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AN ABSTRACT NONLOCAL FUNCTIONAL-DIFFERENTIAL SECOND ORDER
EVOLUTION CAUCHY PROBLEM

ABSTRAKCYJNE NIELOKALNE FUNKCYJNALNO-RÓŻNICZKOWE
EWOLUCYJNE ZAGADNIENIE CAUCHY'EGO RZĘDU DRUGIEGO

Abstract

The aim of the paper is to prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution second order equation together with nonlocal initial conditions. The theory of strongly continuous cosine families of linear operators in a Banach space is applied. The paper is based on publications [1–12] and is a generalization of paper [7].

Keywords: nonlocal, second order, functional-differential, evolution Cauchy problem, Banach spaces

Streszczenie

W artykule udowodniono dwa twierdzenia o istnieniu i jednoznaczności całkowych i klasycznych rozwiązań semiliniowego funkcjonalno-różniczkowego zagadnienia ewolucyjnego Cauchy'ego rzędu drugiego z nielokalnymi warunkami początkowymi. W tym celu zastosowano teorię rodziny cosinus liniowych operatorów w przestrzeni Banacha. Artykuł bazuje na publikacjach [1–12] i jest uogólnieniem publikacji [7].

Słowa kluczowe: nielokalne, rzędu drugiego, funkcjonalno-różniczkowe, zagadnienie ewolucyjne Cauchy'ego, przestrzenie Banacha

1. Introduction

In this paper, we consider the abstract nonlocal semilinear functional-differential second order evolution Cauchy problem

$$u''(t) = Au(t) + f(t, u(t), u(a_1(t)), \dots, u(a_m(t)), u'(t)), \quad t \in (0, T], \quad (1.1)$$

$$u(0) = x_0, \quad (1.2)$$

$$u'(0) + \sum_{i=1}^p h_i u_i(t_i) = x_i, \quad (1.3)$$

where A is a linear operator from a real Banach space X into itself, $u: [0, T] \rightarrow X, f: [0, T] \times X^{m+2} \rightarrow X, a_i: [0, T] \rightarrow [0, T]$ ($i = 1, 2, \dots, m$), $x_0, x_i \in X, h_i \in \mathbb{R}$ ($i = 1, 2, \dots, p$) and

$$0 < t_1 < t_2 < \dots < t_p \leq T.$$

We prove two theorems on the existence and uniqueness of mild and classical solutions of the problem (1.1) - (1.3). For this purpose, we apply the theory of strongly continuous cosine families of linear operators in a Banach space. We also apply the Banach contraction theorem and the Bochenek theorem (see Theorem 1.1 in this paper).

Let A be the same linear operator as in (1.1). We will need the following assumption:

Assumption (A_1). Operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself.

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A : X \supset D(A) \rightarrow X$ defined by

$$Ax := \frac{d^2}{dt^2} C(t)x \Big|_{t=0}, \quad x \in D(A),$$

where

$$D(A) := \{x \in X : C(t)x \text{ is of class } C^2 \text{ with respect to } t\}.$$

Let

$$E := \{x \in X : C(t)x \text{ is of class } C^1 \text{ with respect to } t\}.$$

The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

From Assumption (A_1), it follows (see [12]) that there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t)\| \leq Me^{\omega|t|} \quad \text{for } t \in \mathbb{R}.$$

We also will use the following assumption:

Assumption (A₂). The adjoint operator A^* is densely defined in X^* , that is, $\overline{D(A^*)} = X^*$.

The paper is based on publications [1 - 12] and is a generalization of paper [7] in this sense that, now, a more general functional – differential problem is considered than in [7].

For convenience of the reader, a result obtained by J. Bochenek (see [3]) will be presented here.

Let us consider the Cauchy problem

$$u''(t) = Au(t) + h(t), \quad t \in (0, T], \quad (1.4)$$

$$u(0) = x_0, \quad (1.5)$$

$$u'(0) = x_1. \quad (1.6)$$

A function $u: [0, T] \rightarrow X$ is said to be a classical solution of the problem (1.4) – (1.6) if

$$u \in C^1([0, T], X) \cap C^2((0, T], X), \quad (a)$$

$$u(0) = x_0 \text{ and } u'(0) = x_1, \quad (b)$$

$$u''(t) = Au(t) + h(t) \text{ for } t \in (0, T]. \quad (c)$$

Theorem 1.1. *Suppose that:*

(i) *Assumptions (A₁) and (A₂) are satisfied,*

(ii) *$h: [0, T] \rightarrow X$ is Lipschitz continuous,*

(iii) *$x_0 \in D(A)$ and $x_1 \in E$.*

Then u given by the formula

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)ds, \quad t \in [0, T],$$

is the unique classical solution of the problem (1.4) – (1.6).

2. Theorem on mild solutions

A function u belonging to $C^1([0, T], X)$ and satisfying the integral equation

$$u(t) = C(t)x_0 + S(t)x_1 - S(t) \left(\sum_{i=1}^p h_i u(t_i) \right)$$

$$+ \int_0^t S(t-s) f(s, u(s), u(a_1(s)), \dots, u(a_m(s)), u'(s)) ds, \quad t \in [0, T],$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1) – (1.3).



Theorem 2.1. *Suppose that:*

- (i) *Assumption (A_1) is satisfied,*
(ii) $a_i: [0, T] \rightarrow [0, T]$ ($i=1, 2, \dots, m$) *are continuous on $[0, T]$, $f: [0, T] \times X^{m+2} \rightarrow X$ is continuous with respect to the first variable $t \in [0, T]$ and there exists a positive constant L_1 such that*

$$\|f(s, z_1, z_2, \dots, z_{m+2}) - f(s, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{m+2})\| \leq L_1 \sum_{i=1}^{m+2} \|z_i - \tilde{z}_i\| \text{ for } s \in [0, T], \tilde{z}_i, z_i \in X$$

$$(i=1, 2, \dots, m+2),$$

(iii) $2C \left((m+1)TL_1 + \sum_{i=1}^p |h_i| \right) < 1,$

where $C := \sup \{ \|C(t)\| + \|S(t)\| + \|S'(t)\| : t \in [0, T] \},$

- (iv) $x_0 \in E$ and $x_1 \in X.$

Then, the nonlocal Cauchy problem (1.1) – (1.3) has a unique mild solution.

Proof. Let the operator $F: C^1([0, T], X) \rightarrow C^1([0, T], X)$ be given by

$$(Fu)(t) = C(t)x_0 + S(t)x_1 - S(t) \left(\sum_{i=1}^p h_i u(t_i) \right) + \int_0^t S(t-s) f(s, u(s), u(a_1(s)), \dots, u(a_m(s)), u'(s)) ds, \quad t \in [0, T].$$

Now, we shall show that F is a contraction on the Banach space $C^1([0, T], X)$ equipped with the norm

$$\|w\|_1 := \sup \{ \|w(t)\| + \|w'(t)\| : t \in [0, T] \}.$$

To do this, observe that

$$\begin{aligned} \|(Fw)(t) - (F\tilde{w})(t)\| &= \left\| S(t) \left(\sum_{i=1}^p h_i (\tilde{w}(t_i) - w(t_i)) \right) \right. \\ &+ \int_0^t S(t-s) \left(f(s, w(s), w(a_1(s)), \dots, w(a_m(s)), w'(s)) \right. \\ &\left. \left. - f(s, \tilde{w}(s), \tilde{w}(a_1(s)), \dots, \tilde{w}(a_m(s)), \tilde{w}'(s)) \right) ds \right\| \leq C \left(\sum_{i=1}^p |h_i| \right) \|w - \tilde{w}\| \\ &+ \int_0^t \|S(t-s)\| L_1 \left(\|w(s) - \tilde{w}(s)\| + \|w(a_1(s)) - \tilde{w}(a_1(s))\| + \right. \\ &\left. \dots + \|w(a_m(s)) - \tilde{w}(a_m(s))\| + \|w'(s) - \tilde{w}'(s)\| \right) ds \end{aligned}$$

$$\leq C \left((m+1)TL_1 + \sum_{i=1}^p |h_i| \right) \|w - \tilde{w}\|_1$$

and

$$\begin{aligned} \|(Fw)'(t) - (F\tilde{w})'(t)\| &= \left\| S'(t) \left(\sum_{i=1}^p h_i (\tilde{w}(t_i) - w(t_i)) \right) \right. \\ &\quad \left. + \int_0^t C(t-s) \left(f(s, w(s), w(a_1(s)), \dots, w(a_m(s)), w'(s)) \right. \right. \\ &\quad \left. \left. - f(s, \tilde{w}(s), \tilde{w}(a_1(s)), \dots, \tilde{w}(a_m(s)), \tilde{w}'(s)) \right) ds \right\| \leq C \left(\sum_{i=1}^p |h_i| \right) \|w - \tilde{w}\|_1 \\ &\quad + \int_0^t \|C(t-s)\| L_1 \left(\|w(s) - \tilde{w}(s)\| + \|w(a_1(s)) - \tilde{w}(a_1(s))\| + \dots + \|w(a_m(s)) - \tilde{w}(a_m(s))\| \right. \\ &\quad \left. + \|w'(s) - \tilde{w}'(s)\| \right) ds \leq C \left((m+1)TL_1 + \sum_{i=1}^p |h_i| \right) \|w - \tilde{w}\|_1, \quad t \in [0, T]. \end{aligned}$$

Consequently,

$$\|Fw - F\tilde{w}\|_1 \leq 2C \left((m+1)TL_1 + \sum_{i=1}^p |h_i| \right) \|w - \tilde{w}\|_1 \quad \text{for } w, \tilde{w} \in C^1([0, T], X).$$

Therefore, in space $C^1([0, T], X)$, there is the only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1.1) – (1.3). So, the proof of Theorem 2.1 is complete.

Remark 2.1. The application of a Bielecki norm in the proof of Theorem 2.1 does not give any benefit.

3. Theorem about classical solutions

A function $u: [0, T] \rightarrow X$ is said to be a classical solution of the problem (1.1) – (1.3) if

$$u \in C^1([0, T], X) \cap C^2((0, T], X), \quad (a)$$

$$u(0) = x_0 \quad \text{and} \quad u'(0) + \sum_{i=1}^p h_i u(t_i) = x_1, \quad (b)$$

$$u''(t) = Au(t) + f(t, u(t), u(a_1(t)), \dots, u(a_m(t)), u'(t)) \quad \text{for } t \in [0, T]. \quad (c)$$

Theorem 3.1. *Suppose that:*

- (i) *Assumptions (A_1) and (A_2) are satisfied, and $a_i: [0, T] \rightarrow [0, T]$ ($i=1, 2, \dots, m$) are of class C^1 on $[0, T]$.*

(ii) There exists a positive constant L_2 such that

$$\|f(s, z_1, z_2, \dots, z_{m+2}) - f(\tilde{s}, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{m+2})\| \leq L_2 \left(|s - \tilde{s}| + \sum_{i=1}^{m+2} \|z_i - \tilde{z}_i\| \right)$$

for $s, \tilde{s} \in [0, T]$, $z_i, \tilde{z}_i \in X$ ($i=1, 2, \dots, m+2$).

(iii) $2C \left((m+1)TL_2 + \sum_{i=1}^p |h_i| \right) < 1.$

(iv) $x_0 \in E$ and $x_1 \in X.$

Then, the nonlocal Cauchy problem (1.1) - (1.3) has a unique mild solution u . Moreover, if

$$x_0 \in D(A), \quad x_1 \in E \quad \text{and} \quad u(t_i) \in E \quad (i=1, 2, \dots, p),$$

and if there exists a positive constant κ such that

$$\|u(a_i(s)) - u(a_i(\tilde{s}))\| \leq \kappa \|u(s) - u(\tilde{s})\| \quad \text{for } s, \tilde{s} \in [0, T] \quad (i=1, 2, \dots, m)$$

then u is the unique classical solution of nonlocal problem (1.1) - (1.3).

Proof. Since the assumptions of Theorem 2.1 are satisfied, the nonlocal Cauchy problem (1.1) - (1.3) possesses a unique mild solution, which is denoted by u .

Now, we shall show that u is the classical solution of problem (1.1) - (1.3).

Firstly, we shall prove that $u, u(a_i(\cdot))$ ($i=1, 2, \dots, m$) and u' satisfy the Lipschitz condition on $[0, T]$. Let t and $t+h$ be any two points belonging to $[0, T]$. Observe that

$$\begin{aligned} u(t+h) - u(t) &= C(t+h)x_0 + S(t+h)x_1 - S(t+h) \left(\sum_{i=1}^p h_i u(t_i) \right) \\ &\quad + \int_0^{t+h} S(t+h-s) f(s, u(s), u(a_1(s)), \dots, u(a_m(s)), u'(s)) ds \\ &\quad - C(t)x_0 - S(t)x_1 + S(t) \left(\sum_{i=1}^p h_i u(t_i) \right) \\ &\quad - \int_0^t S(t-s) f(s, u(s), u(a_1(s)), \dots, u(a_m(s)), u'(s)) ds. \end{aligned}$$

Since

$$C(t)x_0 + S(t) \left(x_1 - \sum_{i=1}^p h_i u(t_i) \right)$$

is of class C_2 in $[0, T]$, there are $C_1 > 0$ and $C_2 > 0$ such that

$$\left\| (C(t+h) - C(t))x_0 + (S(t+h) - S(t)) \left(x_1 - \sum_{i=1}^p h_i u(t_i) \right) \right\| \leq C_1 |h|$$

and

$$\left\| \left((C(t+h) - C(t))x_0 \right)' + \left((S(t+h) - S(t)) \left(x_1 - \sum_{i=1}^p h_i u(t_i) \right) \right)' \right\| \leq C_2 |h|.$$

Hence

$$\begin{aligned} & \|u(t+h) - u(t)\| \leq C_1 |h| \\ & + \left\| \int_0^t S(s) \left(f(t+h-s, u(t+h-s), u(a_1(t+h-s)), \dots, u(a_m(t+h-s))), u'(t+h-s) \right) \right. \\ & \quad \left. - f(t-s, u(t-s), u(a_1(t-s)), \dots, u(a_m(t-s))), u'(t-s) \right) ds \Big\| \\ & + \left\| \int_t^{t+h} S(s) \left(f(t+h-s, u(t+h-s), u(a_1(t+h-s)), \dots, u(a_m(t+h-s))), u'(t+h-s) \right) ds \right\| \\ & \leq C_1 |h| + \int_0^t M e^{\omega T} L_2 \left(|h| + \|u(t+h-s) - u(t-s)\| + \|u(a_1(t+h-s)) - u(a_1(t-s))\| + \dots \right. \\ & \quad \left. \dots + \|u(a_m(t+h-s)) - u(a_m(t-s))\| + \|u'(t+h-s) - u'(t-s)\| \right) ds + M e^{\omega T} N |h|, \end{aligned}$$

where

$$N := \sup \left\{ \left\| f(s, u(s), u(a_1(s)), \dots, u(a_m(s))), u'(s) \right\| : s \in [0, T] \right\}.$$

From this, we obtain

$$\|u(t+h) - u(t)\| \leq C_3 |h| + C_4 \int_0^t (\|u(s+h) - u(s)\| + \|u'(s+h) - u'(s)\|) ds. \quad (3.1)$$

Moreover, we have

$$u'(t) = \left(C(t)x_0 + S(t) \left(x_1 - \sum_{i=1}^p h_i u(t_i) \right) \right)' + \int_0^t C(t-s) f(s, u(s), u(a_1(s)), \dots, u(a_m(s))), u'(s) ds.$$

From the above formula, we obtain, analogously,

$$\|u'(t+h) - u'(t)\| \leq C_5 |h| + C_6 \int_0^t (\|u(s+h) - u(s)\| + \|u'(s+h) - u'(s)\|) ds. \quad (3.2)$$

By inequalities (3.1) and (3.2), we get

$$\|u(t+h)-u(t)\|+\|u'(t+h)-u'(t)\|\leq C\cdot|h|+C\cdot\int_0^t(\|u(s+h)-u(s)\|+\|u'(s+h)-u'(s)\|)ds.$$

From Gronwall's inequality, we have

$$\|u(t+h)-u(t)\|+\|u'(t+h)-u'(t)\|\leq\tilde{C}|h|, \quad (3.3)$$

where \tilde{C} is a positive constant.

By (3.3), it follows that $u, u(a_i(\cdot))$ ($i=1,2,\dots,m$) and u' satisfy the Lipschitz conditions on $[0,T]$. This implies that the mapping

$$[0,T]\ni t\rightarrow f(t,u(t),u(a_1(t)),\dots,u(a_m(t)),u'(t))\in X$$

also satisfies the Lipschitz condition.

The above property of f together with the assumptions of Theorem 3.1 imply, by Theorem 1.1 and by Theorem 2.1, that the linear Cauchy problem

$$v''(t)=Av(t)+f(t,u(t),u(a_1(t)),\dots,u(a_m(t)),u'(t)), \quad t\in[0,T],$$

$$v(0)=x_0,$$

$$v'(0)=x_1-\sum_{i=1}^p h_i u(t_i)$$

has a unique classical solution v such that

$$v(t)=C(t)x_0+S(t)\left(x_1-\sum_{i=1}^p h_i u(t_i)\right) + \int_0^t S(t-s)f(s,u(s),u(a_1(s)),\dots,u(a_m(s)),u'(s))ds = u(t), \quad t\in[0,T].$$

Consequently, u is the unique classical solution of the semilinear Cauchy problem (1.1) – (1.3) and, therefore, the proof of Theorem 3.1 is complete.

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