\varphi^1 - x_j)^4;x_j) = 12x_j^2 + 12x_j^3 + 3x_j^4 - 3a_j x_j^3 + \frac{(3x_j - 5)a_j x_j^3}{1+x_j} - \frac{[12(\alpha_j+2)+4]a_j^2 x_j^3}{(1+x_j)^2} \tag{3.1}$$

for $j = 1,2,\ldots,m$.

Theorem 4. *Let* $f \in C_B(\mathbb{R}_+^m)$ and $\boldsymbol{x} \in \mathbb{R}_+^m$. *If f is of the class* $C_B^1(\mathbb{R}_+^m)$ *in a certain neighbourhood of a point* $\boldsymbol{x}$ *and* $f''(\boldsymbol{x})$ *exists (in the Fréchet sense), then for every* $\boldsymbol{x} \in \mathbb{R}_+^m$, *we have*

$$\lim_{n\to\infty} n\{M_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x})\} = \sum_{j=1}^{m}\left(\alpha_j + 1 + \frac{a_j x_j}{1+x_j}\right)\frac{\partial f(\boldsymbol{x})}{\partial x_j} + \sum_{j=1}^{m}\left(x_j + \frac{1}{2}x_j^2\right)\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j^2}.$$

Proof. Let $\boldsymbol{x}$ be a fixed point in $\mathbb{R}_+^m$. By Taylor's formula we get

$$f(\boldsymbol{s}) = f(\boldsymbol{x}) + \sum_{j=1}^{m}(s_j - x_j)\frac{\partial f(\boldsymbol{x})}{\partial x_j} + \frac{1}{2}\left\{\sum_{j=1}^{m}(s_j - x_j)^2\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j^2}\right.$$
$$\left. +2\sum_{i\neq j,i,j=1}^{m}(s_i - x_i)(s_j - x_j)\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}\right\} + \psi_{\boldsymbol{x}}(\boldsymbol{s})\left(\sum_{j=1}^{m}(s_j - x_j)^4\right)^{1/2},$$

where the function ψ_x is uniformly continuous and bounded in $\mathbb{R}_+^m$ and $\lim_{\boldsymbol{s}\to\boldsymbol{x}}\psi_{\boldsymbol{x}}(\boldsymbol{s}) = 0.$ From linearity of $\boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{a}}$, we obtain

$$\begin{aligned} n\{\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x})\} &= n\sum_{j=1}^{m} M_n^{\alpha_j,a_j}(\varphi^1 - x_j;x_j)\frac{\partial f(\boldsymbol{x})}{\partial x_j} \\ &+\frac{1}{2}n\left\{\sum_{j=1}^{m} M_n^{\alpha_j,a_j}((\varphi^1 - x_j)^2;x_j)\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j^2}\right. \\ &\left.+2\sum_{i\neq j,i,j=1}^{m} M_n^{\alpha_j,a_j}(\varphi^1 - x_j;x_j)M_n^{\alpha_i,a_i}(\varphi^1 - x_i;x_i)\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i\partial x_j}\right\} \\ &+n\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\psi_{\boldsymbol{x}}\phi_{\boldsymbol{x}};\boldsymbol{x}), \end{aligned} \tag{3.2}$$

where $\phi_{\boldsymbol{x}}(\boldsymbol{s}) = \left(\sum_{j=1}^{m}(s_j - x_j)^4\right)^{1/2}$. Using the Cauchy-Schwarz inequality we obtain

$$n\left|\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\psi_{\boldsymbol{x}}\phi_{\boldsymbol{x}};\boldsymbol{x})\right| \le \left|\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\psi_{\boldsymbol{x}}^2;\boldsymbol{x})\right|^{1/2}\left|n^2\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\phi_{\boldsymbol{x}}^2;\boldsymbol{x})\right|^{1/2}.$$

Moreover, the function $\psi_{\boldsymbol{x}}^2$ satisfies the assumption of Corollary 1. Hence

$$\lim_{n\to\infty}\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\psi_{\boldsymbol{x}}^2;\boldsymbol{x}) = \psi_{\boldsymbol{x}}^2(\boldsymbol{x}) = 0.$$

Observe that

$$\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\phi_{\boldsymbol{x}}^2;\boldsymbol{x}) = \boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}\left(\sum_{j=1}^{m}(\varphi^1 - x_j)^4;\boldsymbol{x}\right) = \sum_{j=1}^{m} M_n^{\alpha_j,a_j}((\varphi^1 - x_j)^4;x_j).$$

Using (3.1) we obtain

$$\lim_{n\to\infty} n\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(\psi_{\boldsymbol{x}}\phi_{\boldsymbol{x}};\boldsymbol{x}) = 0. \tag{3.3}$$

From (3.2), (3.3) and Lemma 2 we get the assertion. □

Corollary 3. *Let $\boldsymbol{x}\in\mathbb{R}_+^m$. If f satisfies the assumption of Theorem 4, then*

$$\left|\boldsymbol{M}_{\bar{\boldsymbol{n}}}^{\boldsymbol{\alpha},\boldsymbol{a}}(f;\boldsymbol{x}) - f(\boldsymbol{x})\right| = O\left(\frac{1}{n}\right) \quad as \quad n\to\infty.$$

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