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VECTOR BUNDLES ON REAL ALGEBRAIC CURVES

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Abstract. We prove that any topological real line bundle on a compact real algebraic curve X is isomorphic to an algebraic line bundle. The result is then generalized to vector bundles of an arbitrary constant rank. As a consequence we prove that any continuous map from X into a real Grassmannian can be approximated by regular maps.

1. Introduction. Throughout this paper X denotes a compact real algebraic curve, that is, a compact 1-dimensional algebraic subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We refer to [**1**] for terminology and background material on real algebraic geometry. In this paper all vector bundles are real vector bundles. Recall that algebraic vector bundles on X correspond to finitely generated projective modules over the ring of real-valued regular functions on X, cf. [**1**, p. 302]. Our main goal is the following:

Theorem 1.1. *Any topological line bundle on X is isomorphic to an algebraic line bundle.*

Theorem 1.1 is proved in section 2. It can be easily generalized.

Corollary 1.2. *Any topological constant rank vector bundle on X is isomorphic to an algebraic vector bundle.*

Proof. Any topological vector bundle on X of constant rank $r \geq 1$ splits off a trivial vector bundle of rank $r-1$, since $\dim(X) = 1$. Hence it suffices to apply Theorem 1.1. □

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As a consequence of Corollary 1.2, we obtain a counterpart of the classical Weierstrass approximation theorem for maps from X into the Grassmann variety $\mathbb{G}_{n,k}$ of k-dimensional vector subspaces of $\mathbb{R}^n$.

COROLLARY 1.3. *Let $f : X \longrightarrow \mathbb{G}_{n,k}$ be a continuous map. Each neighborhood of f in the compact-open topology contains a regular map.*

PROOF. It suffices to show that the pullback vector bundle $f^*\gamma_{n,k}$ on X, where $\gamma_{n,k}$ is the tautological vector bundle on $\mathbb{G}_{n,k}$, is isomorphic to an algebraic vector bundle, cf. [**1**, Theorem 13.3.1]. This however follows from Corollary 1.2. □

Since the real variety $\mathbb{G}_{2,1}$ is biregularly isomorphic to the unit circle

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

we immediately get:

COROLLARY 1.4. *Let $f : X \longrightarrow \mathbb{S}^1$ be a continuous map. Each neighborhood of f in the compact-open topology contains a regular map.*

All the results above are proved in [**1**] under the assumption that the curve X is nonsingular. The arguments presented in [**1**] do not directly generalize to yield Theorem 1.1.

COROLLARY 1.5. *For every cohomology class u in $H^1(X; \mathbb{Z}/2)$, there exists a regular map $f : X \longrightarrow \mathbb{S}^1$ such that $f^*(s_1) = u$, where s_1 is the unique generator of the cohomology group $H^1(\mathbb{S}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.*

PROOF. There is a one-to-one correspondence between the homotopy classes of continuous maps from X into $\mathbb{S}^1$ and the cohomology classes in $H^1(X; \mathbb{Z})$, cf., [**2**, p. 300]. Since the reduction modulo 2 homomorphism $H^1(X; \mathbb{Z}) \longrightarrow H^1(X; \mathbb{Z}/2)$ is surjective, it follows that each cohomology class in $H^1(X; \mathbb{Z}/2)$ is of the form $f^*(s_1)$ for some continuous map $f : X \longrightarrow \mathbb{S}^1$. According to Corollary 1.4, the map f can be assumed to be regular. □

Let us note that Corollary 1.5 implies Theorem 1.1. Indeed, let ξ be a topological line bundle on X. The first Stiefel–Whitney class $w_1(\gamma_{2,1})$ of the tautological line bundle $\gamma_{2,1}$ on $\mathbb{G}_{2,1}$ generates the cohomology group $H^1(\mathbb{G}_{2,1}; \mathbb{Z}/2)$. According to Corollary 1.5, there exists a regular map $f : X \longrightarrow \mathbb{G}_{2,1}$ satisfying $w_1(\xi) = f^*(w_1(\gamma_{2,1})) = w_1(f^*\gamma_{2,1})$. Since topological line bundles are classified by the first Stiefel-Whitney class (cf. [**3**, Proposition 3.10]), it follows that ξ is isomorphic to the algebraic line bundle $f^*\gamma_{2,1}$. However, we do not know how to prove Corollary 1.5 without making use of Theorem 1.1.

2. Line bundles on real algebraic curves. We first recall a useful construction of algebraic line bundles on an arbitrary affine real algebraic variety V. Lemma 2.1 below is a special case of [**1**, Theorem 12.1.11].

LEMMA 2.1. *Let* $\{U_1, \ldots, U_r\}$ *be a Zariski open cover of* V *and let* $h_{ij} : U_j \longrightarrow \mathbb{R}$ *be a regular function satisfying* $h_{ij}(U_i \cap U_j) \subset \mathbb{R}\setminus\{0\}$ *for* $1 \le i, j \le r$. *Assume that* $h_{ij} \cdot h_{jk} = h_{ik}$ *on* $U_j \cap U_k$ *for all* i, j, k, *and* $h_{ii}(x) = 1$ *for all* i *and* x *in* U_i. *Let*

$$E = \{(x, (v_1, \ldots, v_r)) \in V \times \mathbb{R}^r : v_i = h_{ij}(x)v_j \text{ for } x \in U_j, 1 \le i, j \le r\}$$

and let $p : E \longrightarrow V$ *be defined by* $p(x, (v_1, \ldots, v_r)) = x$. *Then* $\xi = (E, p, V)$ *is an algebraic line subbundle of the product vector bundle on* V *with total space* $V \times \mathbb{R}^r$, *and the map*

$$U_i \times \mathbb{R} \longrightarrow p^{-1}(U_i),\ (x, v) \mapsto (h_{1i}(x)v, \ldots, h_{ri}(x)v))$$

is an algebraic trivialization of ξ *over* U_i *for* $1 \le i \le r$.

For any vector bundle η and any global section s of η, let $Z(s)$ denote the zero locus of s.

The set $\mathrm{Reg}(X)$ of nonsingular points of X in dimension 1 is a Zariski open subset of X, cf. [**1**, p. 69]. Furthermore, $\mathrm{Reg}(X)$ is a 1-dimensional C^∞ manifold.

LEMMA 2.2. *Let* x_0 *be a point in* $\mathrm{Reg}(X)$. *There exists an algebraic line bundle* $\xi = (E, p, X)$ *on* X *which admits an algebraic section* $s : X \longrightarrow E$ *such that* $Z(s) = \{x_0\}$ *and the restriction of* s *to* $\mathrm{Reg}(X)$ *is transverse to the zero section of* ξ.

PROOF. Let $\mathcal{R}_X$ be the sheaf of real-valued regular functions on X. For any point x on X, we identify the stalk $\mathcal{R}_{X,x}$ with the localization of the ring $\mathcal{R}_X(X)$ at the maximal ideal

$$\mathfrak{m}_x = \{f \in \mathcal{R}_X(X) : f(x) = 0\},$$

cf. [**1**, Proposition 3.2.3]. Since the point x_0 is in $\mathrm{Reg}(X)$, the stalk $\mathcal{R}_{X,x_0}$ is a regular local ring of dimension 1 and thus a principal ideal domain. In particular, the ideal $\mathfrak{m}_{x_0}\mathcal{R}_{X,x_0}$ of the ring $\mathcal{R}_{X,x_0}$ is principal. Thus we can find a regular function f_1 in $\mathfrak{m}_{x_0}$ and a Zariski open neighborhood U_1 of x_0 in $\mathrm{Reg}(X)$ such that

$$\mathfrak{m}_{x_0}\mathcal{R}_X(U_1) = (f_1)\mathcal{R}_X(U_1).$$

In particular, $f_1|_{U_1} : U_1 \longrightarrow \mathbb{R}$ is a C^∞ function for which 0 in $\mathbb{R}$ is a regular value and $(f_1|_{U_1})^{-1}(0) = \{x_0\}$.

Let f_2 be any regular function in $\mathfrak{m}_{x_0}$ with $f_2^{-1}(0) = \{x_0\}$, e.g., a polynomial given by the formula $\|x - x_0\|^2$, where $\|\cdot\|$ denotes the euclidean metric in $\mathbb{R}^d$. We have

$$f_2|_{U_1} = h_{21} f_1|_{U_1}$$

for some regular function $h_{21} : U_1 \longrightarrow \mathbb{R}$. If $U_2 = X \setminus \{x_0\}$, then

$$h_{12} = \frac{f_1}{f_2} : U_2 \longrightarrow \mathbb{R}$$

is a regular function on U_2. By construction, the sets $h_{21}(U_1 \cap U_2)$ and $h_{12}(U_1 \cap U_2)$ are contained in $\mathbb{R} \setminus \{0\}$. Define $h_{11} : U_1 \longrightarrow \mathbb{R}$ and $h_{22} : U_2 \longrightarrow \mathbb{R}$ to be constant functions identically equal to 1. Let $\xi = (E, p, X)$ be the algebraic line bundle on X determined, as in Lemma 2.1, by the Zariski open cover $\{U_1, U_2\}$ of X and the regular functions h_{ij}. Note that

$$s : X \longrightarrow E,\ s(x) = (x, (f_1(x), f_2(x)))$$

is an algebraic section of ξ with $Z(s) = \{x_0\}$. On the set U_1, the section s is represented by the map

$$U_1 \longrightarrow U_1 \times \mathbb{R}\,,\ x \mapsto (x, f_1(x)),$$

and hence the restriction of s to $\operatorname{Reg}(X)$ is transverse to the zero section of ξ. □

We will now give a convenient description of the first cohomology group $H^1(X; \mathbb{Z}/2)$ of the curve X. The subset $X \setminus \operatorname{Reg}(X)$ of X is finite. If X has nonsingular connected components, we choose one arbitrary point in each of those and denote the set of such points by Z. The curve X can be regarded as a graph (1-dimensional CW complex) with $(X \setminus \operatorname{Reg}(X)) \cup Z$ as the set of vertices. This assertion is a straightforward consequence of the triangulation theorem for semi-algebraic sets, cf. [**1**, Theorem 9.2.1].

Lemma 2.3. *There exist subgraphs $X_1, \ldots, X_n$ of X such that each X_i is homeomorphic to the unit circle $\mathbb{S}^1$, and the inclusion maps $X_i \hookrightarrow X$ induce an isomorphism*

$$\varphi : H^1(X; \mathbb{Z}/2) \longrightarrow \bigoplus_{i=1}^{n} H^1(X_i; \mathbb{Z}/2)$$

Proof. Let K be a connected 1-dimensional component of X and let T be a maximal tree of the graph K. The quotient map $q : K \longrightarrow K/T$ is a homotopy equivalence and the quotient space K/T is homeomorphic to the wedge sum of a finite number of pointed circles, [**2**, p. 153]. Each such pointed circle corresponds to a subset of K/T of the form $q(C)$, where C is a subgraph of K homeomorphic to the unit circle. The inclusion maps $q(C) \hookrightarrow K/T$ induce an isomorphism

$$\psi : H^1(K/T; \mathbb{Z}/2) \longrightarrow \bigoplus_{C} H^1(q(C); \mathbb{Z}/2)$$

If $q_C : C \longrightarrow q(C)$ is the restriction of the map q, then the homomorphism

$$\alpha = \bigoplus_C q_C^* : \bigoplus_C H^1(q(C); \mathbb{Z}/2) \longrightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)$$

is an isomorphism. The homomorphism

$$q^* : H^1(K/T; \mathbb{Z}/2) \longrightarrow H^1(K; \mathbb{Z}/2)$$

is an isomorphism, the quotient map being a homotopy equivalence. Finally, the inclusion maps $C \hookrightarrow K$ induce a homomorphism

$$\varphi_K : H^1(K; \mathbb{Z}/2) \longrightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)$$

satisfying $\varphi_K \circ q^* = \alpha \circ \psi$. Consequently, φ_K is an isomorphism.

The assertion of the lemma follows, because X has finitely many connected components. □

Proof of Theorem 1.1. The isomorphism classes of topological line bundles on X form a group, denoted $\mathrm{Vect}^1(X)$, with tensor product as the group operation. The first Stiefel–Whitney class gives a group isomorphism between $\mathrm{Vect}^1(X)$ and the first cohomology group $H^1(X; \mathbb{Z}/2)$, cf. [**3**, Proposition 3.10]. Also, note that the isomorphism classes of algebraic vector bundles form a subgroup of $\mathrm{Vect}^1(X)$. Hence, in view of Lemma 2.3, it remains to construct for each $i = 1, \dots, n$ an algebraic line bundle ξ_i on X with $w_1(\xi_i|_{X_i}) \neq 0$ and $w_1(\xi_i|_{X_j}) = 0$ for all $j \neq i$ (note that $H^1(X_i; \mathbb{Z}/2) \cong \mathbb{Z}/2$). Such a line bundle ξ_i can be obtained as follows.

Let x_i be a point in

$$(X_i \cap \mathrm{Reg}(X)) \setminus \bigcup_{j \neq i} X_j$$

and let $\xi = (E, p, X)$ be an algebraic line bundle on X as in Lemma 2.2 with $x_0 = x_i$. There exists an algebraic section $s : X \longrightarrow E$ such that $Z(s) = \{x_i\}$ and the restriction of s to $\mathrm{Reg}(X)$ is transverse to the zero section of ξ. It follows that the line bundle $\xi|_{X_j}$ is trivial and $w_1(\xi|_{X_j}) = 0$ for $j \neq i$.

Suppose for a moment that the line bundle $\xi|_{X_i}$ is trivial, and let

$$\theta : p^{-1}(X_i) \longrightarrow X_i \times \mathbb{R}$$

be a topological trivialization of $\xi|_{X_i}$. Then $\theta(s(x)) = (x, f(x))$ for each x in X_i, where $f : X_i \longrightarrow \mathbb{R}$ is a continuous function. By construction, $f^{-1}(0) = \{x_i\}$. The function f does not change sign on $X_i \setminus \{x_i\}$, the set $X_i \setminus \{x_i\}$ being homeomorphic to $\mathbb{R}$. This however is impossible since s is transverse to the zero section of ξ in a neighborhood of x_i. Consequently, the line bundle $\xi|_{X_i}$ is nontrivial and $w_1(\xi|_{X_i}) \neq 0$.

We complete the proof by setting $\xi_i = \xi$. □

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