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KAROLINA KARPIŃSKA*

TEACHING THINKING IN TERMS OF FUNCTIONS – FULFILLING THE FUNDAMENTAL IDEA OF THE MERANO PROGRAMME AT TORUN CLASSICAL GRAMMAR SCHOOL IN THE EARLY TWENTIETH CENTURY

WYRABIANIE NAWYKÓW MYŚLENIA FUNKCYJNEGO – REALIZACJA POSTULATU PROGRAMU MERAŃSKIEGO W TORUŃSKIM GIMNAZJUM KLASYCZNYM W PIERWSZYCH LATACH DWUDZIESTEGO WIEKU

A b s t r a c t

In this article, one of the main postulates of the Merano Programme for teaching mathematics will be analysed, namely: teaching thinking in terms of functions. This postulate will be discussed in the context of its implementation at Torun Classical Grammar School. Through the detailed analysis of mathematics curricula implemented in the Torun lassical Grammar School at the beginning of the twentieth century, the mathematical textbooks then used and school-leaving examinations problems from the years 1905–1911, the degree of fulfillment of the said postulate of the Merano Programme at the Torun school will be assessed.

Keywords: *mathematics*, *functions*, *dependent variables*, *Merano Programme*, *Torun Grammar School*, *Torun Classical Grammar School*, *Torun Real School*, *school-leaving examinations in the early 20th c.*

S t r e s z c z e n i e

W niniejszym artykule analizie zostanie poddany jeden z głównych postulatów Programu Merańskiego dotyczących nauczania matematyki, czyli wyrabiania nawyków myślenia funkcyjnego. Postulat ten omówimy w kontekście jego realizacji w toruńskim Gimnazjum Klasycznym. Szczegółowa analiza programów nauczania matematyki realizowanych w toruńskim Gimnazjum Klasycznym na początku XX wieku, stosowanych wówczas podręczników do matematyki oraz zestawów zadań maturalnych z lat 1905–1911, pozwoli ocenić stopień realizacji wspomnianego postulatu Programu Merańskiego w szkole toruńskiej.

Słowa kluczowe: *matematyka*, *funkcje*, *zmienne zależne*, *Program Merański*, *Gimnazjum Toruńskie*, *toruńskie Gimnazjum Klasyczne*, *toruńska Szkoła Realna*, *egzaminy maturalne w pierwszych latach XX wieku*

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* Ph.D. student at the L. & A. Birkenmajer Institute for History of Science, Polish Academy of Sciences, Warsaw, Poland; karolinakarpinska001@gmail.com

1. Introduction

The Merano Programme was the reform of teaching mathematics and natural sciences, which has been prepared for secondary schools in Prussia: Classical Grammar Schools, Real Grammar Schools and Higher Real Schools. The Merano Programme was announced in 1905 at the conference of the Society of German Natural Scientists and Physicians (dt.: *Gesellschaft Deutscher Naturforscher und Ärzte*) in Merano[1]. Its main author was Felix Klein – a mathematician of Göttingen.

2. Postulates of the Merano Programme

The Merano Proramme changed the overall organization of secondary education. It stated that since 1905 three types of schools in Prussia: Classical Grammar Schools, Real Grammar Schools and Higher Real Schools, had the same rights. Since 1905, each of these schools had to put equal emphasis on education in terms of both mathematics and natural sciences, as well as philological and historical sciences [17, p. 17]. Thus, Grammar Schools lost their one-sided humanistic direction, and Real School – its mathematical and natural sciences direction. Each school was giving a general education and their graduates would join a university on equal terms[2].

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[1] Merano is a city located in the southern part of Tyrol, northern Italy. Tyrol in the years 1814–1919 was a part of Austria [19, p. 117]. After World War I, under the treaty of St. Germain (20 September 1919), its southern part, along with Merano, was annexed to Italy [13, pp. 69-70].

[2] Up to 1905, the school-leaving certificate obtained in Grammar Schools, Real Grammar Schools and Higher Real Schools, enabled the commencement of study at university or institute of technology under certain conditions [17, p. 11]:

– if a graduate of Grammar School wanted to enter an institute of technology, he had to pass an exam in mathematics and natural sciences; this obligation did not concern Real Grammar Schools and Higher Real Schools graduates,
– if a graduate of Real Grammar School or Higher Real School wanted to start studying at a faculty different from mathematics or natural sciences, he had to provide a certificate of sufficient knowledge of ancient languages,
– medical and theological studies were open only to people who obtained the school-leaving certificate in Grammar Schools.

Additional exams aroused a fear in Grammar School students of studying mathematics and natural sciences [17, p. 11].

The aim of the Merano reform was to provide equal level of education at Grammar Schools, Real Grammar Schools and Higher Real Schools, and thus the elimination of additional entry exams at university and institute of technology.

In the case of mathematics lessons, the Merano Programme recommended the same curricula and timetables (four hours per week) in the corresponding classes of Grammar Schools and Real Grammar Schools. Higher Real Schools had the same curriculum framework as Grammar Schools and Real Grammar Schools, but they discussed certain issues in more detail. The hourly schedule of mathematics classes at Higher Real Schools was as follows (from the lowest class): 5, 5, 6, 6, 5, 5, 5, 5, 5 [11].

A compendium of guidelines of the Merano Programme connected with teaching of mathematics was as follows [17]:

General learning objectives:
- logical thinking,
- ability to think independently,
- ability to model natural phenomena mathematically,
- awareness that mathematics plays an important role in all areas of life and is essential for human and industrial society.

Specific learning objectives:
- development of spatial imagination,
- **teaching thinking in terms of functions**,
- combining various mathematical problems,
- paying attention to the application of mathematics,
- the balance between applications and theory in mathematics,
- a common approach to plane geometry and stereometry,
- emphasis on the history of mathematics.

Teaching methods:
- the genetic method – 'one should connect ideas, put new knowledge in an inseparable relation with the knowledge already gained, eventually associate the knowledge with the rest of the school curriculum, more and more, so that combination of knowledge would grow and students become more aware',
- the psychological principle – the material should be adapted to the course of the intellectual development of students,
- the principle of utility – showing that mathematics is important for everyday life.

Teaching thinking in terms of functions became a symbol of the mathematical part of the Merano Programme [17, p. 14]. In the curricula recommended by the Merano Programme it was clearly stated that up to the Lower Tertia this should be done by educating the student's intuition of variability – students should observe that the change of some values affects changes of others, they should observe the way of changing and notice that each result is in a relationship with changing some values. Ability to distinguish independent and dependent values, understanding the relationship between them and appealing to student's intuition, allowed for making in the Upper Tertia first graphical representations of such relationships (between two variables: the independent variable and the dependent variable) and for simultaneous introduction of the concept of a function and its graph. The students of the highest classes were to learn functions such as the quadratic function, trigonometric functions and the logarithmic function, check their properties and use them, for example, to solve equations. The only issue in which the Merano Programme left freedom to teachers was whether to discuss differential and integral calculus or not [17, p. 20].

Let us see if the curricula implemented at Torun Classical Grammar School after 1905 contained issues related to teaching thinking in term of functions and, if so, whether they were consistent with the guidelines of the Merano Programme.

3. Teaching thinking in terms of functions at Torun Classical Grammar School

3.1. The curricula of mathematics

Analysis of the curricula contained in the school reports of Torun Grammar School leads to the conclusion that in the years 1905–1911 the curricula of mathematics at Torun Classical Grammar School were identical [8] from year to year. Namely:

Sexta:	Arithmetic operations on integer numbers (absolute and denominate numbers). The German measures, weights and coins along with practicing decimal notation and easy calculations in the decimal system. Preparation for making calculations on fractions (4 hours per week).
Quinta:	Divisibility of numbers. Factorization. Common fractions. Four arithmetic operations on numbers written in the decimal system. Simple tasks using the Rule of Three (4 hours per week).
Quarta:	Calculations on decimal numbers. The Rule of Three (simple and composed) with integers and fractions. Tasks related to civic life, the simplest cases of calculating percents, interests and discounts. – Preparation for geometry. Exercises in using the compass and ruler. Learning about straight lines, angles and triangles (4 hours per week).
Lower Tertia:	The introduction of positive and negative numbers. Tasks that are based on solving equations of the first degree with one unknown. – Extending the learning on triangles. Parallelograms. Chords and angles in a circle. Construction problems (3 hours per week).
Upper Tertia:	Review of operations on fractions which are denoted with the use of letters. Proportions. Equations of the first degree with one or more unknowns. Powers whose exponents are positive integers. – Learning about the circle. Equality of figures in terms of the surface area. Calculating the area of straight-line figures. Construction problems (3 hours per week).
Lower Secunda:	Powers, roots, logarithms. Calculations using four-digit logarithmic tables. Simple quadratic equations with one and two unknowns. – The similarity, proportionality of straight lines in a circle, golden section. Regular polygons. Circumference and area of a circle, tasks associated with these problems. Construction problems (4 hours per week).
Upper Secunda:	Equations, in particular quadratic equations with more than one unknown. Harmonic points, radii of circles, secants. Application of algebra in geometry[3].

[3] In the textbooks of the nineteenth and early twentieth century, part of geometry dealing with applications of algebra in geometry was called algebraic geometry. The main task of algebraic

Construction problems, especially with using algebraic analysis[4]. Goniometry. Simple calculations related to solving triangles (4 hours per week).

geometry was to find, through algebraic calculations, specific information about a given geometric figure (solid), taking into consideration the specified information relating to this figure (solid), for example: the length of some of its sides, heights, diagonals, measures of certain angles, or a relationship between some elements of this figure (solid) [1, p. 219].

Example of algebraic geometry problem:

Exercise [1, p. 247, ex. 22]. A triangle with area equal to a^2 is given. One of the sides of this triangle is divided into two parts, which are to each other as 3:7. Then through the dividing point of this side we draw a line parallel to the side of the triangle which is adjacent to the shorter part of the division. How big is each of the two parts of the triangle which are defined by this parallel line?

It was believed that of all the applications of algebra, the most important was its use in geometry: to formulate propositions and solve construction problems, using a method called algebraic analysis – this method was a part of algebraic geometry [1, p. 219].

[4] The following scheme of reasoning was called 'the solution to the construction problem using algebraic analysis method': 'firstly, the unknown lines of the figure should be found using algebraic calculations, then the arithmetic expressions should be constructed and used to construct the whole figure' [1, p. 255].

Solution of the following exercise illustrates the way in which in the nineteenth century construction problems were solved by algebraic analysis:

Exercise [1, p. 255]. Divide a given rectangle into two parts by a line parallel to one side of this rectangle so that the circumferences of these parts are to each other as 2:3.

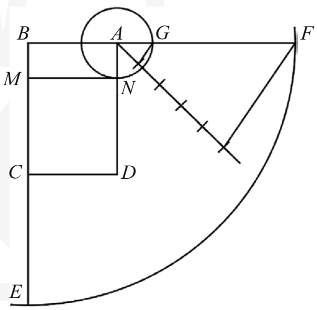
Solution [1, pp. 255-256]:

Analysis: Let a and b be the sides of a given rectangle, let the line dividing the rectangle be parallel to a and located at a distance of x from a; then:

$$2a + 2x : 2a + 2(b - x) = 2 : 3,$$

therefore:

$$x = \frac{2b - a}{5}.$$



Construction: Let $ABCD$ be a given rectangle; let us extend the side BC (through C) to the point E in such a way that: $BF = BE = 2b$, then $AF = 2b - a$. On AF we mark out $AG = \frac{1}{5}AF$; next, on the side AD we mark out $AN = AG$ and draw NM – a line parallel to AB. Thus, $ABCD$ is divided through NM into two parts satisfying the given condition.

Requirements: $2b > a$.

□

Besides the algebraic analysis which was a method of solving construction problems, there also existed a branch of mathematics with the same name: algebraic analysis. The main task of algebraic analysis, as a branch of mathematics, was to solve the three types of tasks ([15, pp. VI– VII).

Assumption: Let an equation $y = f(x)$ be given, where x is an independent variable, f – a function, y – a dependent variable.

Lower Prima: Arithmetic and geometric sequences; civic calculations; quadratic equations; extension of the concept of numbers on imaginary numbers. Apollonius's problems[5] according to the old methods and other construction tasks. Solving triangles using the sum and difference of their sides, radii of tangent circles, angles and heights of these triangles. The main theorems on relative position of points, lines and planes in space. Calculation of the areas and volumes of solids, such as: prism, pyramid, cylinder, cone and sphere. Revision of the material of previous classes (4 hours per week).

Upper Prima: Binomial theorem for exponents that are positive integers. The equations of higher degrees, which can be reduced to quadratic equations. Basic information about the coordinates, the equation of a straight line, a circle and conics. Construction problems. Repetition of stereometry; the introduction of certain formulas of spherical trigonometry in relation to the Earth and Sky[6].

Problem 1. Knowing x and f, determine y.
Problem 2. Knowing y and f, determine x.
Problem 3. Knowing x and y, determine the formula of the function f.

These tasks were related to one of the fundamental concepts of mathematical analysis: that of a function, and their solution required the use of algebraic calculations, hence the name 'algebraic analysis'.

In some cases, to the three problems above the following were appended (if they could be solved algebraically) [15, p. VII].

Problem 4. Knowing the formula of the function f, determine its properties.
Problem 5. Knowing the properties of the function f, determine its formula.

In the nineteenth century many textbooks were written containing an exposition of algebraic analysis, for example [7, 14, 15, 18].

In *Principles of algebraic analysis with supplements to the 'Arithmetic to school- and self-study'* [*Anfangsgründe der algebraischen Analysis nebst Ergänzungen zur Arithmetik für den Schul- und Selbst-Unterricht*] by K. Koppe [7], the chapter devoted to the algebraic analysis included the following topics: general comments about functions and the most important theorems about equations of higher degrees (for example, cubic equations were solved there), series (discussed, for example, were: convergence of series, exponential series, logarithmic series and periodic functions), complex numbers.

[5] Apollonius's problems are problems that require the construction of a circle satisfying three conditions, each of which may have one of the following forms:
- the circle has to pass through a given point;
- the circle has to be tangent to a given line;
- the circle has to be tangent to a given circle [10, p. 16].

For example, the following problem is an Apollonius's problem: Construct a circle which passes through the three given points.

[6] In addition, the curriculum of physics in the Upper Prima contained mathematical geography and mathematical astronomy. A comprehensive article about teaching of mathematical astronomy can be found in the school report of Grammar School in Bydgoszcz from the year 1906/1907 [5, pp. 3-22].

Revision of the previous classes, according to the guidelines of Mehler[7] (4 hours per week).

Note that there was no theory of functions at Torun Classical Grammar School curricula in the years 1905–1911, there were also no clear signs that habits of thinking in terms of functions were developed. On the other hand, it can be seen that the Merano Programme attached a great importance to showing functional relations, discussing the functions, their graphs and applications.

Bearing in mind that the curricula of Torun Grammar School were written concisely, and also that in the curricula of the top class (Upper Prima) of the Classical Grammar School in the years 1902–1905 there were 'remarks about functions' [8], after 1905, we will explore further the implementation of this issue of the Merano Programme (namely: teaching thinking in terms of functions) at Torun Classical Grammar School by analyzing the textbooks from that time and school-leaving examinations problems.

Were habits of thinking in terms of functions formed in mathematics lessons at Torun Classical Grammar School? An answer to this question will be given on the basis of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3] (used in the classes from Lower Tertia to Upper Prima at Torun Classical Grammar School), textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12] (used in the classes from Quarta to Upper Prima at Torun Classical Grammar School)[8] and school-leaving examinations tasks from the years 1905–1911.

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[7] That is, according to the issues contained in the textbook: *Basic theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F. G. Mehler [12].

[8] In the late nineteenth century and the first decade of the twentieth century at Torun Grammar School the following textbooks of mathematics were used [8, 1907, p. 15]:

1. *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12]:
   - classes IV – IU (Quarta – Upper Prima), Classical Grammar School,
   - classes IV – IU (Quarta – Upper Prima), Real Grammar School.
2. *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3]:
   - classes IIIL – IU (Lower Tertia – Upper Prima), Classical Grammar School,
   - classes IV – IU (Quarta – Upper Prima), Real Grammar School.
3. *Four-digit logarithmic tables with mathematical, physical and astronomical tables* [*Vierstellige Logarithmentafeln nebst mathematischen, physikalischen und astronomischen Tabellen*] by A. Schülke [16]:
   - classes IIL – IU (Lower Secunda – Upper Prima), Classical Grammar School,
   - classes IIL – IU (Lower Secunda – Upper Prima), Real Grammar School.
4. *Problems in elementary arithmetic* [*Aufgaben zum Ziffernrechnen*] by J. Blümel and R. E. Pflüger, parts III, IV and V:
   - classes VI – IV (Sexta – Quarta), Classical Grammar School and Real Grammar School.

3.2. Methodically structured collection of problems [Methodisch geordnete Aufgabensammlung] by F. Bardey

Bardey's *Methodically structured collection of problems* includes more than 8000 exercises. Before certain batches of exercises the author placed the theory needed to solve them. However, these cases were only individual and concerned issues that were not discussed in the *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar--Mathematik*] by F. G. Mehler [12]. Analyzing the contents of tasks and the theory in Bardey's collection we can observe that there is no mention of the word 'function' at all. Nevertheless, in a supplement to a collection of problems, on page 326 (the whole collection of problems contains 330 pages) the author took up the explanation of what the independent variables and the dependent variables are. He noted that 'the independent variable' is the name for a variable, denoted generally by x, which may take any value, for example: 0, 1, 2, 3 and so on. 'The dependent variable' is the name for a variable, usually denoted by y, whose value depends on the value of the variable x, and this relationship is defined by a certain equation with two variables x and y.

On the basis of the definition of a function given by T. Gutkowski in *Elementary algebra*, part I (Warsaw 1918), which was evaluated by K. Wuczyńska (in [20]) as a textbook maintaining the spirit of the Merano Programme [20, p. 271, 278], it can be stated that according to Klein 'dependent variable' and 'function' were synonymous.

Thus, it appears that, despite the fact that in *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] the word 'function' does not appear, the author dealt with functions. After defining the independent variable and the dependent variable (or function), he explained in which way the relationship between them can be represented graphically. He began with a graphic representation of first-degree equations with variables x and y, then passed to the equations of the second degree, pointed out that their graphic representations are conics, and using graphs he found minimum and maximum values of both variables [3, pp. 326-330].

Although, in the collection of problems, functions were discussed in a symbolic way (Bardey dealt with graphical representations of functions, but did not use them e.g. to solve equations; he did not explore the properties of the function either), the thinking in the term of function was very often used by the author.

Bardey tried to develop the student's intuition of variability and functional dependence. He prepared tasks that allowed students to observe that the change of some values affects changes of others. Later, he went one step further and demanded that students made analysis of occurring changes. Some other time he demanded to describe mathematically the relationship between values, and finally, he passed to the tasks in which students, knowingly or not, had to write formulas of functions, calculate when the functions take the given values or find their minima and maxima.

Some of the tasks may be treated as strictly preparatory for the introduction of dependent variables. Among them are the following:

– Tasks that point out to the fact that the change in some values causes others to change, for example:

Exercise [3, p. 4, ex. 47]. What values are taken by the following expressions:

1. $a - b + c - d$	9. $a + b(c - d)$
2. $a - (b + c) - d$	10. $a - b(c - d)$
3. $a - b - c + d$	11. $(a - b)c + d$
4. $a - b - (c + d)$	12. $(a + b) : c - d$
5. $(a + b)(c + d)$	13. $a + b : c + d$
6. $(a - b)(c - d)$	14. $(a + b) : (c + d)$
7. $a + bc - d$	15. $a - b : c - d$
8. $a - b + cd$	16. $(a - b) : (c - d)$

1. for $a = 30$, $b = 12$, $c = 3$, $d = 2$;
2. for $a = 96$, $b = 36$, $c = 6$, $d = 3$;
3. for $a = 72$, $b = 12$, $c = 6$, $d = 2$?

Exercise [3, p. 22, ex. 142]. By how much a product 831 · 754 will decrease, if its first factor increases by 1, and the second factor will decrease by 1? How much this product will increase when his first factor will decrease by 1 and the other one will increase by 1?

Exercise [3, p. 22, ex. 143]. By how much a rectangle (meaning: the area of the rectangle), whose sides have lengths 793 and 137 feet, will increase if the longer side of this rectangle is extended by 5 feet and the shorter side is extended by 7 feet (give an answer without calculating the area of the rectangle)?

Exercise [3, p. 83, ex. 1]. What are the logarithms with base 2 of the numbers 2, 4, 64, 16, 128, 32, 1, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$?

Exercise [3, p. 91, ex. 92]. What value does the expression $\sqrt{1-\frac{ac}{b^2}}$ take for $a = 28{,}371$, $b = 39{,}832$ and $c = 41{,}504$?

Exercise [3, p. 91, ex. 93]. The same as above for $a = 173{,}54$, $b = 375{,}42$ and $c = 280{,}19$?

– The change of some values does not always cause others to change, for example:

Exercise [3, p. 239, ex. 14]. The diagonal of a rectangle is 65 feet long. If the smaller side of this rectangle is shortened by 17 feet and the larger side is extended by 7 feet, the diagonal of the rectangle will not be changed. How long are the sides of this rectangle?

– Tasks that point to the fact that some values can depend on others by means of mathematical formulas, for example:

Exercise [3, p. 59, ex. 39]. Knowing that $\sqrt{50} = a$, calculate: $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$, $\sqrt{72}$ and $\sqrt{98}$.

Exercise [3, p. 84, ex. 36]. Knowing that log 2 = 0,30103 and log 3 = 0,47712, calculate: log 4, log 5, log 6, log 8, log 9, log 12, log 15, log 16, log 18, log 20, log 24, log 25, log 27, log 30, log 32, log 36, log 40, where log 10 =1 (the base of the logarithm is 10).

Exercise [3, p. 84, ex. 20]. How big are: log 530, log 5300, log 53000, log 5300000, log 5,3, log 0,53, log 0,0053, when log 53 = 1,7243? (Prove!)

– Certain values are directly or indirectly proportional, for example:

Exercise [3, p. 4, ex. 49]. Calculate in the memory[9]:

1. 1 lot costs respectively: 2, 7, 10, 15 pfennig; how much does 1 gram cost?
2. 1 lot costs respectively: 4, 6, 9, 14 pfennig; how much does 1 pound cost?
3. 1 pound costs respectively: 65, 70, 80, 85 pfennig; how much does 1 Hundredweight cost?
4. 1 litre costs respectively: 10, 15, 20, 24 pfennig; how much does 1 hectolitre cost?
5. 1 metre costs respectively: 3, 5, $7\frac{1}{2}$ Mark, 9 Mark 30 pfennig; how much does 1 centimetre cost?
6. 1 Hundredweight costs respectively: 13, 50, 300, $715\frac{1}{2}$ Mark, 87 Mark 30 pfennig; how much does 1 pound cost?
7. 1 gram costs respectively: 4, 11, 15 Mark, 7 Mark 30 pfennig; how much does 1 lot cost?
8. 1 pound costs respectively: $1\frac{1}{2}$, $2\frac{1}{2}$, 5 Mark, 3 Mark 20 pfennig; how much does 1 lot cost?
9. 1 hectolitre costs respectively: 7, 9, $20\frac{1}{2}$ Mark, 30 Mark 40 pfennig; how much does 1 litre cost?

Exercise [3, p. 22, ex. 153]. Prove the following theorems:

1. 1 pound costs *a* pfennig, therefore *b* hundredweight costs *ab* Mark.
2. 1 lot costs *a* pfennig, therefore *b* grams costs *ab* Mark.
3. 1 lot costs *a* pfennig, therefore *b* pounds costs $\frac{1}{2}ab$ Mark.
4. 1 litre costs *a* pfennig, therefore *b* hectolitres costs *ab* Mark.

Exercise [3, p. 22, ex. 154.2]. How much does respectively: 4, $5\frac{1}{8}$, $6\frac{1}{4}$ pounds, 9 pounds 8 lot, cost, when 1 lot costs respectively: 2, 3, 4, 5 pfennig?

Other tasks, already touching upon the essence of dependent variables, for example:

Exercise [3, p. 271, ex. 85]. An equilateral triangle is given. From the height of this triangle we build another equilateral triangle, from the height of the second triangle we build a third triangle and so on. What is the sum of all these triangles (including the given triangle)?

Exercise [3, p. 92, ex. 106] What is the base of a logarithm in which:

1. log 10 = 2;
2. log 100 = 3;
3. log 3 = 4;
4. log 33 = 3;
5. log 444 = 4;
6. log 666 = 6?

[9] 1 pound = 30 lot = 500 grams; 1 Hundredweight = 100 pounds [10, p. 127]. 1 Mark = 100 pfennig.

Exercise [3, p. 246, ex. 17]. Find one solution of a diophantine equation $13x + 5y = 444$, then determine another solutions and find the general form of solution of this equation.

There are also the tasks that require writing a formula of a function and calculating the minimum or maximum values of this function[10], such as:

[10] In mathematics lessons at Torun Grammar School tasks relating to the calculation of minimum and maximum values of functions were solved without using differential calculus (at that time in the Torun Grammar School curricula there was no differential calculus).

In *Principles of algebraic analysis with supplements to the 'Arithmetic to school- and self-study'* [*Anfangsgründe der algebraischen Analysis nebst Ergänzungen zur Arithmetik für den Schul- und Selbst-Unterricht*] [7] Karl Koppe showed the method of solving this kind of tasks without using the derivatives. This method was based on three fundamental theorems:

Theorem 1 [7, p. 1]. Let m and x are positive numbers, then the following expressions (dependent on x):

1. $2mx - x^2$,
2. $3m^2x - x^3$,
3. $3mx^2 - 2x^3$,
4. $4m^3x - x^4$,
5. $4m^2x^2 - 2x^4$,
6. $4mx^3 - 3x^4$,

attain the greatest value when $x = m$.

Proof. 1 [7, p. 1]. The expression $2mx - x^2$ is equal to m^2 when $x = m$. To demonstrate that this is the greatest value of $2mx - x^2$, we have to add to this expression $m^2 - m^2 = 0$. As a result, we obtain the following equation: $2mx - x^2 = m^2 - (m^2 - 2mx + x^2) = m^2 - (m - x)^2$. The subtrahend $(m - x)^2$ vanishes when $x = m$, whereas in all other cases it assumes a positive value. Thus, the above expression is greatest when $x = m$.

□

Theorem 2 [7, p. 1, 3]. Let m and x be positive numbers, then the following expressions (dependent on x):

1. $\frac{m^2}{x} + x$,
2. $\frac{m^3}{x^2} + 2x$,
3. $\frac{2m^3}{x} + x^2$,
4. $\frac{m^4}{x^3} + 3x$,
5. $\frac{2m^4}{x^2} + 2x^2$,
6. $\frac{3m^4}{x} + x^3$,

attain the smallest value when $x = m$.

Proof. 1 [7, p. 3]. It is easy to see, that: $\frac{m^2}{x} + x = \frac{m^2 + x^2}{x} = 2m + \frac{(m-x)^2}{x}$. Firstly, we have to notice that $\frac{(m-x)^2}{x}$ is always non-negative. Therefore, the expression $\frac{m^2}{x} + x$ is the smallest when the second summand of $2m + \frac{(m-x)^2}{x}$ vanishes, that is, when $x = m$.

□

Exercise [3, p. 217, ex. 130]. A circle of radius r is given. Find a rectangle inscribed in this circle, which:

1. has the largest area,
2. has the greatest perimeter[11].

Theorem 3 [7, p. 4]. If an expression dependent on x (x assumes only positive values) for a specific value of x is the largest or the smallest, the same applies to expression which arises from the previous one by:

– adding, subtracting, multiplying, or dividing the previous expression by an expression independent of x,
– raising the previous expression to any power,
– taking any root of the previous expression.

[11] Solution of this problem consists of two parts. The first part is concerned with finding a rectangle with the largest area, the second one – a rectangle with the greatest perimeter. We will use: Theorem 1 and Theorem 3, which were given in the previous footnote.

Solution [7, p. 6; in original notation]:

Let x and y are the sides of the rectangle. Then, we obtain the following equations:

$$x^2 + y^2 = 4r^2 \quad \text{and}$$

1. xy = maximum.

   From the above equation we eliminate y:

$$x\sqrt{4r^2 - x^2} = \text{maximum.}$$

   On the base of Theorem 3, placed in the previous footnote, we can raise the left hand side of this equation to the square (it does not change the solution of this equation):

$$4r^2x^2 - x^4 = \text{maximum.}$$

   Now, we want to bring the last equation into the form:

$$4m^2x^2 - x^4 = \text{maximum}$$

   (because we want to use Theorem 1 given in the previous footnote), so we multiply by 2 its left hand side:

$$8r^2x^2 - 2x^4 = \text{maximum.}$$

   Hence, we obtain that:

$$m^2 = 2r^2,$$

   or:

$$m = r\sqrt{2}.$$

   On the basis of Theorem 1, we obtain that:

$$x' = r\sqrt{2} = y'.$$

   Thus, of all rectangles inscribed in a circle, the square has the largest area.

2. $2x + 2y$ = maximum, or: $x + y$ = maximum.

   From the above equation we eliminate y:

$$x + \sqrt{4r^2 - x^2} = \text{maximum.}$$

Exercise [3, p. 217, ex. 123]. Of all triangles satisfying the property: the sum of the base and the height is equal to a, find the one which has the greatest area and calculate this area.

Exercise [3, p. 217, ex. 117]. Of all rectangles of the same area, find the one with the smallest circumference.

Moreover, there were more sophisticated problems that required using the formula of velocity, cosine theorem, Pythagoras's theorem, writing a formula of a function and calculating for which arguments function takes a given value, such as:

Exercise [3, p. 215, ex. 106]. In a vertex of an equilateral triangle with a side a, there are two bodies. One of them moves along one side of the triangle at a speed of m m/s and the second body moves along another side that is adjacent to the previous one, at a speed of n m/s. When will the distance between these bodies be equal to the height of the triangle?

Exercise [3, p. 215, ex. 104]. Two circles are given. The centers of these circles are located on two perpendicular lines and move toward the point of intersection of these lines. The first circle has a radius of 100 meters, in each second it covers a distance of 3 meters and the center of this circle is located 247 meters from the intersection of the perpendicular lines. The second circle has a radius of 35 meters, in each second it covers a distance of 2 meters and the center of this circle is located 169 meters from the intersection of perpendicular lines. After how many seconds will the circles be internally tangent?

Exercise [3, p. 210, ex. 41]. Two bodies A and B are situated on different arms of an angle. The body A is located 123 meters from the vertex of the angle and starts to move away from the vertex at a speed of 239 m/s. The body B is located 239 meters from the vertex and starts to move toward to the vertex at a speed of 123 m/s. When will the distance of these bodies be equal to 850 meters?

In *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*], Bardey also puts tasks whose solution requires using the 1-1 property (injectivity) of the power functions, exponential and logarithmic functions, for example:

Exercise [3, p. 193, ex. 191]. Solve the equation: $\sqrt{x+3}+\sqrt{2x-3}=6$.

Exercise [3, p. 119, ex. 7]. Solve the equation: $(a^{x-5})^{x-6}=(a^{x-8})^{x-1}$.

We raise to the square the left hand side of this equation (on the basis of Theorem 3):

$$4r^2+2x\sqrt{4r^2-x^2}=\text{maximum},$$

and remove the constants, which will not change the solution (on the basis of Theorem 3):

$$x\sqrt{4r^2-x^2}=\text{maximum}.$$

Once again, we raise to the square the left hand side of last equation:

$$4r^2x^2-x^4=\text{maximum}.$$

This equation is the same as the equation obtained at the first point of this solution. Therefore, further steps of this reasoning (point 2) are the same as steps described at point 1. Thus, of all rectangles inscribed in a circle, the square has the greatest perimeter.

Exercise [3, p. 202, ex. 156]. Solve the equation: $\log\sqrt{7x+5}+\frac{1}{2}\log(2x+7)=$ $=1+\log 4{,}5.$

In the collection of problems, Bardey did not include the definition of an injective function. Moreover, in the solutions, which were included in Bardey's book, there was no remark that at a certain moment author used the 1-1 property of a function. This suggests that students used the 1-1 property of functions, but they were not aware of it.

3.3. Fundamental theorems of elementary mathematics [Hauptsätze der Elementar--Mathematik] by F. G. Mehler

The textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] [12] was divided by Mehler into five main chapters: planimetry, algebra, trigonometry, series and binomial theorem, stereometry. In this textbook, as the title indicates, only theory was included, namely: definitions, theorems, proofs and certain patterns of reasoning, e.g. methods of solving equations [12, pp. 58–72] or a method of converting numbers into continued fractions [12, pp. 72–80]. Only in a few cases the author did exercises designed to apply theory in practice [12, pp. 13–15, 21, 26–27, 34, 41–45, 50, 75, 77, 78–79, 97, 100, 109–111]. Tasks to be solved independently by students were almost completely ignored by Mehler (one of them can be found: [12, p. 33]).

In Mehler's textbook, functions were discussed only in the context of trigonometric functions – the author gave definitions of trigonometric functions and he thoroughly discussed their properties: the relationship between trigonometric functions, signs of trigonometric functions, their periodicity, evenness and oddness.

In *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] there was no definition of a dependent variable, there was also no evidence of indicating functional relationship between certain values. Furthermore, in the chapter concerning series and binomial theorem, when Mehler develops e^x in a power series, he avoids explaining that e^x is an exponential function [12, p. 104]. Therefore, it can be assumed that Mehler did not intend to make general considerations about functions. He did not apply functional approach to various mathematical problems either. He discussed only trigonometric functions, because they are the basis of planimetric and stereometric considerations (including spherical trigonometry). They are also very important to calculations on complex numbers.

It should be stressed that Mehler's textbook was published in 1869, that is, 36 years before the promulgation of the Merano reform. Therefore, we cannot criticize the author for the lack of implementation of the Merano Programme postulates. Should Torun Grammar School have definitively abandoned Mehler's textbook after 1905? It turns out that not necessarily so. Together with Bardey's collection of problems, this textbook perfectly implemented the postulate of functional approach to various mathematical problems. Definitions and tasks included in Bardey's collection of problems allowed one to analyse the contents of Mehler's textbook in terms of dependent variables and as a result, the vast majority of the material could be read in the terms of function. For example:

- **Ptolemy's Theorem** (if a quadrilateral can by inscribed in a circle, then the product of its diagonals is equal to the sum of the products of the pairs of opposite sides [12, p. 32]) can be understood as follows: the product of the diagonals of a quadrilateral inscribed in a circle is a function of its sides.
- **Exercise** [12, p. 33, §87]. Construct a fourth proportional to the three given lines: a, b and c (it means: the fourth component x, such that $\frac{b}{c} = \frac{c}{x}$).

  The solution of this exercise allowed the students to note that the fourth proportional x is a function of the three given lines: a, b and c.

  In a similar way the following theorems can be read:
- **Theorem** [12, p. 17, §47]. The sum of angles of a polygon with n sides is equal to $2n - 4$ right angles.
- **Theorem** [12, p. 82, §160]. Assume that: the initial capital is equal to c, the annual interest on one thaler is equal to p and n is the number of years after which the capital will increase to the amount of k. Then:

$$k = c(1 + p)^n.$$

- **Theorem** [12, p. 122, §224]. If h is a height of a cone, r a radius of its base, and s its lateral face, then the lateral area of this cone is equal to πrs.

3.4. School-leaving examinations

By analyzing the school-leaving examinations in mathematics which were carried out at Torun Classical Grammar School in the years 1905–1911 [8], we conclude that the teachers of mathematics at this institution had to attach a great importance to construction tasks. As a part of every written school-leaving examination in mathematics at Torun Classical Grammar School in the years 1905–1911, there was a construction task. Some of them required finding the geometric locus of points. Tasks of this type provide excellent opportunities for practicing thinking in terms of functions – students can observe how the change of some values affects changes of others, moreover, they have to look for a relationship between certain values.

Examples of school-leaving examinations problems:

Exercise [8, 1908, p. 9, ex. 1]. Construct a triangle when

1. radius of a circle circumscribed about this triangle,
2. one of angles of this triangle and
3. a square, which is the sum of squares of the sides adjacent to the given angle

are given.

Exercise [8, 1907, p. 9, ex. 1]. Construct a quadrilateral when

1. two osculate sides of the quadrilateral,
2. the ratio of the two other sides and
3. both diagonals

are given.

Exercise [8, 1910, ex. 2]. Draw a rectangle whose circumference and area are equal to the circumference and area of a given triangle[12].

[12] Solution of the exercise:
Given: a triangle *ABC* with sides: *a*, *b* and *c*.
Description of the construction (sketch):

1. We construct a rectangle with the same area as the given triangle:
   - we determine the center of the base *AB* of the given triangle *ABC*, we denote it by *D*,
   - we mark out a perpendicular line to the base *AB* passing through the vertex *A* (line *l*) and perpendicular line to the base *AB* passing through the point *D* (line *u*),
   - we mark out the parallel line to the base *AB* passing through the vertex *C* (line *n*),
   - a point of intersection of *n* with *l* we denote by *E*, a point of intersection of *n* with *u* we denote by *F*,
   - rectangle *ADEF* has the same area as the given triangle *ABC*.
2. We draw a straight line *m*, then select any point on this line and draw the perpendicular line to *m* passing through the selected point – a line *r*.
3. On the same side of the line *m* we draw a few rectangles, each of them satisfying the following properties:
   - the circumference of the rectangle is equal to $a + b + c$,
   - one of its sides lies on the line *m*,
   - line *r* is the axis of symmetry of the rectangle.

   Then, we can notice that the left upper vertices of these rectangles determine a straight line (denote it by *s*), right upper vertices also designate a straight line (denote it by *t*). Lines *s* and *t* are symmetrical with respect to the line *r*. A piece of line *s* restricted by *m* and *r* (denote it by s') is the geometric locus of the left upper vertices of the rectangles with one of the sides lying on *m* and with a circumference equal to $a + b + c$. The piece of line *t* restricted by *m* and *r* (denote it by t') is the geometric locus of the right upper vertices of the rectangles with one of the sides lying on *m* and with a circumference equal to $a + b + c$.
4. On the same side of the line *m* which we chose in the previous point (point 3), we draw a few more rectangles, each of them satisfying the following properties:
   - its area is equal to the area of the rectangle *ADEF*,
   - one of the sides of this rectangle lies on the line *m*,
   - line *r* is the axis of symmetry of this rectangle.

   Then, the left upper vertices of these rectangles determine a piece of a hyperbola (denote it by *p*) and so do the upper right vertices, too (denote this piece of a hyperbola by *q*).
   The pieces *p* and *q* are symmetrical with respect to the line *r*, *p* is the geometric locus of the left upper vertices of the rectangles with one of the sides lying on *m* and with area equal to the area of rectangle *ADEF*; analogous reasoning can be applied to *q*.
5. The point of intersection of s' with *p* (denote it by *X*) determines the left upper corner of the rectangle we are looking for. Then, the right upper corner of this rectangle is the point of intersection of p' with *q* and simultaneously is a reflection of the point *X* in the line *r* (denote it by *Y*). It is not difficult to find the other two vertices of this rectangle: *Z* is the projection of *X* onto *m*, *T* is the projection of *Y* onto *m*.
6. The second point of intersection of s' with *p* will generate a second rectangle satisfying the desired properties.

4. Conclusions

In conclusion, in the curricula of the various classes of Torun Classical Grammar School in the years 1905–1911 there were no functions, but it certainly can be said that functions were discussed there. Teachers of mathematics introduced a definition of a dependent variable, which, according to Klein, was a synonym for a function. In Bardey's collection of problems the definitions of an independent variable and a dependent variable were the foundation for introducing a coordinate system and discuss graphical representation of linear and quadratic equations (here conics were introduced, as a graphical representations of quadratic equations). In the years 1905–1911 in the curricula of top class Prima Classical Grammar School, there was the following issue: 'Basic information about the coordinates, the equation of a straight line, a circle and conics' [8]. Mehler's textbook did not include this issue, therefore, it had to be implemented on the basis of Bardey's collection of problems. Thus, students of top class Prima had to learn the definition of the dependent variable.

The considerations made in this article prove that not all recommendations of the Merano Programme which involved teaching of functions were implemented at Torun Grammar School. On the basis of the contents of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3], the textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12] and school-leaving examinations tasks from the years 1905–1911, it can be assumed that at Torun Grammar School a graph of logarithmic function and monotonicity of functions were not discussed, moreover, equations were not solved by the graphical method. However, we cannot be sure about it. In support of it, let us mention here the situation at Grammar School in Bydgoszcz in 1906/1907. There, like at Torun Classical Grammar School, in the curricula there were no considerations of functions [9]. However, in the curriculum of the Upper Prima, there was an issue of 'information about coordinates and theory of conics' [9, p. 27]. At Bydgoszcz Grammar School, in mathematics lessons in Upper Prima two books were used: *Elementary mathematics* [*Die Elementar Mathematik*] by L. Kambly [6] and *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey[13]. In *Elementary mathematics* neither theory of functions nor dependent variables appeared, there was also no theory of conics. Thus the issue mentioned had to be implemented on the basis of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey. Although in Bardey's collection of problems neither the word 'function' nor consideration of the logarithmic function appeared, mathematics lessons at Bydgoszcz Grammar School filled in these gaps. This is confirmed by one of the school-leaving examinations tasks:

Exercise [9, p. 27, Easter, ex. 1]. Draw a graph of a function $y = \log x$ and discuss its main properties, in particular calculate the logarithm of a given number using the mean value[14].

[13] In addition, in Bydgoszcz Grammar School: *Complete logarithmic and trigonometric tables* [*Vollständige logarithmische und trigonometrische Tafeln*] by E. F. August [2; 9, p. 27] was used.

[14] This means: calculate the logarithm of a given number using the interpolation method.

We can say with certainty that the postulate of teaching thinking in terms of functions was executed very well at Torun Classical Grammar School.

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