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CARATHÉODORY COMPLETENESS ON THE PLANE

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Dedicated to the memory of Professor Józef Siciak

Abstract. M. A. Selby [**8–10**] and, independently, N. Sibony [**11**] proved that on the complex plane c-completeness is equivalent to c-finitely compactness. Their proofs are quite similar and are based on [**4**]. We give more refined equivalent conditions and, along the way, simplify the proofs.

1. Introduction. Let $D \subset \mathbb{C}^n$ be a domain and let $\zeta \in \partial D$ be its boundary point. We denote by $A(D \cup \{\zeta\})$ the set of all bounded holomorphic functions on D which extend continuously to $D \cup \{\zeta\}$. Following [**7**], we say that ζ is a *weak peak point* for D if there exists a function $f \in A(D \cup \{\zeta\})$ such that $|f| < 1$ on D and $f(\zeta) = 1$.

Theorem 1. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. Then the following conditions are equivalent:*
(1) *ζ is a weak peak point for D;*
(2) *there exists no finite Borel measure μ on D such that*

$$|f(\zeta)| \leq \int |f| d\mu \quad \textit{for any } f \in A(D \cup \{\zeta\});$$

(3) *we have*

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta, a) \setminus D) = +\infty,$$

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where $A_n(\zeta) = \{z \in \mathbb{C} : \frac{1}{2^{n+1}} \leq |z - \zeta| \leq \frac{1}{2^n}\}$ *and* γ *is the analytic capacity (see the definition below).*

The equivalency of (1) and (3) in Theorem 1 was proved by M. A. Selby (see [**9**]). Note that the implication (1) $\implies$ (2) is straightforward (also in a higher dimension). The implication (2) $\implies$ (1) in any dimension is claimed in [**2**]. However, the proof is based on a false version of Hahn–Banach theorem, claimed in [**5**]. So, we give a new proof on the complex plane. In a higher dimension, it is still an open problem whether (2) $\implies$ (1).

Let $\mathbb{D}(\zeta, r) = \{z \in \mathbb{C} : |z - \zeta| < r\}$ denote the disk on the complex plane and let $\mathbb{D} = \mathbb{D}(0, 1)$ denote the unit disk. We define the Poincaré function p on $\mathbb{D}$ as

$$p(\lambda_1, \lambda_2) = \frac{1}{2} \log \frac{1 + m(\lambda_1, \lambda_2)}{1 - m(\lambda_1, \lambda_2)}, \quad \lambda_1, \lambda_2 \in \mathbb{D},$$

where $m(\lambda_1, \lambda_2) = \left|\frac{\lambda_1 - \lambda_2}{1 - \bar{\lambda}_1 \lambda_2}\right|$ is the Möbius function.

Let $D \subset \mathbb{C}^n$, $n \geq 1$, be a domain. For $z_1, z_2 \in D$ put

$$c_D(z_1, z_2) = \sup\{p(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D})\}, \tag{1}$$

$$c_D^*(z_1, z_2) = \sup\{m(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D})\}, \tag{2}$$

where $\mathcal{O}(D; \mathbb{D})$ denotes the set of all holomorphic mappings $D \to \mathbb{D}$. c_D is called the Carathéodory pseudodistance for D (see e.g. [**6**]). In case when c_D is indeed a distance we say that D is c-hyperbolic. A c-hyperbolic domain D is called c-complete if any c_D-Cauchy sequence $\{z_\nu\}_{\nu \geq 1} \subset D$ converges to a point $z_0 \in D$ (w.r.t. Euclidean topology).

The aim of this paper is to study more carefully the completeness on the complex plane. Along the way we simplify the proofs by M. A. Selby [**8–10**] and by N. Sibony [**11**].

We say that a measurable set $F \subset \mathbb{C}$ is of positive density at a point $\zeta \in \mathbb{C}$ if

$$\limsup_{r \to 0+} \frac{\mathcal{L}\big(\overline{\mathbb{D}(\zeta; r)} \cap F\big)}{r^2} > 0.$$

First we show the following result.

Theorem 2. *Let* $D \subset \mathbb{C}$ *be a domain and let* $\zeta \in \partial D$ *be its boundary point. If* ζ *is not a weak peak point for* D *then*

$$\lim_{r \to 0+} \frac{\mathcal{L}\big(\mathbb{D}(\zeta; r) \cap D\big)}{\pi r^2} = 1.$$

We have the following inverse of Theorem 2.

THEOREM 3. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. Assume that*

$$\lim_{r \to 0+} \frac{\mathcal{L}\big(\mathbb{D}(\zeta; r) \cap D\big)}{\pi r^2} = 1.$$

Then the following conditions are equivalent to conditions (1), (2), (3) in Theorem 1:

(4) *there exists a set $A \subset D$ of positive density at ζ such that for any sequence $\{z_\nu\}_{n \geq 1} \subset A$ with $z_\nu \to \zeta$ we have $c_D(z_0, z_\nu) \to \infty$;*
(5) *there exists a set $A \subset D$ of positive density at ζ such that for any sequence $\{z_\nu\}_{\nu \geq 1} \subset A$ such that $z_\nu \to \zeta$ there follows that $\{z_\nu\}$ is not a c_D-Cauchy sequence.*

Note that the implications (1) $\implies$ (4) $\implies$ (5) are straightforward. Essentially, the main result of the paper is showing that (5) $\implies$ (2). In case $A = \Omega$ in Theorem 3, the result is proved in [**8**] and [**11**].

2. Proof of Theorem 1. Recall the definition of the analytic capacity (see e.g. Chapter VIII in [**3**]). Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. The analytic capacity of a compact set K is defined by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in \mathcal{O}(\Omega), \|f\| \leq 1, f(\infty) = 0\},$$

where Ω is the unbounded component of $\widehat{\mathbb{C}} \setminus K$ and

$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).$$

For any set $F \subset \mathbb{C}$ we put

$$\gamma(F) = \sup\{\gamma(K) : K \subset F \text{ compact}\}.$$

Recall also the following characterization (see Theorem VIII.4.5 in [**3**]).

THEOREM 4 (Melnikov's criterion). *Let $K \subset \mathbb{C}$ be a compact set and let $\zeta \in K$. Then ζ is a peak point for $R(K)$ if and only if*

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) = +\infty.$$

Note that the implication (1) $\implies$ (2) in Theorem 1 is immediate. Let us show the implication (2) $\implies$ (3).

PROOF OF (2) $\implies$ (3) IN THEOREM 1. Assume that

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus D) < +\infty.$$

Using the continuity property of the analytic capacity (see the proof of Theorem 3.1 in [**4**]) one can show that there exists a compact set $K \subset D \cup \{\zeta\}$ such that

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) < +\infty.$$

By Melnikov's criterion ζ is a not peak point for $R(K)$. Hence, by Bishop's characterization of peak points (see e.g. [**3**]) there exists a Borel probability measure μ on K such that $\mu(\{\zeta\}) = 0$ and

$$f(\zeta) = \int f d\mu \quad \text{for any } f \in R(K).$$

Note that $A(D \cup \{\zeta\}) \subset R(K)$ (see Corollary 8 below). Hence,

$$f(\zeta) = \int f d\mu \quad \text{for any } f \in A(D \cup \{\zeta\}).$$

A contradiction. □

3. Proof of Theorem 2. Let $\mathcal{L}$ denote the Lebesgue measure in $\mathbb{C}$. Recall the following well-known result (see e.g. [**1**], Lemma 1.5).

Proposition 5. *Let $K \subset \mathbb{C}$ be a compact set. Then the function*

$$f(z) = \int_K \frac{d\mathcal{L}(\eta)}{z - \eta}$$

is holomorphic on $\widehat{\mathbb{C}} \setminus K$, continuous on $\widehat{\mathbb{C}}$ and $f(\infty) = 0$. Moreover,

$$|f(z)| \le \int_K \frac{1}{|z-\eta|} d\mathcal{L}(\eta) \le 2\sqrt{\pi \mathcal{L}(K)}. \tag{3}$$

As a corollary of Proposition 5 we get Theorem 2 (cf. Corollary VIII.4.2 in [**3**]).

Proof of Theorem 2. Assume that

$$\limsup_{r \to 0+} \frac{\mathcal{L}(\overline{\mathbb{D}(\zeta; r)} \setminus D)}{r^2} > 0.$$

Choose $r_n \to 0+$ and $b > 0$ such that $\mathcal{L}(K_n) > b r_n^2$, where $K_n = \overline{\mathbb{D}(\zeta; r_n)} \setminus D$. Put

$$g_n(z) = \frac{1}{\mathcal{L}(K_n)} \cdot (z - \zeta) \int_{K_n} \frac{d\mathcal{L}(\eta)}{z - \eta}.$$

From Proposition 5 there follows that g_n is a continuous function on $\widehat{\mathbb{C}}$, holomorphic on $\widehat{\mathbb{C}} \setminus K_n$, $g_n(\infty) = 1$.

Note that for any $z \in \mathbb{C}$ such that $|z-\zeta| \leq r_n$ we have

$$|g_n(z)| \leq \frac{2r_n\sqrt{\pi\mathcal{L}(K_n)}}{\mathcal{L}(K_n)} \leq 2\sqrt{\frac{\pi}{b}}.$$

From the maximum principle we see that the above inequality holds on the whole $\widehat{\mathbb{C}}$. Now we proceed as in the proof of Theorem VIII.4.1 in [**3**] and get a weak peak function for D. □

4. Proof of Theorem 3. We denote by $\mathcal{M}$ the set of all positive finite Borel measures in $\mathbb{C}$. For $\mu \in \mathcal{M}$ we define its Newton potential as

$$M(z) = M_\mu(z) = \int \frac{1}{|z-\eta|} d\mu(\eta).$$

From the inequality (3) we have

$$\frac{1}{\pi r^2} \int_{\mathbb{D}(\eta,r)} |z-\eta| \cdot M(z) d\mathcal{L}(z) \leq 2\mu(\mathbb{C}), \tag{4}$$

and, therefore, $M < \infty$ a.e. on $\mathbb{C}$. The following result, which essentially is a corollary of Fubini's theorem, shows the behaviour of the left side of (4) when $r \to 0$ (see e.g. [**12**], Lemma 26.16).

Proposition 6. *Let $\mu \in \mathcal{M}$. For any $\eta \in \mathbb{C}$ we have*

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{\mathbb{D}(\eta,r)} |z-\eta| \cdot M(z) d\mathcal{L}(z) = \mu(\{\eta\}).$$

In particular, if $\mu(\{\eta\}) = 0$, then for any $\epsilon > 0$ the set

$$\Pi(\epsilon) = \{z \in \mathbb{C} : |z-\eta| \cdot M(z) > \epsilon\}$$

is of zero density at η, i.e.,

$$\lim_{r\to 0} \frac{\mathcal{L}(\Pi(\epsilon) \cap \mathbb{D}(\eta, r))}{r^2} = 0.$$

Recall the following approximation result (see e.g., Theorem 10.8 in Chapter VIII in [**3**]).

Theorem 7. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. For any $f \in H^\infty(D)$ there exists a sequence $\{f_n\}_{n\geq 1} \subset H^\infty(D)$ with $\|f_n\|_D \leq 17\|f\|_D$ such that $f_n \to f$ locally uniformly on D and each f_n extends holomorphically to a neighborhood of ζ. Moreover, if f extends continuously to ζ, then f_n tends to f uniformly on D.*

From this we get.

Corollary 8. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. Then for any compact set $K \subset D \cup \{\zeta\}$ we have $A(D \cup \{\zeta\}) \subset R(K)$.*

The following simple observation holds true.

PROPOSITION 9. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$. Assume that μ is a finite Borel measure in D such that*

$$|f(\zeta)| \leq \int |f| d\mu$$

for any $f \in A(D \cup \{\zeta\})$. Then for any $\eta \in D$ we have

$$|f(\eta) - f(\zeta)| \leq 2\|f\|_\infty M(\eta)|\eta - \zeta|.$$

In particular, for any $\eta_1, \eta_2 \in D$ we have

$$c_D^*(\eta_1, \eta_2) \leq 34\Big(|\zeta - \eta_1|M(\eta_1) + |\zeta - \eta_2|M(\eta_2)\Big). \tag{5}$$

PROOF. Fix $\eta \in D$. Then for any $f \in A(D \cup \{\zeta\})$ we have $\widetilde{f}(z) = \frac{f(z)-f(\eta)}{z-\eta} \in A(D \cup \{\zeta\})$. Then

$$|\widetilde{f}(\zeta)| \leq \int |\widetilde{f}| d\mu.$$

Hence,

$$|f(\zeta) - f(\eta)| \leq |\zeta - \eta| \int \left| \frac{f(z) - f(\eta)}{z - \eta} \right| d\mu(z) \leq 2\|f\|_\infty M(\eta)|\eta - \zeta|.$$

Inequality (5) follows from Theorem 7. □

We have the following corollary, which proofs the implication (5) $\Longrightarrow$ (2).

COROLLARY 10. *Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$. Assume that μ is a finite Borel measure in D such that*

$$|f(\zeta)| \leq \int |f| d\mu$$

for any $f \in A(D \cup \{\zeta\})$. Then for any measurable set $A \subset D$ of positive density at ζ there exists a c-Cauchy sequence $\{\eta_n\}_{n \geq 1} \subset A$ such that $\eta_n \to \zeta$.

PROOF. If

$$\liminf_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} < 1$$

then by Theorem 2 ζ is a weak peak point, which contradicts the existence of the measure μ. So,

$$\lim_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} = 1.$$

Hence,

$$\limsup_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap A)}{r^2} > 0.$$

Then by Proposition 6 there exists a sequence $\{\eta_n\}_{n\geq 1} \subset D$ with $\eta_n \to \zeta$ such that $|\zeta - \eta_n| M(\eta_n) \leq \frac{1}{2^n}$. From Theorem 9 we get the result. □

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