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THE NOTION OF CONNECTEDNESS IN MATHEMATICAL ANALYSIS OF XIX CENTURY

POJĘCIE SPÓJNOŚCI W ANALIZIE MATEMATYCZNEJ XIX WIEKU

Abstract

The notion of connectedness was introduced by Listing in 1847 and was further developed by Riemann, Jordan and Poincaré. The notion and rigorous definition of metric and topological space were formed in Fréchet's works in 1906, and in Hausdorff's works in 1914. The notion of continuum could be traced back to antiquity, but its mathematical definition was formed in XIX century, in the works of Cantor and Dedekind, later of Hausdorff and Riesz. Karl Weierstrass (1815–1897) brought mathematical analysis to a rigorous form; also, the notions of future areas of mathematics – functional analysis and topology – were formed in his reasoning. Weierstrass's works were not translated into Russian, and his lectures were not published even in Germany. In 1989, synopses of his lectures devoted to additional chapters of the theory of functions were published. Their material served as the basis for this article.

Keywords: connectedness, Weierstrass, Cantor

Streszczenie

Pojęcie spójności zostało wprowadzone w 1847 roku przez Listinga, a dalej zostało opracowane przez Riemanna, Jordana i Poincarégo. Pojęcie i rygorystyczna definicja przestrzeni metrycznej i topologicznej pojawiły się w pracach Frécheta w 1906 roku i Hausdorffa w 1914 roku. Pojęcie kontinuum sięga starożytności, ale jego matematyczna definicja powstała w XIX wieku, w pracach Cantora i Dedekinda, później Hausdorffa i Riesza. Karl Weierstrass (1815–1897) przedstawił analizę matematyczną w rygorystycznej formie; również pojęcia przyszłych dziedzin matematyki – analizy funkcjonalnej i topologii – zostały zapoczątkowane w jego rozumowaniu. Prace Weierstrassa nie zostały przetłumaczone na język rosyjski, a jego wykłady nie były publikowane nawet w Niemczech. W 1989 roku ukazały się streszczenia jego wykładów poświęconych dodatkowym rozdziałom teorii funkcji. Opublikowany w nich materiał jest podstawą tego artykułu.

Słowa kluczowe: spójność, Weierstrass, Cantor

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1. Introduction

The history of topology dates to the Königsberg bridge problem, formulated and solved by L. Euler in 1736 [1].

The first work from which topology got its name was written by Listing in 1848 [2]. J.B. Listing (1808–1882) was a professor at the University of Göttingen, where Gauss was a lecturer, and where Riemann was a student. Like Riemann, Listing paid attention mainly to combinatorial properties of transformations, and he did not have an idea of a domain as of a set of points yet. In 1862 Listing continued combinatorial topological topics in his work “Description of spatial manifold, or generalization of the Euler’s polyhedron theorem” [3]. That was an early period of development of topology, still before the works of G. Cantor (1845–1918) on the set theory.

For the first time, the notion of connectedness was used by B. Riemann (1826–1866) in his dissertation titled “Fundamentals of the Theory of Functions of a Complex Variable” (1851), in his report “On the Hypotheses which lie at the Foundation of Geometry” (1854) and in the “Theory of Abelian functions” (1857). Riemann considered the space as having a real physical meaning, but not as a set of points; he considered a surface as a sheet spread out on a plane, or over the plane [4, p. 52]. He applied the notion of ‘1-connected’ to a “piece” of surface bounded by a closed non-self-intersecting curve. In 1851 in the “Fundamentals of the theory of functions of a complex variable”, Riemann wrote: “We shall consider two parts of a surface to be connected if a curve belonging to the surface can connect a point of one part to a point of the other part; otherwise, two parts of the surface shall be defined as disconnected, or separately located” [4, p. 54].

2. The notion of connectedness in the works of Georg Cantor

Cantor started from the analysis of convergence of trigonometric series and analysis of points on a straight line, and in his initial works he created the theory of point domains. He introduced the notion of real number on the basis of a fundamental sequence; he developed the notion of accumulation point proposed by Weierstrass in 1865; on its basis he formulated the notions of derived set and uniform convergence. Then he constructed a hierarchy of infinite sets, which led him to transfinite numbers. Cantor was interested in the nature of continuum, and many of his investigations gave topological results, for example, the issue of the possibility of a one-to-one mapping of a two-dimensional continuum on the domain of real numbers (1878). During the period from 1879 till 1884, Cantor published a cycle of six articles “On the infinite linear point manifolds” [5, p. 40-139], which contained his main results on the set theory. Cantor defined the sets of the first kind, which had empty n -derived set, and all the others – sets of the second kind. He introduced the notion of density within an interval, and demonstrated that the sets of the first kind were not dense anywhere within the interval; he demonstrated denumerability of the sets of first kind and some of the sets of second kind. He introduced the notion of an isolated set, as a set not containing its accumulation points, and proved denumerability of isolated sets in R^n . In his fifth work, Cantor introduced transfinite ordinals, and formulated the hypothesis of continuum, and

also considered an issue of when a subset of R^n could be defined as a “continuum”. For that purpose, he defined the notion of a perfect set and a connected point set. A perfect set coincides with its derived set by definition. The set T would be connected by definition if for any $\varepsilon > 0$ and any t and t' in T , a finite number of points t_1, t_2, \dots, t_n , exist in T , so that all the distances $tt_1, t_1t_2, t_2t_3, \dots, t_{n-1}t_n, t_nt'$ do not exceed ε . If a subset R^n is perfect and connected, then it would be called a ‘continuum’.

In 1880s, many new results and notions appeared in German mathematicians’ works devoted to mathematical analysis. These notions needed to be uniformized and made rigorous. Ketsier wrote: “Cauchy has created a new conceptual apparatus to ensure a strong basis for the existing analysis, and in his mathematics a function would always be linked to a formula. In the second half of XIX century the conceptual apparatus itself became an object of research. This happened due to generalization of the notion of function: it started to mean an arbitrary correspondence between numbers” [6, p. 3]. At the same time, the theory of real numbers was still insufficiently developed – though there were different definitions of irrational number, it was not known how many irrational numbers exist, in comparison with rational numbers, or whether there are other numbers, non-definable with the help of sequences of number, and, above all, how the irrational numbers are distributed on the complete number scale. The uniform continuity theorem was formulated for a function ranging between two rational limits. Weierstrass realized that construction of real numbers takes place speculatively, “in the world of our thoughts”, and tried to harmonize arithmetical conception of a number with general conception of a value as a result of measurement of a geometrical or physical object. Alongside with the growth of importance of the notion of an irrational number, criticism was also growing, regarding extension of the notion of a real number. A colleague of Weierstrass at the University of Berlin, L. Kronecker (1823–1891), came out strongly against the theories of Weierstrass and Cantor, and asserted that all numbers should be expressible through natural numbers and their relations. His harsh words, both in his publications and in conversations among the circle of his colleagues, as well as in his lectures delivered to students, were meant to prove that Weierstrass’s theory of functions was groundless [7, p. 327]. Klein related emotional experiences of Weierstrass, as described in the letter from Weierstrass to S. Kovalevskaya, on March, 24, 1885, regarding the malicious attacks of Kronecker. Probably it was an aspiration to defend himself and to demonstrate validity of the theory of functions in the light of new concept of real numbers that resulted in Weierstrass’s intention to deliver an additional course of lectures devoted to the grounds of mathematical analysis. He implemented that in 1886.

3. Lectures of Weierstrass delivered in 1886

Karl Weierstrass (1815–1897) delivered lectures at the Königlichem Gewerbeinstitut of Berlin and the University of Berlin, starting from 1856. He systematized the course of mathematical analysis and introduced the notion of continuous function in the language of “ $\varepsilon - \delta$ ”. He did a lot for the theory of real numbers. It was he who ensured a rigorously substantiated form of mathematical analysis. He aspired to arrange in order new discoveries made during the 1870s by Charles Méray [8], Edward Heine [9], Richard Dedekind and Georg

Cantor [10], while striving to put them in classical terms and harmonize with traditional notion of a number as a ratio of values.

Especially for that purpose, he delivered a special course of lectures devoted to basic notions of mathematical analysis during the summer semester of 1886. The lectures were delivered thrice a week, in May and June. Synopsis of these lectures prepared by his students was published relatively recently, in 1989 [11].

Weierstrass started the course of lectures with the words: “These lectures have been compiled in order to supplement the lectures on the theory of analytical functions delivered during the winter semester of 1884/1885. The intended aim was achieved, using however a more synthetic method, and for some of the results no desirable generalization was attained; the quality of proof was not fully satisfactory. Hence after delivering those lectures it appears useful to recount in detail the various methods underlying the theory of functions, outline them historically and critically, in order to demonstrate the various points of views, and make an attempt to reconcile them. In short, to demonstrate the tendency of historical development of mathematical science, especially in the field of mathematical analysis, and thus, to explain the fundamental notions of science. Our aim will be to demonstrate that the principles of mathematical science are based on an actually solid foundation” [11, p. 20].

For the purpose of substantiation of representability of a function, Weierstrass used the notion of real number, including his own theory about a number as an aggregate (visible totality), i.e. finite or infinite decimal (or other) record, which in the infinite case would represent an absolutely convergent series satisfying the relation of equality (equivalence for infinite representations) and order.

Like Dedekind, Weierstrass separated physical reality and “the world of our thoughts”, in which an idea of number is formed for the description of numbers and functions; and therefore, a set of numbers can be extended by means of passage to the limit. Hence, any lengths could be represented by numbers; however, the length would not correspond to any number.

For him, all the values expressed through proportions (ratios) are limited; an infinite numerical value is called defined when each element of its underlying convergent series is specified. Definite points correspond to each of the summands of a well ordered series, but only in case of absolute convergence of the series. Condition for that consists in uniform convergence. Then, with any rearrangement of terms in series, the limit would be the same numerical value, extending the notion of a number.

Weierstrass introduced the notion of a variable, associating the notion of accumulation point with it, and, following Cantor, he defined an irrational number as an accumulation point of rational numbers. He gave an example: “the number e , consisting of the elements

$1, \frac{1}{2}, \frac{1}{6}, \dots, \frac{1}{n!}, \dots$, is a well ordered series, which defines a particular numerical value;

however, it can be shown that there is no rational numerical value equal to it by definition, which shows that numerical domain of rational numbers is not full” [11, p. 58]. Weierstrass solved this problem in relation to analytical continuation of a function.

Weierstrass referred to Bolzano’s theorem of the least upper bound of a variable, but he doubted whether a numerical value always corresponds to a point. For him, the entire totality

of positive numerical values is wider than the entire totality of all possible segments between A and B , up to AB . If a numerical value is defined, i.e. expressed by absolutely converging series, it would undoubtedly be in correspondence with, e.g., a geometrical decimal segment, in the form of $a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots$, where $0 \leq a_k \leq 10$, when $k \geq 1$; a series can represent any numerical value on the segment, and such values appear in the above expression of the series as the first, the second (subtotal) and so on. By interrupting separation of elements, we shall thus define infinitely many numerical values in whose neighborhood infinitely many definite points are condensed, and a continuous sequence can be formed without definite points. In that way, one may define position of the point D , subdividing the segment into two continuous segments, “so that it would correspond somehow to our innate, natural notion of limit, whereupon we could imagine that a straight line is not limited by anything except points, so one can suppose that D represents a definite value” [11, p. 63].

4. Cut sets and their extension to a plane and multidimensional domains. Neighborhood

Hence, there should be one and only one point of separation of two segments from each other, and this point is the numerical value under consideration. Having defined cut set on segment, as exact upper bound of the subtotals of convergent series, Weierstrass passed to determination of a neighborhood and a bound in the space of points (x_1, x_2, \dots, x_n) , having defined the point of n -multiple manifold as (a_1, a_2, \dots, a_n) :

“Let us assume that in an n -multiple point manifold defined in an arbitrary way, however in the way that infinitely many points comply with this definition, at least one point exists in whose direct neighborhood infinitely many definite points are concentrated. Let us consider the totality of all the points created in case where x_1 can possess all the values from $a_1 - d$ to $a_1 + d$, x_2 can possess all the values from $a_2 - d$ to $a_2 + d$, and so on, and assume that, in that way, one can form all possible numerical combinations of n values in the **neighborhood** of the point (a_1, a_2, \dots, a_n) ; thus we affirm, that if d is arbitrarily small, then in every arbitrarily small neighborhood of at least one point, infinitely many points exist which comply with the definition.

Now it is possible to define a neighborhood of the point (a_1, a_2, \dots, a_n) in the domain of n -multiple manifold of real variables through inequalities $\sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2} \leq d$, in that way, for n -multiple complex manifold of the values in the form of

$$x_k = \xi_k + \eta_k i, \quad k = 1, 2, \dots, n \quad \text{there will be} \quad \sqrt{\sum_1^n (\xi_k - a_k)^2 + \sum_1^n (\eta_k - b_k)^2} \leq d.$$

The value d is referred to as the radius of neighborhood of the point under consideration (a_1, a_2, \dots, a_n) ; the expression for the space is applicable to arbitrary n -multiple manifolds.

Now, two approaches are possible: either we relate a point whose each neighborhood contains infinitely many definite points, or we do not relate such a point, to definite points. In the latter case, the point shall be referred to as a frontier point (limit point, *Grenzstelle*)” [11, p. 65].

5. Definition of continuum

This Section is devoted to Weierstrass’s attempt to accomplish connectedness and separation of plane domain with the help of removable sets. Weierstrass defined continuum in an n -multiple manifold. Initially, he considered a function of two independent variables whose domain of definition was a part of a plane with some excluded points. “Passage from one non-excluded point to another similar point is possible through a continuously connected path. One could always separate a part of a plane which connects the first point to the second. This is possible with the help of a sequence of the circles for which the center of the next circle is located within the previous circle, and radii are selected so that all discarded points would be left outside. If the number of excluded points is infinitely large, there is no need in constructing a line. E.g., the excluded points are located on a circumference, so that an arbitrary initial point is projected onto the arc u' towards some direction; this circle has a unit radius for $u' = 2\xi\pi$, where ξ runs through all rational values from 0 to 1, despite the fact that the set of removable points is not a continuous line (thus, a certain part of the plane without points located continuously on the line is removed). No excluded points are located inside the circle; we shall take any point as the center of a circle whose all points are to be defined; one can make sure that its radius does not exceed a certain limit; again, we shall take a new point in the circle and circumscribe a circle around it in a similar way, like around the first point. One can draw a conclusion that if we continue such a procedure for an unlimited number of times, we could never get out from the interior through discarded points bounding the circle; just as by analogy, from beyond the bounds of the circle, a point from outside would never be able to get into the circle. Hence, we see that a continuous sequence of points is insufficient for separation of a two-dimensional manifold into parts. As we see from the example, it is not possible, even a priori, to define the kinds of separation of the plane into parts” [11, p. 66].

Professor E. Mioduszewski kindly commented on this place in the work of Weierstrass: “Prehistory of Weierstrass’s connectedness is rather interesting. That notion for modern topologists is a pure property of figures and spaces, but in case of Weierstrass, his research was motivated by the tasks facing him. The sets considered by Weierstrass were the sets of points *excluded* from definitional domain of a function (singularities of function), or their complements. In this example, the set of points $u' = 2\xi\pi$ of unit circumference with rational ξ means the set of excluded points. It’s countable, but sufficient to prevent analytical extension from the interior of a circle to exterior, since analytical extension is made with the help of finite chain of open discs, each of which has a common point with the previous one. This is a very strong condition of connectedness, considerably stronger than the Cantor’s condition, where we only require connectedness for any ε , of a finite sequence of *points*, each of which is remote at a distance ε from the next one. By means of such sequences, we can

pass through from the interior of a circle to the exterior. In order to prevent such free passage from definitional domain into the complement of the set of discarded points, a set would be required, containing nontrivial continuums. This is just the case described by Phragmen. It is not known, whether somebody considered the notion of connectedness as applicable to the function theory. Perhaps, only Mittag-Leffler was able to consider connectedness as an autonomous tool in mathematical study of functions”.

Weierstrass; “We shall now pass to consideration of an n -multiple manifold; we can define it as a set of definite points, and discuss the main theorem, a detailed substantiation of which we saw in the case of a prime manifold. A point set is referred to as **closed** if any neighborhood of each of its definite points contains infinitely many definite points. If we define, for example, all points of a circular domain, then each point of the circumference would be defined, and simultaneously it would be a frontier point, whereas outside the circular domain not a single point would be found about which it could be said that any of its small neighborhood would have definite points. Now, we can make any point set P closed, by adding its accumulation points P' to it. Assuming that the point belongs neither to P nor to P' , we could in any case circumscribe a circle of finite radius around it, the circle not containing P ; otherwise, it would contain P' . But it cannot contain P' either, because by definition, P' contains infinitely many points of P .

With the help of such closed set of points, we shall now separate either a **single** continuum from an n -multiple manifold, **or** a **number** of continua, located at a distance from each other. Let the point A be given, not belonging to the set of points; we could circumscribe it by a neighborhood of a radius ρ such that no definite (definierten) points would fall within it; ρ is a variable having an upper bound, to the effect that it possesses arbitrarily smaller values. **An absolute neighborhood** of A would be a neighborhood whose radius ρ_0 just presents this upper bound. Starting from the point A , we should construct a sequence of points A_1, A_2, \dots, A_n in such a manner that every next one would be contained within the neighborhood of the previous one. Then, two cases could be possible: either we would pass from A to any point that does not belong to the point set P , or we would not. In the first case, all those points not included in the point set P would form a continuum; in the second case, because of our assumption, the point A would only define the part of n -multiple manifold remaining after removal of P . Now, in the remaining part of n -multiple manifold, we shall select the point B , again, in order to define a new continuum in that manifold with the help of this point. Thus, we see that by virtue of the definition of the point set, an n -multiple manifold can be separated into infinitely many parts”.

6. Connectedness of continuum

“Now, a question arises whether two points initially belonging to the same defined continuum always result in that same continuum or not. To answer this question, let us imagine that between A and B there is a sequence of points such that we can pass from A to B via these points, whereas each successive point is located within the neighborhood of the previous one. Then it appears immediately that from A we could reach all the points which we can get from B , because we simply can get at A after passing B . It is not quite clear

how we can reach from B all the points reachable from A . To show that, it would be necessary to prove that it would be possible to return to A from B , and namely, that it would be possible to pass through all the points reachable from A after leaving B for A . To prove that, we shall connect A and B using a sequence of points, which could provide passage from A to B . We shall connect these points using a broken line; i.e. $A_{n-1}A_n$, completely in the neighborhood of A_{n-1} , and so on. Let this line go through a point in such a manner that each position be in correspondence with **absolute radius of neighborhood** ρ , which is a variable quantity having a lower bound not equal to zero; and if we use the principle of classification which we applied in the proof of the main theorem, then, first of all, we would be certain that it would be necessary to get at one or a number of **particular** locations for which the lower bound is actually attained. But now, at any point of the broken line, ρ would have a finite value, because otherwise the mentioned location (point) would be an accumulation point for A , i.e. it would belong to the point set P , so that it could be used as an extension of the continuum.

We would like to present another proof, which is based on the following argument. If two points A and B are located at a distance δ , and ρ is the radius of neighborhood of A , then between $\rho - \delta$ and $\rho + \delta$ a radius ρ' of a neighborhood of B would exist. Now it is possible to select ρ small enough so that the difference between $\rho - \delta$ and $\rho + \delta$ would be arbitrarily small. Let a point run along the segment from A_1 to A_n ; then the respective value of ρ would attain its smallest value, which is not equal to zero, as we mentioned above. Let the point run along the segment backwards, A_n, A_{n-1}, \dots, A_1 , and we shall select the distance $A_{v-1}A_v$ to be small enough so that A_v would be placed within the neighborhood A_{v-1} , also, A_{v-1} would be placed within the neighborhood of A_v , which could be achieved if $A_{v-1}A_v < \frac{1}{2}\rho$; this is the same radius of the neighborhood of A_v that we have referred to as being $> \frac{1}{2}\rho$; thus, A_{v-1} would be inside the circumference circumscribed around A_v . If we select intermediate points so that all distances between them be $< \frac{1}{2}\rho_0$, where ρ_0 is the lowest bound of ρ , then we would be certain that it could be possible to return to A from A_n as mentioned above, i.e., from any point we could only get to the points of continuum whose points belong to the same one, and it is actually so; or, in other words, a point can only belong to the continuum, which has been proved completely”.

“Wednesday, 6.23.1886.

This theorem is applicable, first of all, to a plane, i.e., to a two-dimensional manifold; and now, our task is to extend the proof onto an arbitrary n -multiple manifold. Here we shall use geometrical expressions reduced to the following form: let $x_k = a_k + t(b_k - a_k)$, $k = 1, \dots, n$, where t is an unlimited real variable. Then we shall give the name of a **line** in the respective manifold, to a totality of points (x_1, x_2, \dots, x_n) complying with this expression. The grounds for such name do not require comments. The totality of values of the system (x_1, x_2, \dots, x_n) , which follow from this expression, where t possesses the values from 0 to 1, has the name of **segment** ab . By giving t the values > 1 , we would obtain an extension of the segment ab beyond the bounds of b ; by giving t negative values, we would obtain an extension beyond

the bounds of a . The expression $\sqrt{\sum_1^n (b_k - a_k)^2}$ has the name of distance from the point b to the point a . If the distance = r , then we understand the neighborhood of the point a , having the radius r ; as a totality of all the system's values for which $\sqrt{\sum_1^n (b_k - a_k)^2} < r$ □

7. Triangle inequality, calculus of variation

“Now we shall prove the theorem, which has been a direct generalization of the following statement for a space: If a, b, c are three points within a space, then $ac < ab + bc$, unless the points are collinear. Now, let c vary, but always in the way that bc would have the same value, so that the positions could exist where ac reaches its maximum, and those where ac becomes minimal. For an n -multiple manifold this theorem is formulated as follows: If a, b, x are three points, then $ax < ab + bx$, or $\sqrt{\sum(x_\lambda - a_\lambda)^2} < \sqrt{\sum(b_\lambda - a_\lambda)^2} + \sqrt{\sum(x_\lambda - b_\lambda)^2}$, except for those cases where $x_\lambda = a_\lambda + t(b_\lambda - a_\lambda)$. In the latter case, the above inequality reduces to equality. Now we can use a purely algebraic proof, but first we shall do the following. Let a, b be fixed, x be a variable, however, such that bx has constant value ρ , so that at first it would be necessary to show that a point exists for which $\sqrt{\sum(x_\lambda - a_\lambda)^2} = R$ takes the maximum, and another point, for which R is minimal. It shall also satisfy $\sum(b_\lambda - a_\lambda)^2 = r^2$. We shall apply a common approach and form the following expression: $\sum(x_\lambda - a_\lambda)^2 - \varepsilon \sum(x_\lambda - b_\lambda)^2$; this should possess minimal or maximal value, and in addition we multiply the item by an indefinite constant ε , because this problem is related to **calculus of variations**. In this way, we combine equations $(x_\lambda - a_\lambda) - \varepsilon(x_\lambda - b_\lambda) = 0, (\lambda = 1, \dots, n)$ or $x_\lambda - a_\lambda = -\frac{\varepsilon}{1-\varepsilon}(b_\lambda - a_\lambda)$, which proves that the desired points are actually on a line. Then we would obtain: $R = \pm \varepsilon \rho = \pm \frac{\varepsilon}{\varepsilon - 1} r$, and

$$\text{also, } \varepsilon = 1 \pm \frac{r}{\rho}.$$

Thus we obtain two points, and not more, satisfying the requirement. Then, in the first case, $R = \left(1 + \frac{r}{\rho}\right)\rho = r + \rho$, in the second case, $R = \left(\frac{r}{\rho} - 1\right)\rho = r - \rho$. In effect, in one case we have maximum, in another case we have minimum, so in fact in other cases $ax < ab + bx$. This theorem will serve as a basis for the proof of our theorems for n -multiple manifolds. Let a closed point set be given in a closed manifold, but so that there exist points not belonging

to the set. Then, for each point not belonging to the set, a neighborhood will certainly be defined. Definitely, the points of the set P are not in an arbitrary proximity to the said points, i.e., there may be a point, denoted by a , with a neighborhood having radius ρ , inside of which there are no points of P . Undoubtedly, this radius doesn't exceed the finite bound r , i.e., the neighborhood of r is defined so that with a small increase the point P would fall inside the neighborhood. By marking some point b within the neighborhood of the point a , one could show that in case where $ab = d$, the radius of the neighborhood of b would not be less than $r - d$, and would not exceed $r + d$. If c is a point from the neighborhood of b , then, by virtue of the theorem just proved, first of all $ab + bc > ac$ or $bc > r - d$, and also, $bc < ab + ac$, $bc < r + d$, which was to be proved. In the second part, c stands for a point on the boundary of the neighborhood of b , located within the neighborhood of a . So, we can formulate the following definition: if the point b exists within the neighborhood of the point a , the point c exists within the neighborhood of the point b , and so on, then any point s through which we could move from a to c would be called a point **connected** (adjacent) to the point a . We now show that if s is connected to a , then a is also connected to s . Then it would be proved, obviously, that having started from any point of a continuum, we would always stay within the same.

Now, let the point b exist within the neighborhood of the point a , so that $ab < \frac{1}{2}r$, where r is the neighborhood (radius of the neighborhood) of the point a ; then, the neighborhood (radius of the neighborhood) of the point b , by virtue of the theorem proved, will be $> \frac{1}{2}r$.

Let us connect ab by a segment of straight line, then, as soon as $ab = d$, a neighborhood could always be defined for any point, such neighborhood would be of radius $> r - d$; we only need – which is always possible – to connect a and b by a sequence of points of such a kind that at passage from a to b every point would be within the neighborhood of the previous one, to be sure that it could be possible to return back from b to a ; however, it would always be possible, since the distance between the points is $< r - d$. But a is connected to b , and starting from b , we can arrive at any point at which we can also get if we start moving from a ; therefore, it would be necessary to depart from b through a . This theorem is true for any n -multiple manifold”.

8. Bounds of continuum

“Friday, 6. 25. 1886. Now we shall put the question about creation of the concept of the **bounds of a continuum**; that is, about whether all the points of manifold belong to continuum, or not. In the latter case, there should exist points in whose arbitrary proximity both the points belonging to the continuum and those not belonging to the continuum would exist. We shall refer to the totality of these points as the **bound (limit)** of the continuum. This represents just the largest possible manifold. For example, a single point, an infinite number of points located discretely, and finally, a continuous contour can represent a delimiter on a plane. We have to prove two things:

- 1) that with the exception of the above simple case, bounds exist in a continuum;
- 2) that all this can be found in a closed point set, with the help of which we defined continuum from the very beginning.

It is absolutely unnecessary for all the points belonging to a closed set of points to be limit points (frontier points); for example, we shall define a closed set in a space with the help of points existing in a sphere, or on its surface; in this regard, if we separate the continuum of frontier points from the sphere, then the remaining internal points of the sphere would not represent a closed point set anymore.

Therefore, in the first place, it would be necessary to prove the existence of accumulation points. Let a be a point within a continuum, b would then be any other point; according to the definition given earlier for an n -multiple manifold: $x_\lambda = a_\lambda + t(b_\lambda - a_\lambda)$, ($\lambda = 1, \dots, n$), where t takes all positive and negative real values; then a and b could be connected by a straight line. Now, let τ be an arbitrary positive value, i.e. such a value that, when used as a substitute for t in the expression for coordinates of the segment, would result in the point P on the segment, located within the continuum, and it would be such a point that all the points from a to this point would belong to the segment (part) of the continuum; then the distance (segment) aP would be a variable quantity, and as such, it would have an upper bound. Should it be infinitely large, then all the points of the straight line would belong to the continuum; however, this is not the case, and hence, for P_0 , there should be an extreme location for the given P ; thus, P_0 would simply be a bound for P , as it follows from the definition of the bound. The segments for which these frontier points P_0 actually exist should be available in any case; indeed, one could mark any point for which a point set connected to a has been defined in the continuum; of course, this point has been a point of the class of P_0 , otherwise, there would be no points on the segment preceding the point in connection to the point a , i.e., those corresponding to a smaller value of t .

And finally, one can easily see that all the frontier points belong to a closed set; because any point which does not belong to the point set belongs to the continuum; otherwise, if the manifold is located within point sets in a number of continua, then it belongs to one of these continua.

A continuum like one that has just been described shall be referred to as an **open continuum**; we think that in case when all accumulation points are added to such a continuum, we could obtain a structure which could rightfully be referred to as a **closed continuum**. In fact, the continuum that we defined before actually represents a closed set" [11, p. 63-73].

In her thesis about Mittag-Leffler, Laura Turner describes two different definitions of the continuum in the following way: "Central to Weierstrassian analysis was Weierstrass' notion of a continuum. These are unions of open disks in the complex plane which arise as domains in which the power series representation of a function is uniformly convergent. Weierstrass' continua are thereby closely connected to his concept of monogenicity; the domain of a function is divided into continua, and if there is only one such continuum the function is monogenic. However, if singularities block analytic continuation, the domain will be divided into multiple continua at the boundary they form. Cantor, on the other hand, in connection with his advocacy of a general shift from functions – and in particular the representations of functions – to dealing with set of points of their domain and range, had a different definition. For him, a continuum was a perfect connected point set.

Mittag-Leffler claimed to have found both notions important, and Weierstrass' in particular for its connections to his own work. On 27 February 1883 he wrote to Cantor: "I very much agree with your definition of continuum, and would however like to refer to what Weierstrass calls a continuum as a "completely connected point set" (...). It will follow sufficiently from my work that such completely connected point sets have their necessary place in the theory of analytic functions and can't be replaced by your continua.

Mittag-Leffler was interested, moreover, in the relative relationship between the two definitions within a three dimensional space, and he took this issue as a potential starting point for some of Phragmén's earliest work. This is the first of two instances for which there exists evidence that Mittag-Leffler explicitly requested that Phragmén consider a particular topic. In a letter to Phragmén dated 7 December 1883 Mittag-Leffler indicated that a line in the plane can be defined as a perfect connected point set — a Cantorian continuum — of which no part comprises a continuum in the plane according to Weierstrass' definition. However, if one considers this definition within a three dimensional space, the distinguishing characteristics between a surface and a line are not immediately obvious. Mittag-Leffler added: "It is well worth speculating over this issue. I believe that it is closely connected to some of the most important questions in the theory of functions". The next documented instance of Mittag-Leffler's proposal of a specific problem to Phragmén concerns work on Weierstrass' notion of the continuum. A letter written by Phragmén to Mittag-Leffler dated 29 December 1883 concerned the relationship between Weierstrassian continua and particular points sets in the plane. Phragmén published this work in January of 1884 in an article entitled «En ny sats inom teorien för punktmängder» ("A new theorem within the theory of point sets") [24, p. 114–115].

9. Further development of the ideas of Weierstrass

As we can see, the notions of measurable sets, metric and topological space have already been stipulated in the lectured delivered by Weierstrass. The theorem stipulating that each bounded infinite subset in R^n has an accumulation point was discussed by Weierstrass for $n = 2$ in his course of lectures delivered in 1865; a general proof was proposed by him in 1874. However, Weierstrass did not distinguish the notion of accumulation point as a basic one: this was done in 1872 by Cantor, who constructed the hierarchy of derived sets. The lectures delivered by Weierstrass were not published; however, his ideas were disseminated by his audience: German, Italian and Russian students. In Italy, his ideas were developed by G. Ascoli (1843–1896), C. Arzela (1847–1912), and U. Dini (1845–1918).

10. Weierstrass's ideas in Italy

The courses of mathematical analysis in Italy were brought into accord with achievements of German and French mathematicians. For that purpose, F. Brioschi, E. Betti, F. Casorati (more than once) visited Germany to meet Weierstrass, Kummer, Kronecker and other mathematicians [15, p. 221]. Due to an intensive exchange of letters between H. Schwarz,

Casorati and U. Dini, Italian mathematicians were well informed about all German mathematical news. The course of lectures by Ulisse Dini, “Fundamentals of the theory of functions of real variable” [12, 16], which he delivered at the University of Pisa from 1871 till 1915, was considered to be the best in Italy. New results obtained by Weierstrass, Cantor and Dedekind were included in that course of lectures [14]. During the years of 1877–1878, S. Pincherle (1853–1936) was among Weierstrass’s students, and in 1880 Pincherle started to deliver the course of lectures “Theory of analytical functions according to Weierstrass” (Teoria delle funzioni analitiche secondo Weierstrass), at the University of Pavia, and later on he published the course [17]. In this way, the ideas of Weierstrass were disseminated in Italy.

In 1883 Volterra started creating the theory of functionals, or “functions of lines with real values”. These functions were considered as the elements of a set for which the notion of neighborhood and limit of a sequence can be defined. Volterra gave the definition of continuity and derivative of a line function, and made an attempt to construct the theory of line functions by analogy with Riemann’s theory of complex functions.

In 1884 Giulio Ascoli (1843–1896) extended Bolzano–Weierstrass theorem to sets of functions. In 1889 C. Arzelà generalized the theorem and proved that an equicontinuous set F of uniformly bounded functions on $[a, b]$ has a limit function.

Then Arzelà discussed continuous real-valued functionals defined on an equicontinuous set of functions F , and demonstrated that, if F is closed, i.e., it contains all its limit functions, with the lower bound of the set of values of the functionals, then the upper bound would be reachable, along with all the intermediate values.

Now the Ascoli-Arzelà fundamental theorem in mathematical analysis is formulated in terms of compactness, in the language created by Fréchet in 1904.

11. M. Fréchet, F. Riesz and F. Hausdorff

In 1906, Maurice R. Fréchet (1878–1973), in his dissertation “On some questions of functional calculus” [18] introduced the notion of a metric domain [18, p. 30], formalized by Hausdorff in 1914, with identity, symmetry axioms, and triangle inequality.

In 1908, F. Riesz (1880–1956), in his report “Continuity and abstract theory of sets” [19] delivered at IV International mathematical congress in Rome, characterized continuum using the notion of accumulation point satisfying the three basic axioms: each element representing an accumulation point of a set M also represents accumulation point of any set that contains M ; if a set is divided into two subsets, then each accumulation point would be an accumulation point of at least one of these subsets; a subset that consists of one element only does not have an accumulation point. For the purpose of strengthening, Riesz added the fourth axiom: each accumulation point of a set is defined uniquely through a totality of all its subsets whose accumulation point it is.

Felix Hausdorff (1868–1942), who graduated from the University of Leipzig in 1891, attended the lectures at the University of Freiburg and Berlin. Probably, he was aware of the Weierstrass’s lectures delivered in 1886. Yet in 1912, Hausdorff, while delivering lectures on the set theory at the University of Bonn, introduced the notion of neighborhood U_x of the point x as a set of all the points y for which $xy < \rho$, where ρ is a positive number:

the interior of a sphere having the radius ρ , $xy = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2} \geq 0$ is the distance between the points x and y . A neighborhood, according to him, possesses the following properties: Each U_x contains x and is contained within r (where r is any n -dimensional space, e.g., a plane). For two neighborhoods of the same point: $U'_x \supseteq U_x$ or $U_x \supseteq U'_x$. If there exists y within U_x , then a neighborhood U_y exists which is contained within U_x ($U_x \supseteq U_y$). If $x \neq y$, then two neighborhoods, U_x, U_y , having no common points would exist [20]. In 1914, Hausdorff wrote one of first methodical accounts of the set theory and the theory of topological spaces "Grundzüge der Mengenlehre" (Essentials of Set Theory) [21], where he introduced the notion of topological space. In that book, Hausdorff used the notion of upper bound, introduced by Weierstrass.

12. Conclusion

The notion of connectedness is primarily used in topology [22]. Weierstrass, while developing his method of analytic extension, created his own notion of connectedness for the purpose of the theory of analytic functions; the notion that proved to be stronger than the one of Cantor's. The intention of Weierstrass to substantiate and systematize the contemporaneous mathematical analysis resulted in his creation of new trends and notions in analysis and topology. As Weierstrass used to say, "even an introduction to mathematical sciences requires study of various problems, which, in the first place, shows us significance and consistency of science. One should bear in mind that the ultimate aim of the study of foundations of science is the striving for confidence in objectivity of science" [11, p. 20]. Weierstrass analyzed the methods and notions of the classical analysis so deeply that his constructions resulted in the notion of metric and topological space, on the basis of which functional analysis was formed.

We know the early history of the notion of connectedness because of a trace left by B. Bolzano (1817) – G. Cantor (1879–1883) – C. Jordan (1893) – N.J. Lennes (1905– 1911) – A. Schoenflies (1904) – F. Riesz (1906) – W.H. and G.C. Young (1906) – F. Hausdorff (1914) [23, 24, 25]. In [23, 25] there are nothing about Weierstrass' ideas. At present, Weierstrass occupies a deserved place in the history of evolution of the notion of connectedness.

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