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## USAGE OF A ZERO-SUM DIFFERENTIAL GAME IN THE OPTIMAL CONTROL OF AN OBJECT DESCRIBED BY A NONLINEAR MODEL

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## WYKORZYSTANIE GRY RÓŻNICZKOWEJ O SUMIE ZEROWEJ W OPTYMALNYM STEROWANIU OBIEKTEM OPISANYM MODELEM NIELINIOWYM

### Abstract

This article presents the usage of a zero-sum differential game to control a nonlinear object, which, in the analysed problem, was a mathematical pendulum. The obtained control was optimal with regard to adopted quality indicator for the worst interference. The two-point boundary value problem was solved numerically by means of the Dircol software application. Numerical solutions, meeting all the necessary optimality conditions, were obtained for different values of the rough parameter and for different values of damping.

**Keywords:** robust control, optimal control, two-point boundary value problem, minimum principle

### Streszczenie

W artykule wykorzystano grę różniczkową o sumie zerowej do sterowania obiektem nieliniowym, jakim w analizowanym problemie jest wahadło matematyczne. Uzyskano sterowanie optymalne ze względu na przyjęty wskaźnik jakości, przy najgorszym zakłóceniu. Dwupunktowy problem brzegowy został rozwiązany numerycznie przy wykorzystaniu programu Dircol. Rozwiązania numeryczne spełniające wszystkie warunki konieczne optymalności zostały otrzymane dla różnych wartości parametru szorstkości oraz przy różnych wartościach tłumienia.

**Słowa kluczowe:** sterowanie typu robust, optymalne sterowanie, dwupunktowy problem brzegowy, zasada minimum

## 1. Introduction

Structure control and regulation problems should take into account different criteria. One of the crucial criteria is the stability of control, specifically the expectation that a dynamic system is near its ideal state or is approaching it. Optimisation with regard to the chosen objective function is also significant requirement for different technical issues. The formal description of such criteria within the confines of modelling is an element of the present study.

In this paper, we deal with the optimal robust control. Unknown and unpredictable interferences are analysed; however, their influence on system dynamics is known. A well-designed rough process must be capable of limiting interferences, but at the same time, can be very far from the optimal solution. However, there exists a certain concept of compromise between optimality and roughness. A differential game with the null sum can be use in the robust-optimal control, which leads to a saddle point problem.

## 2. Dissipative systems controlled optimally

Let us consider the following dynamical dissipative system [10]:

$$\begin{aligned}x' &= f(x, u(x), w) \\ v &= l(x, u(x), w)\end{aligned}\tag{1}$$

with the control  $u \in U \subset L_2^{n_u}$  and the interference  $w \in W \subset L_2^{n_w}$ .

System (1) is dissipative if functions  $S, s$  exist with the following properties:

$$S(x(t_1)) < S(x(t_0)) + \int_{t_0}^{t_1} s(w, v) dt\tag{2}$$

for all  $w \in W$  and  $t > t_0$ .

The choice  $s(w, v) = \gamma^2 \|w\|^2 - \|v\|^2$  leads to the limited strengthened stable system  $L_2$ .

The expression (3) satisfies property (2).

$$S^*(x, u) = \max_{w \in W, t_1 > t_0} \int_{t_0}^{t_1} \left\{ \|l(x, u, w)\|^2 - \gamma^2 \|w\|^2 \right\} dt\tag{3}$$

In addition to the system's dissipativity, the robust type optimal control  $u^* = u^*(x)$  should provide potentially good optimal properties, which means that  $u$  should be chosen for arbitrary interferences  $w$  such that:

$$\begin{aligned}J(x, u) &= \int_{t_0}^{t_f} \|l(x, u, w)\|^2 dt \Rightarrow \min \\ l(x, u, w) &= \|l(x, u, w)\|^2\end{aligned}\tag{4}$$

If we assume a pessimistic interference in the system controlled by  $u^*$ , we obtain:

$$S^*(x, u^*) = \max_w \min_u \int_{t_0}^{t_f} \left\{ \|l(x(t), u(x(t)), w(t))\|^2 - \gamma^2 \|w(t)\|^2 \right\} dt \quad (5)$$

and thus, the saddle point for problem (4) is described by functional [5]:

$$J(x, u, w) = \int_{t_0}^{t_f} \left\{ \|l(x(t), u(x(t)), w(t))\|^2 - \gamma^2 \|w(t)\|^2 \right\} dt \Rightarrow \max_w \min_u \quad (6)$$

The form of quality indicator (6) results from the dissipative systems theory.

Remarks:

1. Control  $u^*$  guarantees the roughness of the system. Optimal properties resulting from the minimisation of functional (4) play a secondary role. The degree of roughness represented by parameter  $\gamma$  indirectly describes the value of the functional (the rougher the system is, the less optimal it is).
2. The sequence of mathematical operations, specifically min and max, occurring in (6) is important. Swapping these operations can facilitate the obtaining of the solution considerably.
3. Parameter  $\gamma$  indirectly gives information on the dynamic system. The optimisation of roughness is very important  $\gamma^* = \inf \{\gamma\}$ .

The optimisation of roughness is the fundamental task. The saddle point for functional (6) can be determined for a given  $\gamma \in R^+$ . The determination of  $\gamma^*$  can be approximated by the strategy  $\gamma$ .

The formulation of the min/max problem (6) can be interpreted as a two-player differential game. A two-player zero-sum differential game is described by:

1. The two players who are represented by  $u, w$ , which in the time interval  $[t_0, t_f]$  influence the dynamic system

$$\begin{aligned} x' &= f(x, u, w) \\ x(t_0) &= x_0, \quad r(x(t_f)) = 0 \\ u(t) &\in U, \quad \forall t \in [t_0, t_f] \\ w(t) &\in W, \quad \forall t \in [t_0, t_f] \end{aligned} \quad (7)$$

where  $f \in C^1(R^{n_x} \times R^{n_u} \times R^{n_w}, R^{n_x})$ ,  $r \in C^1(R^{n_x}, R^{n_r})$ ,  $u \subset R^{n_u}$ ,  $w \subset R^{n_w}$ ,  $x \subset X$ ;

2. The functional

$$J(x, u, w) = \int_{t_0}^{t_f} l(x(t), u(t), w(t)) dt \Rightarrow \max_w \min_u \quad (8)$$

where  $l \in C^1(R^{n_x} \times R^{n_u} \times R^{n_w}, R)$ .

3. The strategy class  $u = \Gamma^u(t, x)$ ,  $w = \Gamma^w(t, x)$

Information about the structures of controls  $u(t)$ ,  $w(t)$  is significant for solving the saddle problem.

In the dynamic system, these magnitudes take the roles of players in the differential game and are described by strategies  $\Gamma^u$ ,  $\Gamma^w$ .

The fundamental questions are as follows: What strategy should be assumed for rough control  $u(t)$ ? What strategy and interference structure should one account for?

The solution of this game is described by the Nash equilibrium [8].

### 3. Necessary conditions for the solution of the saddle point problem

Pair  $(u^*, w^*)$  defines the saddle point of the zero-sum game if:

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*) \quad (9)$$

for all allowed strategies  $u, w \in U \times W$ .

Let us define:

$$J^+ = \min_u \left\{ \max_w J \right\}$$

$$J^- = \max_w \left\{ \min_u J \right\}.$$

Using this definition, the quality indicator will be named the value of the zero-sum game

if it satisfies  $J^* = J^+ = J^-$ .

The problem of the zero-sum differential game can be written down with the Hamilton-Jacobi-Bellman equation:

$$0 = \min_u \max_w \frac{\partial J}{\partial x} f(x, u, w) + l(x, u, w)$$

$$= \max_w \min_u \frac{\partial J}{\partial x} f(x, u, w) + l(x, u, w) \quad (10)$$

The general solution of this partial equation provides the sought controls. It is worth noting that the solution can be obtained only for simple systems with small dimension of the state space.

If the formalism of the minimum principle is used for the problem described by equations (1–6), the necessary conditions for optimality can be set together:

Let the Hamilton function be defined:

$$H(x, \lambda, u, w) = \lambda^T \cdot f(x, u, w) + l(x, u, w) - \gamma^2 \cdot w^2 \quad (11)$$

then the boundary problem takes the form (12).

$$\begin{aligned}
x' &= f(x, u, w) \\
\lambda' &= -\left. \frac{\partial f}{\partial x} \right|^T (x, u^*, w^*) \cdot \lambda - \left. \frac{\partial l}{\partial x} \right|^T (x, u^*, w^*) \\
x(0) &= x_0, \quad x(t_f) = 0, \quad H(x, \lambda, u, w) \Big|_{t_f} = 0 \\
u^* &= \arg \min_{u \in U} H(x, \lambda, u, w^*) \\
w^* &= \arg \max_{w \in W} H(x, \lambda, u^*, w) \\
\frac{\partial^2 H}{\partial u^2} &> 0 \quad \frac{\partial^2 H}{\partial w^2} < 0 \quad \frac{\partial^2 H}{\partial u \partial w} = 0
\end{aligned} \tag{12}$$

In order to determine the saddle point, we have to continuously solve the canonical system of differential equations (12) and thus have assured access to control  $u^*$  and interference  $w^*$ . The determination of saddle points in technical applications are thoroughly described in works [2, 3, 7].

According to the suggestion from [1], if the worst interference  $w = w^*$  is assumed, control  $u$  is determined, such that it minimises the given functional (13) by including functions  $\lambda_1, \lambda_2$  in state variables.

$$\int_{t_0}^{t_1} (l(x, u, w^*) - \gamma^2 w^{*2}) dt \rightarrow \min_u \tag{13}$$

The system of equations for the determination of interference  $w^*$  has the following form [6]:

$$\begin{aligned}
x' &= f(x, u, w^*) \\
\lambda' &= -\left. \frac{\partial f}{\partial x} \right|^T (x, u, w^*) \cdot \lambda - \left. \frac{\partial l}{\partial x} \right|^T (x, u, w^*) \\
w^* &= \arg \max_{w \in W} H(x, \lambda, u, w) \\
u &\in U, \quad x(0) = 0, \quad x(t_f) = 0, \quad H(x, \lambda, u, w^*)(t_f) = 0 \\
H(x, \lambda, u, w) &= \lambda^T \cdot f(x, u, w) + l(x, u, w) - \gamma^2 \cdot w^2
\end{aligned} \tag{14}$$

Assuming that  $w \in L_2^{n_w}$  and taking into consideration condition (14)

$$\frac{\partial H}{\partial w} = \frac{\partial f}{\partial w}(x, u, w^*) \cdot \lambda + \frac{\partial l}{\partial w}(x, u, w^*) - 2\gamma^2 \cdot w = 0, \tag{15}$$

function  $w^* = w^*(x, \lambda)$  is obtained.

In the next step, the state vector will have the form:  $(x, \lambda)$ . Accordingly, the Hamilton function and the conditions for the determination of  $u^*$  can now be written as:

$$\begin{aligned}
H_{new}(x, \lambda, \psi, \theta) &= \psi^T \cdot f(x, u, w^*) + \theta^T \lambda'(x, u, w^*) + l(x, u, w^*) - \gamma^2 \cdot w^{*2} \\
\theta' &= -\frac{\partial H_{new}}{\partial \lambda} \\
\psi' &= -\frac{\partial H_{new}}{\partial x} \\
u^* &= \arg \min_{u \in U} H_{new}(x, \lambda, \psi, \theta, w^*)
\end{aligned} \tag{16}$$

#### 4. Controlled and interfered movement of the mathematical pendulum

Let us consider a physical system (the mathematical pendulum) consisting of a concentrated mass  $m = 1$  attached to a weightless member of length  $l = 1$ . Let it be assumed that the dissipation of energy is in the bearing with a linear relation of the dumping moment on the angular velocity. Additionally, interference  $w$  and control  $u$  are introduced. The optimisation problem is thus formulated as follows:

Functional (17) is minimised:

$$\int_0^{t_f} (u^2 + x_2^2 - \gamma^2 w^{*2}) dt \Rightarrow \min \tag{17}$$

with the limitations:

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= -(ax_2 + b \sin x_1) - u \cos x_1 + w^* \cos x_1
\end{aligned} \tag{18}$$

and for which  $w^*$  is determined from the condition:

$$\begin{aligned}
H &= \lambda_1 x_2 - \lambda_2 (ax_2 + b \sin x_1) - \lambda_2 u \cos x_1 + \lambda_2 w^* \cos x_1 + (u^2 + x_2^2 - \gamma^2 w^{*2}) \\
\frac{\partial H}{\partial w^*} &\rightarrow \lambda_2 \cos x_1 - 2\gamma^2 w^* = 0 \\
w^* &= \frac{\lambda_2 \cos x_1}{2\gamma^2} \\
\lambda_1' &= -\frac{\partial H}{\partial x_1} = \lambda_2 b \cos x_1 + \lambda_2 w^* \sin x_1 - \lambda_2 u \sin x_1 \\
\lambda_2' &= -\frac{\partial H}{\partial x_2} = a\lambda_2 - \lambda_1 - 2x_2
\end{aligned} \tag{19}$$

If we denote  $\lambda_1 = x_4$ ,  $\lambda_2 = x_5$ , we can write state equations  $(x^T, \lambda^T)$  (19) in the form:

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= -ax_2 - b \sin x_1 - u \cos x_1 + w^* \cos x_1 \\
x_3' &= u^2 + x_2^2 - \gamma^2 w^{*2} \\
x_4' &= bx_5 \cos x_1 + w^* x_5 \sin x_1 - ux_5 \sin x_1 \\
x_5' &= -x_4 + ax_5 - 2x_2 \\
w^* &= \frac{x_5 \cos x_1}{2\gamma^2}
\end{aligned} \tag{20}$$

Finally, the system of state equations can be written as follows:

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= -ax_2 - b \sin x_1 - u \cos x_1 + \frac{x_5 (\cos x_1)^2}{2\gamma^2} \\
x_3' &= u^2 + x_2^2 - \frac{x_5^2 (\cos x_1)^2}{4\gamma^2} \\
x_4' &= bx_5 \cos x_1 + \frac{x_5^2 \sin x_1 \cos x_1}{2\gamma^2} - ux_5 \sin x_1 \\
x_5' &= -x_4 + ax_5 - 2x_2
\end{aligned} \tag{21}$$

The following boundary conditions apply for state equations (20):

$$\begin{aligned}
x_1(0) &= x_{10}, \quad x_2(0) = x_{20}, \\
x_1(t_f) &= 0, \quad x_2(t_f) = 0, \quad H(t_f) = 0, \\
x_1(t_f)x_5(t_f) - x_1(t_f)x_4(t_f) &= 0.
\end{aligned} \tag{22}$$

The boundary conditions for functions  $x_i, x_5$  are unknown and the final time  $t_f$  will be

determined from the condition  $H(t_f) = \sum_{i=1}^{nx} \lambda_i \cdot x_i' + u^2 + x_2^2 - \gamma^2 w^{*2} = 0$ .

We can define the Hamilton function for state equations (21) in the following way [4]:

$$\begin{aligned}
H &= \psi_1 x_2 - \psi_2 (ax_2 + b \sin x_1) - \psi_2 u \cos x_1 + \psi_2 \frac{x_5 (\cos x_1)^2}{2\gamma^2} + \\
&\psi_3 \left\{ u^2 + x_2^2 - \left[ \frac{x_5^2 (\cos x_1)^2}{4\gamma^2} \right] \right\} + \psi_4 bx_5 \cos x_1 + \psi_4 \frac{x_5^2 \sin x_1 \cos x_1}{2\gamma^2} - \\
&\psi_4 ux_5 \sin x_1 + \psi_5 (-x_4 + ax_5 - 2x_2)
\end{aligned} \tag{23}$$

The optimal control  $u$  is determined from the condition  $\frac{\partial H}{\partial u} = 0$ :

$$\begin{aligned} -\Psi_2 \cos x_1 + 2\Psi_3 u - \Psi_4 x_5 \sin x_1 &= 0 \\ u &= \frac{\Psi_2 \cos x_1 + \Psi_4 x_5 \sin x_1}{2\Psi_3} \end{aligned} \quad (24)$$

Employing the condition  $\Psi_i' = -\frac{\partial H}{\partial x_i}$ , the system of equations for adjoint functions  $\Psi_i$  can be finally written in the form:

$$\begin{aligned} \Psi_1' &= \Psi_2 b(\cos x_1) - \Psi_2 u \sin x_1 + \frac{\Psi_2 (\sin 2x_1)}{2\gamma^2} - \Psi_2 \frac{x_5^2 (\sin 2x_1)}{4\gamma^2} + \Psi_4 b x_5 (\sin x_1) + \\ & - \Psi_4 \frac{x_5^2 (\sin 2x_1)}{2\gamma^2} + \Psi_4 u x_5 (\cos x_1) \\ \Psi_2' &= \Psi_2 a - 2\Psi_3 x_2 + 2\Psi_5 - \Psi_1 \\ \Psi_3' &= 0 \\ \Psi_4' &= \Psi_5 \\ \Psi_5' &= -\frac{\Psi_2 (\cos x_1)^2}{2\gamma^2} + \frac{\Psi_3 x_5 (\cos x_1)^2}{2\gamma^2} - \Psi_4 b \cos x_1 - \frac{\Psi_4 x_5^2 (\sin 2x_1)}{2\gamma^2} + \Psi_4 u \sin x_1 - \Psi_5 a \end{aligned} \quad (25)$$

Optimal control  $u(t)$  of the robust type minimises functional (17), specifically, variable  $x_3$ , and satisfies the extended state equations with optimally determined interference  $w^*$ , which in turn, maximises functional (17). State equations (21) and adjoint equations (25) allow determining the optimal control (24) which minimises the objective function.

## 5. Numerical results

The optimal controls were determined in accordance with the Pontryagin minimum principle. Program Dircol-2.1 [9] was used to numerically solve the problem formulated in the previous section. This required the preparation of input subprograms: user.f, DATDIM, DATLIM in which the state equations, boundary conditions, limitations, objective function and start values were defined. The calculation results were obtained in a graphic form as diagrams, and as a set of data. Solutions meeting all necessary optimality conditions were found for different values of rough parameter and for different values of damping.

Figures 1–4 depict complete solutions of the  $\min_u \max_w$  problem: state variables  $x_p$ , respective adjoint variables  $\Psi_p$ , control  $u$ , interference  $w$  and the phase diagram  $x_2 = x_2(x_1)$  for a value of damping established as  $a = 1$  and a value of rough parameter of  $\gamma = 0.7$ .



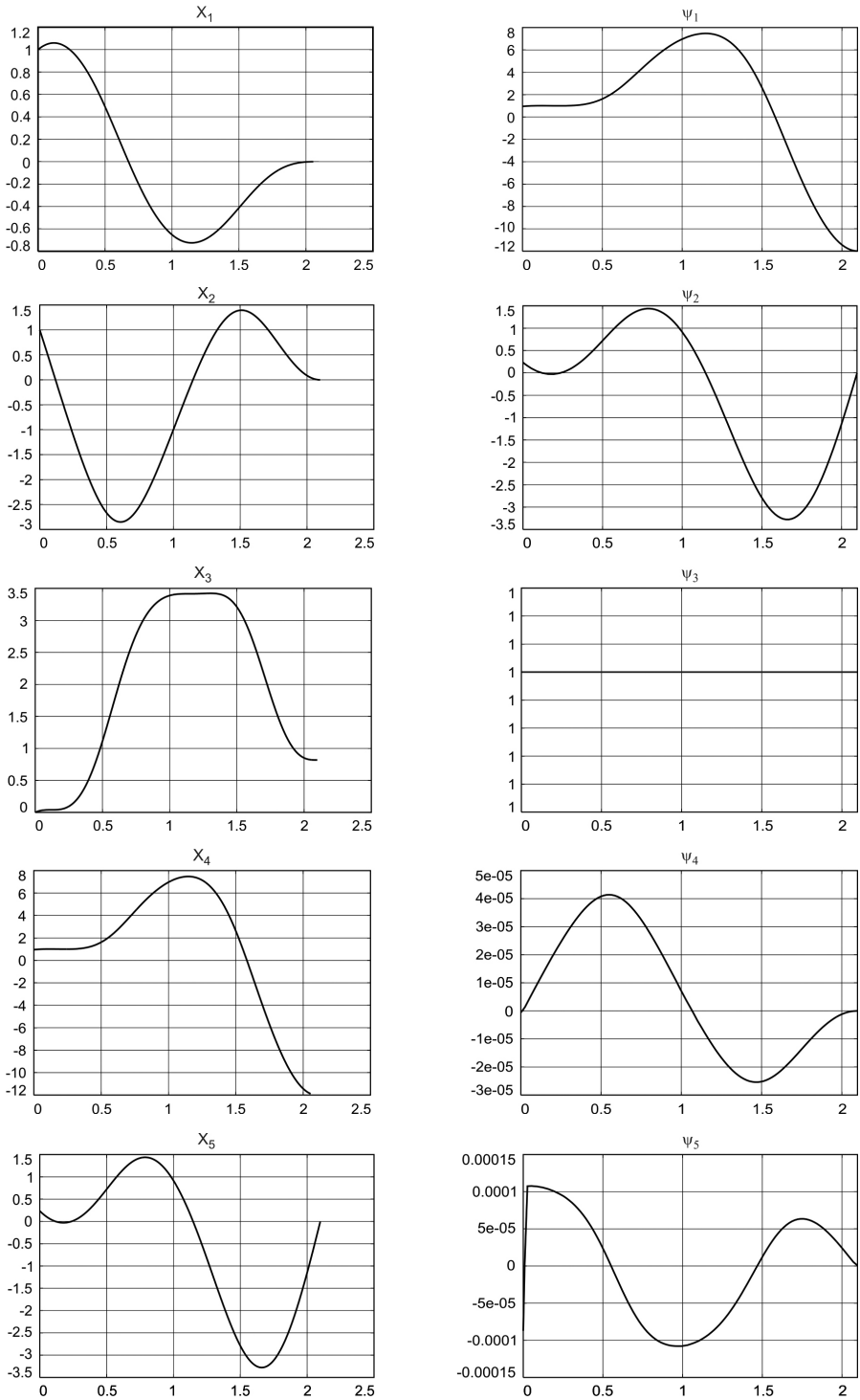


Fig. 1. State variables and corresponding adjoint variables for  $a = 1$  and  $\gamma = 0.7$



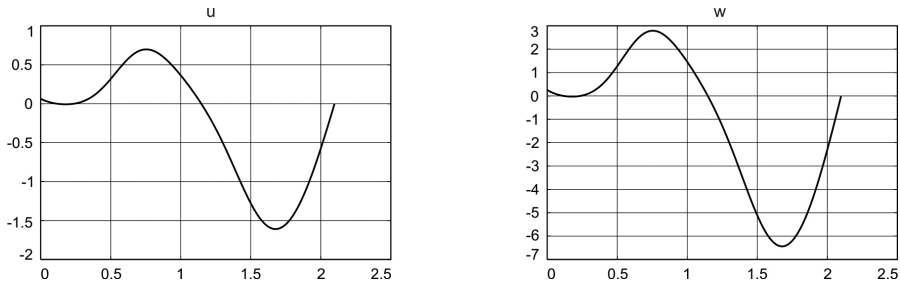


Fig. 2. Control and interference function for  $a = 1$  and  $\gamma = 0.7$

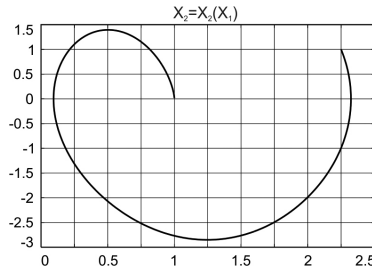


Fig. 3. Diagram  $X_2$  over  $X_1$  in phase-space for  $a = 1$  and  $\gamma = 0.7$

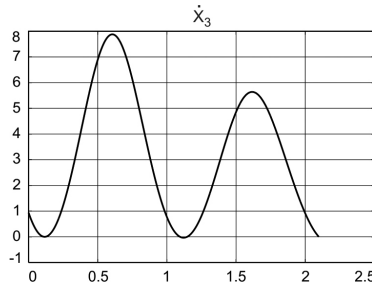


Fig. 4. The  $\dot{x}_3$  function for  $a = 1$  and  $\gamma = 0.7$

Of the numerous results obtained for different values of damping and rough parameters, the solution for the case of large damping with  $a = 5$  and a rough parameter value of  $\gamma = 0.7$  are shown as diagrams in Figs. 5 to 8.

The values of objective function  $J(x, u, w) = x_3$ , time of motion  $t_f$  and the information whether the necessary conditions for optimisation are fulfilled are all set in Table 1 for the selected tasks with damping taking the value  $a = 1$ .

Table 1. Objective function  $x_3$  and final time  $t_f$ , depending on  $\gamma$

$\gamma$	$J(x, u, w) = x_3$	$t_f$	IFAIL
0.5	5.5169	3.2231	0
0.6	1.5371	2.1847	0
0.7	0.8139	2.0977	0
0.8	3.1888	3.2205	0

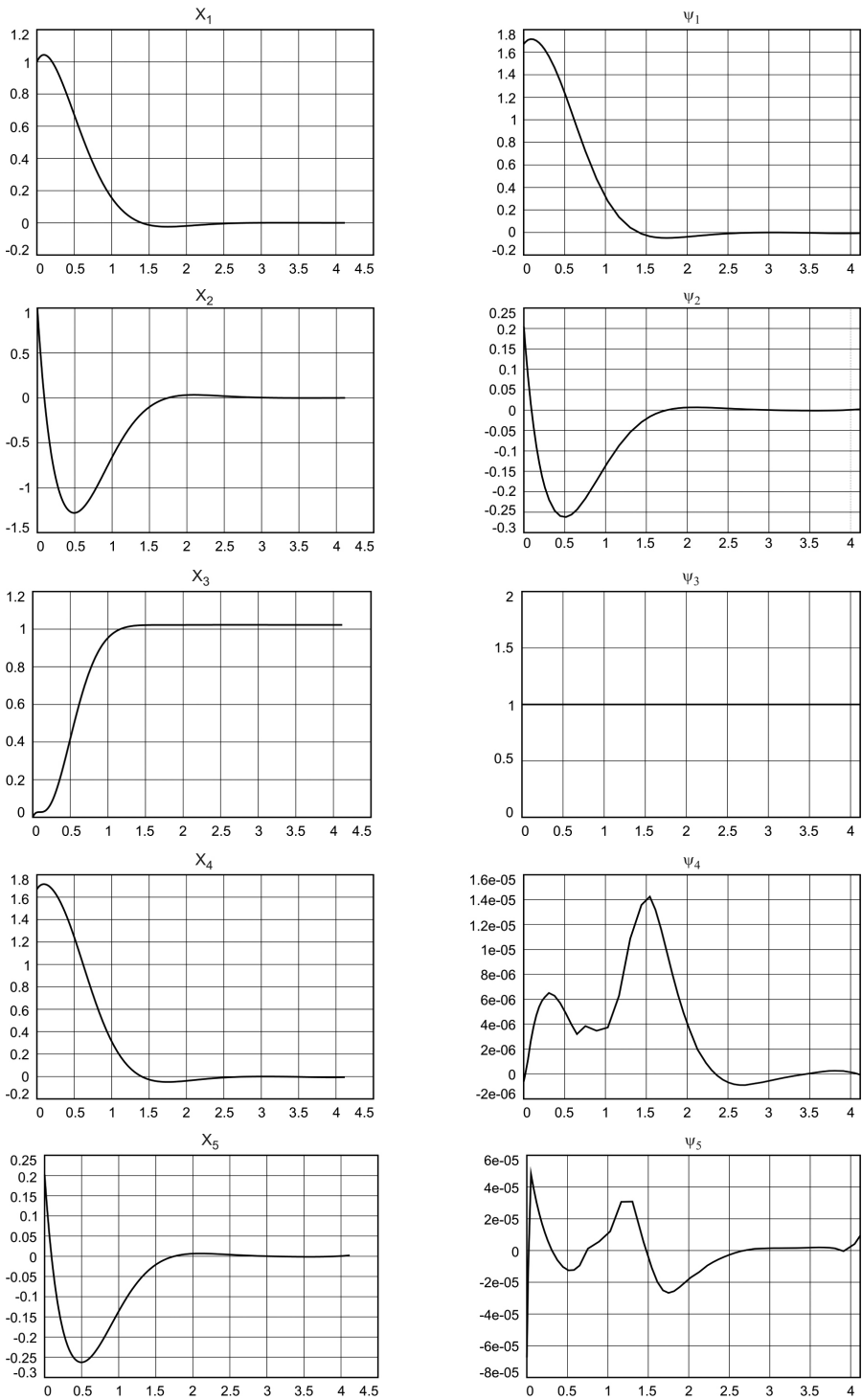


Fig. 5. State variables and corresponding adjoint variables for  $a = 5$  and  $\gamma = 0.5$



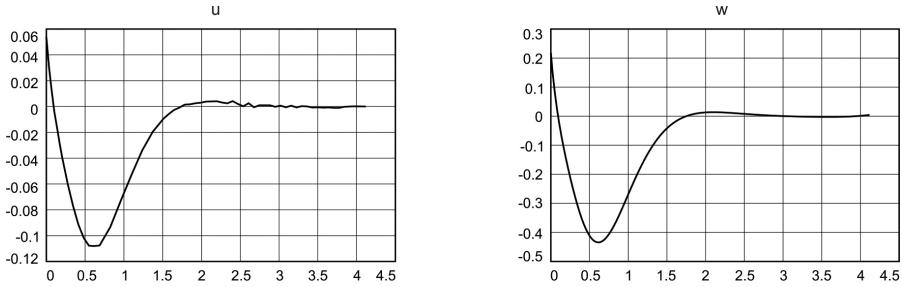


Fig. 6. Control and interference function for  $a = 5$  and  $\gamma = 0.5$

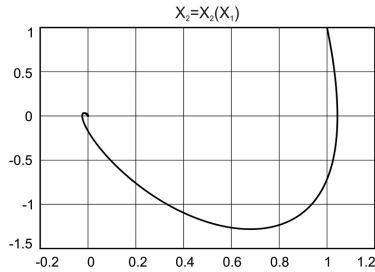


Fig. 7. Diagram  $X_2$  over  $X_1$  in phase-space for  $a = 5$  and  $\gamma = 0.5$

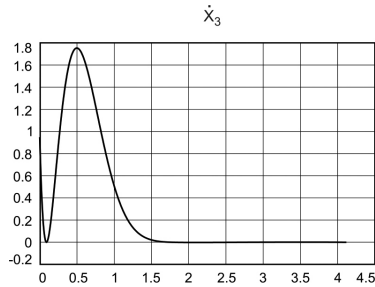


Fig. 8. The  $\dot{X}_3$  function for  $a = 5$  and  $\gamma = 0.5$

## 6. Summary

The problem of control a nonlinear object, which was the mathematical pendulum, has been discussed in this article. Optimal solutions with regard to the adopted objective function were obtained for the worst interference and for different values of the rough parameter and damping. The boundary value problem resulting from the minimum principle was solved numerically by means of the Dircol software. Complete solutions of the  $\min_u \max_w$  problem have been depicted in the figures. The obtained results, meeting all necessary optimality conditions, confirm that optimal control theory may be effectively used to solve problems of nonlinear object optimisation with the use of a zero-sum differential game.

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