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A GENERALISATION OF THE CHIRKA-SADULLAEV THEOREM FOR COMPLEX MANIFOLDS

UOGÓLNIENIE TWIERDZENIA CHIRKI-SADULLAEVA DLA ROZMAITOŚCI ZESPOLONYCH

Abstract

The aim of this paper is to generalize the Chirka-Sadullaev theorem, fundamental in the theory of extension of separately holomorphic functions with singularities, to the case of a σ-compact Josefson manifold.

Keywords: *separately holomorphic function*, *extension with singularities*, *pluripolar set*, *complex manifold*

Streszczenie

Celem pracy jest uogólnienie twierdzenia Chirki-Sadullaeva, podstawowego twierdzenia w teorii przedłużania funkcji oddzielnie holomorficznych z osobliwościami, na przypadek σ-zwartych rozmaitości Josefsona.

Słowa kluczowe: *funkcja oddzielnie holomorficzna*, *przedłużanie z osobliwościami*, *zbiór pluripolarny*, *rozmaitość zespolona*

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1. Introduction and prerequisites

In [1], the authors proved a deep result that is now considered fundamental in the theory of extensions of separately holomorphic functions with singularities.

Theorem 1.1. (Theorem 1 from [1]) Let *f* be a holomorphic function on the polydisk $U'\times U_n$ in $\mathbb{C}^n=\mathbb{C}^{n-1}\times\mathbb{C},$ and for each fixed a' in some nonpluripolar set $E\subset U',$ the function $f(a',z_n)$ can be continued holomorphically to the whole plane with the exception of some polar set of singularities $M(a')\subset\mathbb{C},$ then *f* can be continued holomorphically to $(U'\times\mathbb{C})\setminus S,$ where *S* is a closed pluripolar subset of $(U'\times\mathbb{C}).$

Theorem 1.1 remains true if U' is a domain of holomorphy in $\mathbb{C}^{n-1}$ and U_n is a domain in $\mathbb{C}$. Moreover, a set *S* in the conclusion can be minimalised in the sense that for any $a'\in E,$ the fiber $S_{(a',\cdot)}:=\{z_n\in\mathbb{C}:(a',z_n)\in S\}$ is contained in $M(a')$ and for any $z'\in U',$ the fiber $S_{(z',\cdot)}$ is polar (see, for instance, Theorem 9.2.24 in [5] for details).

Applications in mathematical tomography found by O. Öktem (see [7], [8]) revived an interest in the theory of extensions of separately holomorphic functions with singularities (see [2], [3], [4]) and possible generalisations of the theory to the general case of complex manifolds (see [6], [9]). However, one of the main problems in the development of the theory in the case of manifolds is the lack of analogs of many fundamental results. In this paper, Theorem 1.1 is generalised to the case of σ-compact Josefson manifolds.

Recall that a manifold is called σ-*compact* (or *countable at infinity*) if it is a union of countably many compact subsets. A complex manifold *X* is called a *Josefson manifold* if every locally pluripolar set in *X* is globally pluripolar.

2. Main Theorem

Theorem 2.1. Let *D* be a σ-compact connected Josefson complex manifold of dimension *n* and let $A\subset D$ not be pluripolar. Let Δ be a nonempty domain in $\mathbb{C}$. For any $a\in A$, let $M(a)\subset\mathbb{C}$ be a closed polar set such that $\Delta\cap M(a)=\varnothing.$ Let $S\subset\mathcal{O}(D\times\Delta)$ be such that for any $f\in S$ and $a\in A$, there exists a function $\tilde{f}_a\in\mathcal{O}(\mathbb{C}\setminus M(a))$ such that $\tilde{f}_a(\cdot)=f(a,\cdot)$ on $\Delta.$

Then there exists a closed pluripolar set $\widehat{M}\subset D\times\mathbb{C}$ such that:

(a) $\widehat{M}\cap(D\times\Delta)=\varnothing,$

(b) $\widehat{M}_{(a,\cdot)}\subset M(a), a\in A,$

(c) $\widehat{M}_{(z,\cdot)}\subset\mathbb{C}$ is polar, $z\in D,$

(d) any $f\in S$ extends holomorphically to $\hat{f}\in\mathcal{O}((D\times\mathbb{C})\setminus\widehat{M}).$

Proof of Theorem 2.1.

Step 1.

Let U be a domain in D such that there exists a biholomorphism $\Phi : U \to \Omega_U$, where Ω_U is a domain in $\mathbb{C}^n$. Assume that $A_U := A \cap U$ is not pluripolar. Then $B := \Phi(A_U) \subset \Omega_U$ is not pluripolar. For any $b \in B$, define $M(b) := M(a)$, where $a = \Phi^{-1}(b) \in A_U$. Thus $M(b)$ is closed, polar and $M(b) \cap \Delta = \varnothing$. Define a family $S_U := \left\{ f|_{U\times\Delta} : f \in S \right\}$. From the assumptions for any $f \in S_U$ and $a \in A_U$ there exists a function $\tilde{f}_{U,a} := \tilde{f}_a \in \mathcal{O}(\mathbb{C} \setminus M(a))$ such that $\tilde{f}_{U,a}(w) = f(a,w)$, $w \in \Delta$. Define a new family $\mathcal{F} := \{g(b,w) = f(\Phi^{-1}(b), w) : f \in S_U\}$. Then $\mathcal{F} \subset \mathcal{O}(\Omega_U \times \Delta)$ and for each $g \in \mathcal{F}$ and $b \in B$, a function $\tilde{g}_b(w) := \tilde{f}_{U,\Phi^{-1}(b)}(w)$ is, from its definition, holomorphic on $\mathbb{C} \setminus M(b)$ and $\tilde{g}_b(w) = f(\Phi^{-1}(b), w) = g(b,w)$ for $w \in \Delta$.

Hence, from Theorem 1.1, there exists a relatively closed pluripolar set $\widetilde{M} \subset \Omega_U \times \mathbb{C}$ such that:

- $\widetilde{M}_{(b,\cdot)} \subset M(b), b \in B$,
- $\widetilde{M}_{(b,\cdot)} \subset \mathbb{C}$ is polar, $b \in \Omega_U$,
- $\widetilde{M} \cap (\Omega_U \times \Delta) = \varnothing$,
- for any $g \in \mathcal{F}$, there exists a function $\hat{g} \in \mathcal{O}((\Omega_U \times \mathbb{C}) \setminus \widetilde{M})$ such that $g = \hat{g}$ on $\Omega_U \times \Delta$.

Define a set $\widehat{M} := \{(\Phi^{-1}(b), w) : (b,w) \in \widetilde{M}\}$. Fix a $z \in U$ and let $b = \Phi(z)$. Then

$$\widehat{M}_{(z,\cdot)} = \{w \in \mathbb{C} : (z,w) \in \widehat{M}\} = \{w \in \mathbb{C} : (\Phi^{-1}(b), w) \in \widehat{M}\} = \{w \in \mathbb{C} : (b,w) \in \widetilde{M}\} = \widetilde{M}_{(b,\cdot)}$$

Hence, $\widehat{M}_{(a,\cdot)} \subset M(a)$ for each $a \in A_U$ and $\widehat{M}_{(z,\cdot)}$ is polar for $z \in U$. Now, assume that there exists a point $(a,w) \in \widehat{M}$ such that $(a,w) \in U \times \Delta$. Let $b = \Phi(a)$. Then $(b,w) \in \Omega_U \times \Delta$ and, since $(a,w) \in \{(\Phi^{-1}(b), w) : (b,w) \in \widetilde{M}\}$, $(b,w) \in \widetilde{M}$. Thus, $\widetilde{M} \cap (\Omega_U \times \Delta) \neq \varnothing$ – a contradiction.

For fixed $f \in S_U$, define $\hat{f}(a,w) := \hat{g}(\Phi(a), w)$. Because $(a,w) \in \widehat{M}$ if, and only if, $(\Phi(a), w) \in \widetilde{M}$, $\hat{f} \in \mathcal{O}((U \times \mathbb{C}) \setminus \widehat{M})$. Moreover, $\hat{f}(z,w) = \hat{g}(\Phi(z), w) = g(\Phi(z), w) = f(z,w)$ for $(z,w) \in U \times \Delta$, where last equality follows from the definition of the family $\mathcal{F}$. Thus, any function $f \in S$ has an extension

$$\widehat{f_U} := \begin{cases} \widehat{f} & \text{on} \quad (U\times\mathbb{C})\setminus \widehat{M} \\ f & \text{on} \quad D\times\Delta \end{cases},$$

which is well defined and holomorphic on $(D\times\Delta)\cup((U\times\mathbb{C})\setminus\widehat{M})$.

Step 2.

Let U denote a domain in D such that there exists a relatively closed pluripolar set $\widehat{M}_U \subset U\times\mathbb{C}$ with the following properties:

(1) $(\widehat{M}_U)_{(a,\cdot)} \subset M(a),\ a\in A\cap U,$

(2) $(\widehat{M}_U)_{(z,\cdot)} \subset \mathbb{C}$ is polar, $z\in U,$

(3) $\widehat{M}_U \cap (U\times\Delta)=\varnothing,$

(4) for any $f\in S$, there exists an extension $\widehat{f_U}\in\mathcal{O}((D\times\Delta)\cup((U\times\mathbb{C})\setminus\widehat{M}_U))$ such that $f=\widehat{f_U}$ on $D\times\Delta$.

Let V be a domain in D, biholomorphic to a domain in $\mathbb{C}^n$, such that $V\cap U\neq\varnothing$. Define a family $S_V=\left\{f|_{V\times\Delta} : f\in S\right\}$. For any $a\in U\cap V$, define $M(a):=(\widehat{M}_U)_{(a,\cdot)}$. If the set $(V\cap A)\setminus U$ is not empty, for $a\in(V\cap A)\setminus U$ let $M(a)$ be as in the assumptions of Theorem 2.1. For any $f\in S_V$ for $a\in U\cap V$ define $\widetilde{f}_{V,a}(\cdot):=\widehat{f_U}(a,\cdot)$ and for $a\in(V\cap A)\setminus U$ (if not empty) let $\widetilde{f}_{V,a}:=\widetilde{f}_a$ from the assumptions. From Step 1, with V playing the role of U and A_U replaced by $(V\cap U)\cup(A\cap V)$ [1] there exists a relatively closed, pluripolar set $\widehat{M}_V\subset V\times\mathbb{C}$ such that:

– $(\widehat{M}_V)_{(a,\cdot)}\subset(\widehat{M}_U)_{(a,\cdot)}\subset M(a),\ a\in A\cap V\cap U,$

– $(\widehat{M}_V)_{(a,\cdot)}\subset M(a),\ a\in(A\cap V)\setminus U,$

– $(\widehat{M}_V)_{(z,\cdot)}\subset\mathbb{C}$ is polar, $z\in V,$

– $\widehat{M}_V\cap(V\times\Delta)=\varnothing,$

– for any $f\in S$, there exists $\widehat{f_V}\in\mathcal{O}((D\times\Delta)\cup((V\times\mathbb{C})\setminus\widehat{M}_V))$ such that $f=\widehat{f_V}$ on $D\times\Delta$.

Then a set $\widehat{M}:=\widehat{M}_U\cup\widehat{M}_V\subset(U\cup V)\times\mathbb{C}$ is relatively closed and pluripolar, $\widehat{M}\cap(D\times\Delta)=\varnothing$, for each $a\in A$, the fiber $\widehat{M}_{(a,\cdot)}=(\widehat{M}_U)_{(a,\cdot)}\cup(\widehat{M}_V)_{(a,\cdot)}\subset M(a)$ and for any $z\in D$, $\widehat{M}_{(z,\cdot)}$ is polar. Fix $f\in S$. The function f has two extensions:

[1] $V\cap U$ is necessary, because $A\cap V$ alone could be pluripolar.

$\widehat{f}_U \in \mathcal{O}((D\times\Delta)\cup((U\times\mathbb{C})\setminus \widehat{M}_U))$ and $\widehat{f}_V \in \mathcal{O}((D\times\Delta)\cup((V\times\mathbb{C})\setminus \widehat{M}_V))$ such that on $D \times \Delta$ the equality $\widehat{f}_U = \widehat{f}_V = f$ holds. Define a function

$$\widehat{f} := \begin{cases} \widehat{f}_U & \text{on} \quad (U\times\mathbb{C})\setminus \widehat{M} \\ \widehat{f}_V & \text{on} \quad (V\times\mathbb{C})\setminus \widehat{M} \\ f & \text{on} \quad D\times\Delta \end{cases}.$$

Observe that any connected component of $((U\cap V)\times\mathbb{C})\setminus \widehat{M}$ is a domain intersecting $D\times\Delta$. Thus, from the identity principle, the functions $\widehat{f}_U$ and $\widehat{f}_V$ agree also on $((U\cap V)\times\mathbb{C})\setminus \widehat{M}$. Hence, $\widehat{f}$ is well defined and holomorphic on $(D\times\Delta)\cup$ $\cup(((U\cap V)\times\mathbb{C})\setminus \widehat{M})$.

Step 3.

Since D is σ-compact, there exists a countable covering $\{U_j\}_{j=1}^{\infty}$ such that each U_j is biholomorphic to a domain in $\mathbb{C}^n$. Because A is not pluripolar, there exists j_0 such that $A\cap U_{j_0}$ is not pluripolar. From Step 1, we obtain a relatively closed pluripolar set $\widehat{M}_{U_{j_0}} \subset U_{j_0}\times\mathbb{C}$ having properties (1) to (4). From Step 2, for any $U_j \neq U_{j_0}$ such that $U_j\cap U_{j_0}\neq\varnothing$ there exists a relatively closed pluripolar set $\widehat{M}_{U_j}\subset U_j\times\mathbb{C}$ having same properties. Define $D_1 := \bigcup\{U_j : U_j\cap U_{j_0}\neq\varnothing\}$, a set $\widehat{M}_{D_1} := \bigcup\{\widehat{M}_{U_j} : U_j\in D_1\}$ and a function

$$\widehat{f}_{D_1} := \begin{cases} \widehat{f}_{U_j} & \text{on} \quad (U_j\times\mathbb{C})\setminus \widehat{M}_{D_1},\ j\neq j_0 \\ \widehat{f}_{U_{j_0}} & \text{on} \quad (U_{j_0}\times\mathbb{C})\setminus \widehat{M}_{D_1} \\ f & \text{on} \quad D\times\Delta \end{cases}$$

From Step 2, $\widehat{f}_{D_1}$ is well defined and holomorphic on $(D\times\Delta)\cup((D_1\times\mathbb{C})\setminus \widehat{M}_{D_1})$ and $\widehat{f}_{D_1} = f$ on $D\times\Delta$. Thus, $\widehat{M}_{D_1}$ has properties (1) to (4). Now, for any $U_j\notin D_1$ such that $U_j\cap D_1\neq\varnothing$ from Step 2 there exists a relatively closed pluripolar set $\widehat{M}_{U_j}\subset U_j\times\mathbb{C}$ having properties (1) to (4). Define $D_2 := \bigcup\{U_j : U_j\cap D_1\neq\varnothing\}$, $\widehat{M}_{D_2} := \bigcup\{\widehat{M}_{U_j} : U_j\in D_2\}$ and

$$\widehat{f}_{D_2} := \begin{cases} \widehat{f}_{U_j} & \text{on} \quad (U_j \times \mathbb{C}) \setminus \widehat{M}_{D_2},\ U_j \notin D_1 \\ \widehat{f}_{D_1} & \text{on} \quad (D_1 \times \mathbb{C}) \setminus \widehat{M}_{D_2} \\ f & \text{on} \quad D \times \Delta \end{cases}$$

Once again, $\widehat{f}_{D_2}$ is well defined and holomorphic on $(D\times\Delta)\cup(D_2\times\mathbb{C})\setminus\widehat{M}_{D_2}$), $\widehat{f}_{D_2} = f$ on $D\times\Delta$ and $\widehat{M}_{D_2}$ has all properties (1) to (4).

We obtain an open covering $\{D_k\}_{k=1}^{\infty}$ of D such that for each $k=1,2,\ldots$ there exists a relatively closed pluripolar set $\widehat{M}_{D_k}$ having properties (1) to (4). Define $\widehat{M} := \bigcap_{k=1}^{\infty} \widehat{M}_{D_k}$ and a function $\widehat{f} := \widehat{f}_{D_k}$ on $(D_k\times\mathbb{C})\setminus\widehat{M}$. Thus, $\widehat{f}$ is well defined and holomorphic on $(D\times\mathbb{C})\setminus\widehat{M}$ and $\widehat{f} = f$ on $D\times\Delta$. Hence, the set $\hat{M}$ has all properties (a) to (d) and proof of Theorem 2.1 is finished.

□

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