MOVABLE INTERSECTION AND BIGNESS CRITERION

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Abstract. In this note, we give a Morse-type bigness criterion for the difference of two pseudo-effective $(1,1)$-classes by using movable intersections. As an application, we give a Morse-type bigness criterion for the difference of two movable $(n-1, n-1)$-classes.

1. Introduction. Let X be a smooth projective variety of dimension n, and let A, B be two nef line bundles. Then we have the fundamental inequality

$$\mathrm{vol}(A - B) \geq A^n - nA^{n-1} \cdot B,$$

which was first discovered as a consequence of Demailly's holomorphic Morse inequalities (see [**8, 19, 20**]). Thus the above inequality is usually called algebraic Morse inequality for line bundles. Recall that the volume of a holomorphic line bundle L is defined by

$$\mathrm{vol}(L) := \limsup_{k \to +\infty} \frac{n!}{k^n} h^0(X, \mathcal{O}(kL)).$$

We call L a big line bundle if $\mathrm{vol}(L) > 0$. In particular, the Morse-type inequality for A, B implies that: $A - B$ must be a big line bundle if $A^n - nA^{n-1} \cdot B > 0$. This provides a very effective way to construct holomorphic sections; see [**10, 14**] for related applications.

Assume that L is a holomorphic line bundle over a compact Kähler manifold X. It has been proved [**2**, Theorem 1.2] that then the volume of L can be characterized as the maximum of the Monge–Ampère mass of the positive curvature currents contained in the class $c_1(L)$. This naturally extends the volume function $\mathrm{vol}(\cdot)$ to transcendental $(1,1)$-classes (i.e., classes in $H^{1,1}(X, \mathbb{R})$) over compact Kähler manifolds (see [**2**, Definition 1.3] or [**4**, Definition 3.2]).

Recall that Demailly's conjecture on the (weak) transcendental holomorphic Morse inequality over compact Kähler manifolds is the following statement.

CONJECTURE 1.1. (see [**4**, Conjecture 10.1])[1] Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two nef classes. Then we have
$$\operatorname{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$
In particular, if $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, then there exists a Kähler current in the class $\alpha - \beta$.

Based on the method of [**7**], in our previous work [**23**], it was proved that: if $\alpha^n - 4n\alpha^{n-1} \cdot \beta > 0$, then there exists a Kähler current in the class $\alpha - \beta$. Recently, by keeping the same method as in [**7, 23**] and using the new technique introduced by [**17**], D. Popovici has proved that the constant $4n$ can be improved to the natural and optimal constant n. Thus we have a Morse-type bigness criterion for the difference of two transcendental nef classes – indeed, our results in this note depend mainly on this important improvement.

It is natural to ask whether the above Morse-type bigness criterion "$\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \Rightarrow \operatorname{vol}(\alpha - \beta) > 0$" for nef classes can be generalized to pseudo-effective $(1,1)$-classes. Towards this generalization, we apply the movable intersection products (denoted by $\langle - \rangle$) of pseudo-effective $(1,1)$-classes developed in [**1, 4, 5**]. Then our problem can be stated as follows:

PROBLEM 1.1. Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Does $\operatorname{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ imply that there exists a Kähler current in the class $\alpha - \beta$?

Unfortunately, a very simple example due to [**20**] implies that the above generalization does not always hold.

EXAMPLE 1.1. (see [**20**, Example 3.8]) Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at a point p. Let $R = \pi^* H$, where H is the hyperplane line bundle on $\mathbb{P}^2$. Let $E = \pi^{-1}(p)$ be the exceptional divisor. Then for every positive integer k, the space of global holomorphic sections of $k(R - 2E)$ is the space of homogeneous polynomials in three variables of degree at most k and vanishing up to order $2k$ at p; hence $k(R - 2E)$ does not have any global holomorphic section. Since $H^0(X, \mathcal{O}(k(R - 2E))) = \{0\}$, $R - 2E$ cannot be big. However, we have $R^2 - 2R \cdot 2E > 0$ as $R^2 = 1$ and $R \cdot E = 0$.

As the first result of this note, we show that the answer to Problem 1.1 is positive if β is movable. Here, β being movable means that the negative part of β vanishes in its divisorial Zariski decomposition (see [**3**]). In particular,

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[1]For projective manifolds, this conjecture has been confirmed by [**22**].

if $\beta = c_1(L)$ for some pseudo-effective line bundle, then β being movable is equivalent to the base locus of $mL + A$ being of codimension at least two for a fixed ample line bundle A and for large m.

THEOREM 1.1. *Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes with β movable. Then $\mathrm{vol}(\alpha) - n\langle\alpha^{n-1}\rangle \cdot \beta > 0$ implies that there exists a Kähler current in the class $\alpha - \beta$.*

REMARK 1.1. In the case of $\beta = 0$, Theorem 1.1 is just [**2**, Theorem 4.7], and when α is also nef, it is [**12**, Theorem 0.5].

Though not stated explicitly, by [**6**, Section 3] (or [**4**]), we know that Demailly's conjecture on the weak transcendental holomorphic Morse inequality over compact Kähler manifolds is equivalent to the $\mathcal{C}^1$ differentiability of the volume function for transcendental $(1,1)$-classes.

PROPOSITION 1.1. (*cf.* [**6**]) *Let X be a compact Kähler manifold of dimension n. Then the following statements are equivalent:*

1. *For any two nef classes $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$, we have*
$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

2. *For any two pseudo-effective classes $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ with β movable, we have*
$$\mathrm{vol}(\alpha - \beta) \geq \langle\alpha^n\rangle - n\langle\alpha^{n-1}\rangle \cdot \beta.$$

3. *Let $\alpha, \gamma \in H^{1,1}(X, \mathbb{R})$ be any two $(1,1)$-classes with α big. Then*
$$\frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle\alpha^{n-1}\rangle \cdot \gamma.$$

REMARK 1.2. Here, we point out the equivalence (1)$\Leftrightarrow$(2). It can be derived either from some basic properties of movable intersections, or from Theorem 1.1 and the equivalence (1)$\Leftrightarrow$(3).

REMARK 1.3. It is shown in [**13**] that the $\mathcal{C}^1$ differentiability of the volume function for transcendental $(1,1)$-classes holds on compact Kähler surfaces. It is used to construct the Okounkov bodies of transcendental $(1,1)$-classes over compact Kähler surfaces.

As a direct application of the equivalence (1)$\Leftrightarrow$(2) in Proposition 1.1, the algebraic Morse inequality can be generalized as follows. It slightly generalizes the previous result [**21**, Corollary 3.2].

THEOREM 1.2. *Let X be a smooth projective variety of dimension n, and let α, β be the first Chern classes of two pseudo-effective line bundles with β movable. Then we have*
$$\mathrm{vol}(\alpha - \beta) \geq \mathrm{vol}(\alpha) - n\langle\alpha^{n-1}\rangle \cdot \beta.$$

REMARK 1.4. In particular, if α is nef and β is movable then we have $\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$ which is just [**21**, Corollary 3.2].

Finally, as an application of Theorem 1.1, we give a Morse-type bigness criterion for the difference of two movable $(n-1, n-1)$-classes. In particular, this can be applied to the study of positivity of cohomology classes of curves.

THEOREM 1.3. *Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Then $\mathrm{vol}(\alpha) - n\alpha \cdot \langle \beta^{n-1} \rangle > 0$ implies that there exists a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$.*

In Section 2, we collect some preliminaries. Section 3 presents the proofs of the main results.

2. Preliminaries.

2.1. *Resolution of singularities of positive currents.* Let X be a compact Kähler manifold of dimension n. Let T be a d-closed almost positive $(1,1)$-current on X, that is $T \geq \gamma$ for some smooth $(1,1)$-form γ. If $\gamma = 0$, then T is called a positive $(1,1)$-current and the class $\{T\}$ is called pseudo-effective. If γ is a hermitian metric, then T is called a Kähler current and the class $\{T\}$ is called big. (See [**11**] for the basic theory of positive currents.)

Assume that $T = \theta + dd^c\varphi$ is an almost positive current, where θ is smooth. Then we say that it has analytic singularities, if locally $\varphi = a\log(\sum_{j=1}^m |f_j|^2)$ modulo smooth functions, where $a > 0$ and $f_1, .., f_m$ are holomorphic functions. Demailly's regularization theorem (see [**9**]) implies that one can always approximate the almost positive $(1,1)$-current T by a family of almost positive closed $(1,1)$-currents T_k with analytic singularities such that $T_k \geq \gamma - \varepsilon_k\omega$, where $\varepsilon_k \downarrow 0$ is a sequence of positive constants and ω is a fixed hermitian metric. In particular, when T is a Kähler current, it can be approximated by a family of Kähler currents with analytic singularities.

When T has analytic singularities along an analytic subvariety $V(\mathcal{I})$ where $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf, by blowing up along $V(\mathcal{I})$ and then resolving the singularities, we get a modification $\mu : \widetilde{X} \to X$ such that $\mu^*T = \widetilde{\theta} + [D]$, where $\widetilde{\theta}$ is an almost positive smooth $(1,1)$-form with $\widetilde{\theta} \geq \mu^*\gamma$ and D is an effective $\mathbb{R}$-divisor; see e.g. [**4**, Theorem 3.1]. In particular, if T is positive, then $\widetilde{\theta}$ is a smooth positive $(1,1)$-form. We call such a modification the log-resolution of singularities of T.

For an almost positive $(1,1)$-current T, we can always decompose T with respect to the Lebesgue measure. We write $T = T_{ac} + T_{sg}$, where T_{ac} is the absolutely continuous part and T_{sg} is the singular part. The absolutely continuous part T_{ac} can be seen as a form with L^1_{loc} coefficients, and the wedge product $T_{ac}^k(x)$ makes sense for almost every point x. We always have $T_{ac} \geq \gamma$,

since γ is smooth. If T has analytic singularities along V, then $T_{ac} = \mathbf{1}_{X \setminus V} T$ (see [**2**, Section 2.3] for the proof). However, in general T_{ac} is not closed even if T is closed (see [**1**]). We have the following proposition.

PROPOSITION 2.1. *Let X be a compact Kähler manifold of dimension n. Let $T_1, ..., T_k$ be k almost positive closed $(1, 1)$-currents with analytic singularities on X and let ψ be a smooth $(n - k, n - k)$-form. Let $\mu : \widetilde{X} \to X$ be a simultaneous log-resolution with $\mu^* T_i = \widetilde{\theta}_i + [D_i]$. Then*

$$\int_X T_{1,ac} \wedge ... \wedge T_{k,ac} \wedge \psi = \int_{\widetilde{X}} \widetilde{\theta}_1 \wedge ... \wedge \widetilde{\theta}_k \wedge \mu^* \psi.$$

PROOF. Since μ is an isomorphism outside a proper analytic subvariety and $\widetilde{\theta}_i$ is smooth on $\widetilde{X}$ for $i = 1, \ldots, k$, we only need to check that $T_{1,ac} \wedge ... \wedge T_{k,ac}$ puts no mass on proper analytic subvarieties. Without loss of generality, we may assume that the T_i are positive. Since the currents have analytic singularities, by the discussion above, $T_{1,ac} \wedge ... \wedge T_{k,ac} = \mathbf{1}_{X \setminus V} T_1 \wedge ... \wedge T_k$ where V is the union of singularities of the currents. The wedge product on the right hand side is just the non-pluripolar product of positive currents with analytic singularities, thus it puts no mass on proper subvarieties (see [**5**, Section 1.2]). $\square$

2.2. *Movable cohomology classes.* We first briefly recall the definition of divisorial Zariski decomposition and the definition of movable $(1, 1)$-class on a compact complex manifold (see [**3**], and also [**16**] in the algebraic setting).

Let X be a compact complex manifold of dimension n and let α be a pseudo-effective $(1, 1)$-class over X. Then one can always associate an effective divisor $N(\alpha) := \sum \nu(\alpha, D) D$ to α, where the sum is taken among all prime divisors on X. The class $\{N(\alpha)\}$ is called the negative part of α. The class $Z(\alpha) = \alpha - \{N(\alpha)\}$ is called the positive part of α. The decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ is then the divisorial Zariski decomposition of α.

DEFINITION 2.1. Let X be a compact complex manifold of dimension n, and let α be a pseudo-effective $(1, 1)$-class. Then α is called movable if $\alpha = Z(\alpha)$.

PROPOSITION 2.2. (*see* [**3**, *Proposition 2.3*]) *Let α be a movable $(1, 1)$-class and let ω be a Kähler class. Then for any $\delta > 0$ there exist a modification $\mu : Y \to X$ and a Kähler class $\widetilde{\omega}$ over Y such that $\alpha + \delta \omega = \mu_* \widetilde{\omega}$.*

REMARK 2.1. In [**3**], α is called modified nef if $\alpha = Z(\alpha)$ (see [**3**, Definition 2.2 and Proposition 3.8]). Here we call it movable. In the algebraic geometry setting let L be a line bundle over a smooth projective variety and let $\alpha = c_1(L)$. Then α is modified nef if and only if L is movable.

2.3. *Movable intersections.* We take the opportunity to point out the well-known fact: the several definitions of movable intersections of pseudo-effective $(1, 1)$-classes [**1, 4, 5**] over a compact Kähler manifold coincide. They also coincide with the algebraic construction of [**6**] on smooth projective varieties for specified degrees. One can refer to [**18**, Proposition 1.10] for a detailed proof.

We just recall the analytic definition from [**5**].

DEFINITION 2.2. Let X be a compact Kähler manifold of dimension n, and let $\alpha_1, ..., \alpha_k$ be big $(1, 1)$ classes on X. Then the cohomology class of the non-pluripolar product $\langle T_{\min,1} \wedge ... \wedge T_{\min,k} \rangle$ is independent of the choice of $T_{\min,i} \in \alpha_i$ with minimal singularities. This cohomology class is called the movable intersection (or positive product) of the α_i, and it is denoted by $\langle \alpha_1 \cdot ... \cdot \alpha_k \rangle$. If $\alpha_1, ..., \alpha_k$ are merely pseudo-effective, then the positive product is defined by

$$\langle \alpha_1 \cdot ... \cdot \alpha_k \rangle = \lim_{\varepsilon \to 0} \langle (\alpha_1 + \varepsilon\omega) \cdot ... \cdot (\alpha_k + \varepsilon\omega) \rangle,$$

where ω is an arbitrary Kähler class.

For the non-pluripolar product, we refer the reader to [**5**].

REMARK 2.2. For any pseudo-effective $(1, 1)$-class α, the current with minimal singularities in the class α always exists. For example, assume that $\theta \in \alpha$ is a smooth $(1, 1)$-form, and assume that $\varphi_{\min} = \sup\{\varphi \leq 0 | \varphi \in \mathrm{PSH}(\theta)\}$. Then $T_{\min} := \theta + dd^c\varphi_{\min}$ is a current with minimal singularities.

REMARK 2.3. By [**5**, Proposition 1.12], for any proper modification $\pi : Y \to X$, the movable intersection of n pseudo-effective classes satisfies $\langle \pi^*\alpha_1 \cdot ... \cdot \pi^*\alpha_n \rangle = \langle \alpha_1 \cdot ... \cdot \alpha_n \rangle$.

In [**5**], it is proved that: if $\alpha_1, ..., \alpha_p$ are big classes, then there exists a sequence of Kähler currents $T_i^{(k)} \in \alpha_i$ with analytic singularities such that

$$\lim_{k \to \infty} \{\langle T_1^{(k)} \wedge ... \wedge T_p^{(k)} \rangle\} = \langle \alpha_1 \cdot ... \cdot \alpha_p \rangle.$$

In particular, by considering a sequence of simultaneous log-resolutions of $T_i^{(k)} \in \alpha_i$, one gets a sequence of modifications $\mu_k : X_k \to X$ such that $\mu_k^*\alpha_i = \widehat{\omega}_{i,k} + [E_{i,k}]$ and

$$\lim_{k \to \infty} (\mu_k)_*(\widehat{\omega}_{1,k} \cdot ... \cdot \widehat{\omega}_{p,k}) = \langle \alpha_1 \cdot ... \cdot \alpha_p \rangle,$$

where the notation $\cdot$ in the left-hand side limit is the intersection of cohomology classes, and the $\widehat{\omega}_{i,k}$ are postive classes upstairs and the $E_{i,k}$ are real effective divisors. By [**5**, Theorem 1.16], this is exactly the definition of [**4**].

By [**5**, Theorem 1.16] again, it is easy to see that the movable intersection is monotonous with respect to the positivity of the α_i.

By [**4**, Definition 1.3, Theorem 1.5 and Conjecture 2.3], the definition of movable $(n-1, n-1)$-classes in the Kähler setting can be formulated as follows.

DEFINITION 2.3. Let X be a compact Kähler manifold of dimension n, and let $\gamma \in H^{n-1,n-1}(X, \mathbb{R})$. Then γ is called a movable $(n-1, n-1)$-class if it is in the closure of the convex cone generated by cohomology classes of the form $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle$ with every α_i pseudo-effective.

REMARK 2.4. When X is a smooth projective variety of dimension n, [**4**, Theorem 1.5] implies that the rational movable $(n-1, n-1)$-classes are in the cone of movable curves.

3. Proof of the main results. Now let us begin to prove our main results. We first give a Morse-type bigness criterion for the difference of two pseudo-effective $(1, 1)$-classes by using movable intersections. To this end, we need some properties of movable intersections.

LEMMA 3.1. *Let X be a compact Kähler manifold of dimension n, and let $\alpha_1, ..., \alpha_{n-1}, \beta \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes with β nef. Then we have*

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta.$$

PROOF. By the continuity of positive products, by taking limits, we can assume that $\alpha_1, ..., \alpha_{n-1}$ are big and β is Kähler.

By [**5**] or [**4**, Theorem 3.5], there exists a sequence of simultaneous log-resolutions $\mu_m : X_m \to X$ with $\mu_m^* \alpha_i = \omega_{i,m} + [D_{i,m}]$ and $\mu_m^* \beta = \gamma_m + [E_m]$ such that

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \limsup_{m \to \infty} (\omega_{1,m} \cdot ... \cdot \omega_{n-1,m} \cdot \gamma_m),$$

and

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta = \limsup_{m \to \infty} (\omega_{1,m} \cdot ... \cdot \omega_{n-1,m}) \cdot \mu_m^* \beta,$$

where the $\omega_{i,m}, \gamma_m$ are $(1, 1)$ cohomology classes which can be represented by positive $(1, 1)$ forms.

First, note that by definition we always have $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle \leq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta$, when β is only assumed to be big (or pseudo-effective).

When β is Kähler, the class γ_m is taken to be $\mu_m^* \beta$. Hence, if β is Kähler or nef, then we have the required equality

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta.$$

$\square$

LEMMA 3.2. *Let X be a compact Kähler manifold of dimension n. Let $\alpha_1, ..., \alpha_{n-1}$ be pseudo-effective $(1,1)$-classes, and let $\pi : Y \to X$ be a modification. Then for any Kähler class $\widehat{\omega}$ on Y we have*

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \pi_* \widehat{\omega} \geq \langle \pi^* \alpha_1 \cdot ... \cdot \pi^* \alpha_{n-1} \rangle \cdot \widehat{\omega}.$$

PROOF. First, we have $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \pi_* \widehat{\omega} \geq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \pi_* \widehat{\omega} \rangle$ as noted in the proof of Lemma 3.1. On the other hand, we have $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \pi_* \widehat{\omega} \rangle = \langle \pi^* \alpha_1 \cdot ... \cdot \pi^* \alpha_{n-1} \cdot \pi^*(\pi_* \widehat{\omega}) \rangle$. Assume that E is the exceptional divisor of π and use the same symbol $\widehat{\omega}$ to denote a Kähler metric in the class $\widehat{\omega}$. Then by Siu's decomposition $\pi^*(\pi_* \widehat{\omega}) = \widehat{\omega} + c[E]$ for some $c \geq 0$. In particular, the class $\pi^*(\pi_* \widehat{\omega}) - \widehat{\omega}$ is pseudo-effective, hence

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \pi_* \widehat{\omega} \geq \langle \pi^* \alpha_1 \cdot ... \cdot \pi^* \alpha_{n-1} \cdot \widehat{\omega} \rangle$$
$$= \langle \pi^* \alpha_1 \cdot ... \cdot \pi^* \alpha_{n-1} \rangle \cdot \widehat{\omega}. \qquad \square$$

3.1. Theorem 1.1.

PROOF OF THEOREM 1.1. Fix a Kähler metric ω on X, and denote its Kähler class by the same symbol. By continuity of movable intersections, we have

$$\lim_{\delta \to 0} \langle (\alpha + \delta\omega)^n \rangle - n \langle (\alpha + \delta\omega)^{n-1} \rangle \cdot (\beta + \delta\omega) = \langle \alpha^n \rangle - n \langle \alpha^{n-1} \rangle \cdot \beta.$$

Thus $\langle (\alpha + \delta\omega)^n \rangle - n \langle (\alpha + \delta\omega)^{n-1} \rangle \cdot (\beta + \delta\omega) > 0$ for small $\delta > 0$. Note also that $\alpha - \beta = (\alpha + \delta\omega) - (\beta + \delta\omega)$. Thus, to prove the bigness of the class $\alpha - \beta$, we may assume that α is big, and that $\beta = \mu_* \widetilde{\omega}$ for some modification $\mu : Y \to X$ and some Kähler class $\widetilde{\omega}$ on Y from the start. By Lemma 3.1 and Lemma 3.2, our assumption then implies that

$$\langle (\mu^*\alpha)^n \rangle - n \langle (\mu^*\alpha)^{n-1} \cdot \widetilde{\omega} \rangle = \langle (\mu^*\alpha)^n \rangle - n \langle (\mu^*\alpha)^{n-1} \rangle \cdot \widetilde{\omega} > 0.$$

Claim: there exists a Kähler current in the class $\mu^*\alpha - \widetilde{\omega}$. The claim then implies the bigness of the class $\alpha - \beta = \mu_*(\mu^*\alpha - \widetilde{\omega})$.

Now it is reduced to prove the case when α is big and β is Kähler. Let ω' be a Kähler metric in the class β. The definition of movable intersections (see the discussion in Section 2.3) implies that there exists some Kähler current $T \in \alpha$ with analytic singularities along some subvariety V such that

$$\int_{X \setminus V} T^n - n \int_{X \setminus V} T^{n-1} \wedge \omega' > 0.$$

Let $\pi : Z \to X$ be the log-resolution of the current T with $\pi^*T = \theta + [D]$, where θ is a smooth positive $(1,1)$-form on Z. By Proposition 2.1 we have

$$\int_Z \theta^n - n \int_Z \theta^{n-1} \wedge \pi^*\omega' > 0.$$

The result of [**17**] then implies that there exists a Kähler current in the class $\{\theta - \pi^*\omega'\}$. As $\pi^*\alpha = \{\theta + [D]\}$, this proves the bigness of the class $\alpha - \beta$.

Thus we finish the proof of the result. $\qquad\square$

REMARK 3.1. Indeed, by the above argument we have the following bigness criterion: for any pseudo-effective $(1, 1)$-classes α, β,

$$\langle \alpha^n \rangle - n \langle \alpha^{n-1} \cdot \beta \rangle > 0 \Rightarrow \mathrm{vol}(Z(\alpha) - Z(\beta)) > 0.$$

Since the movable intersections $\langle \alpha^n \rangle$ and $\langle \alpha^{n-1} \cdot \beta \rangle$ depend only on the positive parts of α, β (see e.g. [**1**, Proposition 3.2.10]), we may assume that α, β are movable from the start. By taking limits, we may also assume that $\beta = \pi_* \widehat{\omega}$ for some modification π. Then we have

$$\begin{aligned}
\langle (\pi^*\alpha)^n \rangle - n \langle (\pi^*\alpha)^{n-1} \cdot \widehat{\omega} &= \langle (\pi^*\alpha)^n \rangle - n \langle (\pi^*\alpha)^{n-1} \cdot \widehat{\omega} \rangle \\
&\geq \langle (\pi^*\alpha)^n \rangle - n \langle (\pi^*\alpha)^{n-1} \cdot \pi^*\beta \rangle \\
&> 0,
\end{aligned}$$

which implies that $\alpha - \beta = \pi_*(\pi^*\alpha - \widehat{\omega})$ is big.

In the case of Example 1.1, since R is nef and E is exceptional, we have $\langle R^2 \rangle - 2 \langle R \cdot 2E \rangle = R^2 > 0$, and we get the bigness of $Z(R) - Z(2E) = R$.

3.2. *Proposition 1.1 and Theorem 1.2.*

PROOF OF PROPOSITION 1.1. It is obvious that $(2) \Rightarrow (1)$. The equivalence $(1) \Leftrightarrow (3)$ is proved in [**6**]. It remains to verify $(1) \Rightarrow (2)$.

To prove (2), we only need to consider the case when β is big and movable.

First assume that β is Kähler. Take a sequence of suitable modifications μ_m satisfying $\mu_m^*\alpha = \omega_m + [E_m]$ and

$$\langle \alpha^n \rangle = \limsup_{m \to \infty} \omega_m^n, \quad \langle \alpha^{n-1} \rangle = \limsup_{m \to \infty} (\mu_m)_*(\omega_m^{n-1}).$$

By (1), we have $\mathrm{vol}(\omega_m - \mu_m^*\beta) \geq \omega_m^n - n\omega_m^{n-1} \cdot \mu_m^*\beta$. Then taking limits implies that

$$\mathrm{vol}(\alpha - \beta) \geq \mathrm{vol}(\alpha) - n \langle \alpha^{n-1} \rangle \cdot \beta$$

holds true when β is Kähler.

In the general case, we may assume that $\beta = \pi_* \widehat{\omega}$ for some Kähler class $\widehat{\omega}$ upstairs, so $\pi_*(\pi^*\alpha - \widehat{\omega}) = \alpha - \beta$. By the property of volume function (see e.g. [**1**]), we get $\mathrm{vol}(\alpha - \beta) = \mathrm{vol}(\pi_*(\pi^*\alpha - \widehat{\omega})) \geq \mathrm{vol}(\pi^*\alpha - \widehat{\omega})$. Then we have

$$\begin{aligned}
\mathrm{vol}(\alpha - \beta) \geq \mathrm{vol}(\pi^*\alpha - \widehat{\omega}) &\geq \mathrm{vol}(\pi^*\alpha) - n \langle (\pi^*\alpha)^{n-1} \rangle \cdot \widehat{\omega} \\
&\geq \mathrm{vol}(\pi^*\alpha) - n \langle \alpha^{n-1} \rangle \cdot \pi_* \widehat{\omega} \quad \text{by Lemma 3.2} \\
&= \mathrm{vol}(\alpha) - n \langle \alpha^{n-1} \rangle \cdot \beta.
\end{aligned}$$

This then finishes the proof of the result. $\qquad\square$

REMARK 3.2. Since (1)$\Leftrightarrow$(3), there is an alternative way to see (1)$\Rightarrow$(2). Note that the implication (1)$\Rightarrow$(2) is equivalent to (3)$\Rightarrow$(2). This is an easy consequence of Theorem 1.1. To prove (2), we only need to verify the case when $\langle \alpha^n \rangle - n \langle \alpha^{n-1} \rangle \cdot \beta > 0$. By Theorem 1.1, this implies that $\alpha - \beta$ is big. Applying (3), we have

$$\text{vol}(\alpha - \beta) - \text{vol}(\alpha) = \int_0^1 \frac{d}{dt} \text{vol}(\alpha - t\beta) dt$$

$$= -n \int_0^1 \langle (\alpha - t\beta)^{n-1} \rangle \cdot \beta dt \geq -n \langle \alpha^{n-1} \rangle \cdot \beta,$$

which is statement (2).

PROOF OF THEOREM 1.2. We apply the equivalences in Proposition 1.1. Since the inequality holds true for nef line bundles, by the equivalence (1)$\Leftrightarrow$(2), we get the result. $\square$

3.3. *Theorem 1.3.* Finally, inspired by the method in our previous work [**23**] (see also [**7**]), we show that Theorem 1.1 gives a Morse-type bigness criterion for the difference of two movable $(n-1, n-1)$-classes.

PROOF OF THEOREM 1.3. Denote the Kähler cone of X by $\mathcal{K}$, and denote the cone generated by pseudo-effective $(n-1, n-1)$-classes by $\mathcal{N}$. Then by the numerical characterization of Kähler cone in [**12**] (see also [**4**, Theorem 2.1]), we have the cone duality

$$\overline{\mathcal{K}}^\vee = \mathcal{N}.$$

Without loss of generality, we may assume that α and β are big. Then the existence of a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$ is equivalent to the existence of some positive constant $\delta > 0$ such that

$$\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle \succeq \delta \langle \beta^{n-1} \rangle,$$

or equivalently,

$$\langle \alpha^{n-1} \rangle \succeq (1 + \delta) \langle \beta^{n-1} \rangle,$$

or equivalently by cone duality,

$$\langle \alpha^{n-1} \rangle \cdot N \geq (1 + \delta) \langle \beta^{n-1} \rangle \cdot N,$$

for any non-zero nef class N. Here we denote $\gamma \succeq \eta$ if $\gamma - \eta$ contains a positive current.

In what follows, we will argue by contradiction.

By the above discussion, the statement "the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$ does not contain any strictly positive $(n-1, n-1)$-current" is then equivalent to the

statement "for any $\epsilon > 0$ there exists some non-zero class $N_\epsilon \in \overline{\mathcal{K}}$ such that

$$\langle \alpha^{n-1} \rangle \cdot N_\epsilon \leq (1 + \epsilon) \langle \beta^{n-1} \rangle \cdot N_\epsilon."$$

On the other hand, we claim that Theorem 1.1 implies

$$n(N \cdot \langle \alpha^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle) \geq \langle \alpha^n \rangle(N \cdot \langle \beta^{n-1} \rangle)$$

for any nef $(1,1)$-class N. (Actually, by [**17**], a priori, in the above inequality the class $\langle \beta^{n-1} \rangle$ can be replaced by more general positive classes.) First note that, for each cohomology class, both sides of the above inequality are of the same homogeneous degree. After scaling, we can assume

$$\alpha \cdot \langle \beta^{n-1} \rangle = N \cdot \langle \beta^{n-1} \rangle.$$

Then we need to prove $nN \cdot \langle \alpha^{n-1} \rangle \geq \langle \alpha^n \rangle$. Otherwise, we have $nN \cdot \langle \alpha^{n-1} \rangle < \langle \alpha^n \rangle$. Then Theorem 1.1 implies that there must exist a Kähler current in the class $\alpha - N$. Then we must have

$$\langle \beta^{n-1} \rangle \cdot (\alpha - N) > 0,$$

which contradicts our assumption $\langle \beta^{n-1} \rangle \cdot (\alpha - N) = 0$.

Let $N = N_\epsilon$, we get

$$(1 + \epsilon)n(N_\epsilon \cdot \langle \beta^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle) \geq n(N_\epsilon \cdot \langle \alpha^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle)$$
$$\geq \langle \alpha^n \rangle(N_\epsilon \cdot \langle \beta^{n-1} \rangle).$$

This implies

$$(1 + \epsilon)n\alpha \cdot \langle \beta^{n-1} \rangle \geq \langle \alpha^n \rangle.$$

Since $\epsilon > 0$ is arbitrary, this contradicts our assumption $\langle \alpha^n \rangle - n\alpha \cdot \langle \beta^{n-1} \rangle > 0$. Thus there must exist a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$.

This finishes the proof of the result. $\square$

REMARK 3.3. Let X be a smooth projective variety of dimension n and let $\mathrm{Mov}_1(X)$ be the closure of the cone generated by movable curve classes. In [**15**], we show that any interior point in $\mathrm{Mov}_1(X)$ is of the form $\langle L^{n-1} \rangle$ for a unique big and movable divisor class. Hence, Theorem 1.3 applies to the classes of movable curves.

References

1. Boucksom S., *Cônes positifs des variétés complexes compactes*, PhD thesis, Université Joseph-Fourier-Grenoble I, 2002.
2. Boucksom S., *On the volume of a line bundle*, Internat. J. Math., **13**(**10**) (2002), 1043–1063.
3. Boucksom S., *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4), **37**(**1**) (2004), 45–76.

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4. Boucksom S., Demailly J.-P., Păun M., Peternell T., *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom., **22(2)** (2013), 201–248.

5. Boucksom S., Eyssidieux P., Guedj V., Zeriahi A., *Monge–Ampère equations in big cohomology classes*, Acta Math., **205(2)** (2010), 199–262.

6. Boucksom S., Favre C., Jonsson M., *Differentiability of volumes of divisors and a problem of Teissier*, J. Algebraic Geom., **18(2)** (2009), 279–308.

7. Chiose I., *The kähler rank of compact complex manifolds*, The Journal of Geometric Analysis, **26(1)** (2016), 603–615.

8. Demailly J.-P., *Champs magnétiques et inégalités de Morse pour la d″-cohomologie*, Ann. Inst. Fourier (Grenoble), **35(4)** (1985), 189–229.

9. Demailly J.-P., *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom., **1(3)** (1992), 361–409.

10. Demailly J.-P., *Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture*, Pure Appl. Math. Q., **7(4)**, Special Issue: In memory of Eckart Viehweg (2011), 1165–1207.

11. Demailly J.-P., *Complex analytic and differential geometry.* Online book available at `www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`, Institut Fourier, Grenoble, 2012.

12. Demailly J.-P., Păun M., *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2), **159(3)** (2004), 1247–1274.

13. Deng Y., *Transcendental Morse inequality and generalized Okounkov bodies*, `arXiv:1503.00112`, 2015.

14. Diverio S., Merker J., Rousseau E., *Effective algebraic degeneracy*, Invent. Math., **180(1)** (2010), 161–223.

15. Lehmann B., Xiao J., *Positivity functions for curves on algebraic varieties*, `arXiv:1607.05337`, 2016.

16. Nakayama N., *Zariski-decomposition and abundance*, volume **14** of *MSJ Memoirs*, Mathematical Society of Japan, Tokyo, 2004.

17. Popovici D., *Sufficient bigness criterion for differences of two nef classes*, Math. Ann., **364(1-2)** (2016), 649–655.

18. Principato M., *Mobile product and zariski decomposition*, `arXiv:1301.1477`, 2013.

19. Siu Y. T., *An effective Matsusaka big theorem*, Ann. Inst. Fourier (Grenoble), **43(5)** (1993), 1387–1405.

20. Trapani S., *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z., **219(3)** (1995), 387–401.

21. Trapani S., *Divisorial Zariski decomposition and algebraic Morse inequalities*, Proc. Amer. Math. Soc., **139(1)** (2011), 177–183.

22. Witt Nyström D., *Duality between the pseudoeffective and the movable cone on a projective manifold*, with an appendix by S. Boucksom, `arXiv:1602.03778`, 2016.

23. Xiao J., *Weak transcendental holomorphic Morse inequalities on compact Kähler manifolds*, Ann. Inst. Fourier (Grenoble), **65(3)** (2015), 1367–1379.

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