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SOME REMARKS OF JAN ŚLESZYŃSKI
REGARDING FOUNDATIONS OF MATHEMATICS
OF STANISŁAW LEŚNIEWSKI

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DOTYCZĄCYCH PODSTAW MATEMATYKI
STANISŁAWA LEŚNIEWSKIEGO

Abstract

Jan Śleszyński, a great mathematician, is considered a pioneer of Polish logic; however, he was not connected with the famous Warsaw School of Logic (WSL). He believed that his mission was a critical evaluation of work of other logicians in the field of foundations of mathematics and proof theory. Among his writings we find several notes regarding the work of Stanisław Leśniewski (the co-founder of the WSL) and his collective set theory. These remarks are the subject of investigation of the presented paper.

Keywords: foundations of mathematics, set theory, element, set

Streszczenie

Jan Śleszyński, wielki matematyk, uważany jest za pioniera polskiej logiki, chociaż nie był związany ze słynną Warszawską Szkołą Logiki (WSL). Śleszyński uważał, że jego misją była krytyczna ocena prac dotyczących podstaw matematyki oraz teorii dowodu autorstwa innych logików. Wśród jego zapisków znajdujemy między innymi uwagi dotyczące teorii zbiorów kolektywnych Stanisława Leśniewskiego, współzałożyciela WSL. Uwagi te są przedmiotem analizy niniejszego artykułu.

Słowa kluczowe: podstawy matematyki, teoria zbiorów, element, zbiór

DOI: 10.4467/235373XCT.15.219.4424

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1. Introduction

The beginning of the twentieth century was incredibly rich in significant discoveries in exact sciences. Well known all over the world are the achievements of Polish scholars of the Lvov School of Mathematics, Warsaw School of Logic or Warsaw School of Mathematics. Jan Śleszyński is one of few mathematicians living in this period who studied logic and foundations of mathematics, and who does not belong to any of these schools. Even though he is considered a pioneer of Polish logic, Śleszyński is barely present in the consciousness of Poles. He is well known for his famous two-volume work: *The Proof Theory*, but almost nobody knows that he left a fairly rich heritage: 4 thousand pages of manuscripts (some edited by Stanisław Krystian Zaremba) containing analysis, lectures, speeches and comments. Among them one can find remarks dealing with the early work of Stanisław Leśniewski dedicated to the foundations of set theory, called *mereology*, which is still the subject of scientific investigations of logicians and philosophers¹. Therefore, in this paper, we would like to focus on Leśniewski's foundations of mathematics and Śleszyński's remarks dealing with this work.

2. Jan Śleszyński (1854–1931) and Stanisław Leśniewski (1886–1939)

Jan Śleszyński, the Polish mathematician and logician, was born in 1854 in Łysianka (Ukraine). In 1871–1875 he studied mathematics at the University of Odessa. After finishing his studies, he moved from Odessa to Kiev, where he taught mathematics in secondary schools. In 1880 he received a scholarship from the Russian government and went to Berlin, where he attended the lectures of L. Kronecker, E.E. Kummer and K. Weierstrass. In Berlin he prepared his thesis on the calculus of variations and in 1882 he came back to Odessa to work at the University. In his master's thesis he gave a precise proof of the restricted form of the Central Limit Theorem. In 1898 he became a full professor and retired in 1909.

At that time, the Polish Academy of Arts and Sciences received a donation from Władysław Kretkowski, designed, among other things, to increase the number of lectures in mathematics at the Jagiellonian University. The Foundation invited Śleszyński to come as a professor to Krakow. He accepted the invitation to take the chair and in October 1911 moved permanently to Krakow.

Stanisław Leśniewski was much younger than Śleszyński. He was born in Serpukhov (near Moscow) in Russia, 30 March 1886. Because his father was a railway engineer, they often changed their place of residence. After graduating from a high school in 1904, he went to Germany to study philosophy. He studied in Leipzig and Heidelberg. In 1909 he appeared in Munich, and in 1910 went to Lvov to write his PhD dissertation under the supervision of Kazimierz Twardowski. In Lvov he attended the lectures of Waław Sierpinski and Józef Puzyna on set theory and there he met Jan Łukasiewicz. After obtaining the PhD, he travelled a little: he went to France, Italy and St. Petersburg (1912/1913) and in the middle of 1913 he moved to Warsaw.

¹ For ex. D. Mieville – Switzerland, A.J. Cotnoir, B – UK, R. Poli; L. Biacino – Italy, A. Varzi – USA, A. Betti – NL, P. Forrest – Australia, etc.

In Lvov Leśniewski was influenced by the philosophy of Twardowski, especially by the theory of subjects of Husserl, which he applied to his set theory.

A very important event for Leśniewski was the reading of the work of Łukasiewicz *On Aristotle's principle of contradiction* (in 1911). For the first time he learned of the existence of symbolic logic and of Russell's famous antinomy. However, the reading of this work filled Leśniewski with a long-lasting aversion to logic, which he broke only in 1918. That time, since 1914, he had devoted to a detailed analysis of the work of English logicians – *Principia Mathematica*.

A very fruitful discussion around the work of Łukasiewicz resulted in various works published within the *Review of Philosophy (Przegląd Filozoficzny)* in the period 1911–1913², one of which is especially important for us. In 1913 Leśniewski wrote a paper: *Is the class of classes not subordinated to themselves, subordinated to itself?*, which was a prelude to the work on general set theory (*Podstawy ogólnej teorii mnogości*) called by Leśniewski: mereology, and published in Moscow in 1916. This work closed the so-called grammatical and philosophical period of Leśniewski's creativity and opened a new chapter in his life dedicated to the foundations of mathematics³. These two papers were subjected to investigations of Jan Śleszyński and they will be discussed in detail later in the text.

In that period, around 1915, Śleszyński actively participated in Polish academic life, giving lectures on mathematical logic, proof theory, probability. He participated in meetings of the Polish Mathematical Society, Polish Philosophical Society, etc. In the Krakow period he did not publish a lot. Beside *The Proof Theory* (already mentioned), in 1926 *The Theory of Determinants* was published. Both works are based on Śleszyński's lectures. They were written up by his students and not by the author himself. In addition, in 1923 in volume 3 of the *Guide for Self-Taught (Poradnik dla samouków)* two papers of Śleszyński appeared: *The importance of logic to mathematics* and *On the first stages of development of infinitary concepts*. In 1919 K. Żorawski took the position at the Warsaw University of Technology, and Śleszyński became a professor of mathematics and mathematical logic at the Jagiellonian University. In 1924 he completely retired from the academic life.

As A. Hoborski wrote, it was characteristic of Śleszyński to explain any doubts of logical and mathematical nature in a way that any mathematical reasoning could become complete, i.e. there shouldn't be any hidden rules applied in deductive reasoning. For this reason Śleszyński studied the work of British philosophers, Bertrand Russell and Alfred N. Whitehead "Principia Mathematica" (PM) very deeply and applied their ideograms to rewrite all doubtful proofs he found in literature. He verified theories of other authors and pointed out all shortcomings and mistakes, because, as he believed, his mission was a critical evaluation of work in the foundations of mathematics. Therefore, we find in his notes detailed remarks, written by him or by S. K. Zaremba, regarding not only the early

² One can find a detailed discussion in [7]. We have to add here that many very important ideas, developed later, have their roots in Leśniewski's intuitions from that period: e.g., many-valued logics of Łukasiewicz; the distinction between language and metalanguage – the research that later on was continued by A. Tarski, etc.

³ The complete presentation of his general set theory took place only in years 1927–1931, when it was published in the periodical *Philosophical Reviews (Przegląd Filozoficzny)*.

mathematical works of Leśniewski, but also the works of Łukasiewicz, and perhaps other authors.

Leśniewski, after four years spent in Moscow, returned to Warsaw in 1919 and became the Chair at the Department of Philosophy of Mathematics. Since that time he had been teaching at the University of Warsaw. During this period he only worked on the foundations of mathematics; he constructed his own symbolic language applied to the construction of his systems: Protothetics, Ontology and Mereology⁴, and created his own theory of semantic types applied broadly in the Warsaw School of Logic for a certain period of time. In 1936 he became a full professor.

3. Stanisław Leśniewski and his foundations of mathematics

The work of S. Leśniewski *On the foundations of general set theory*, published in Moscow in 1916, is the first attempt to formalize the general set theory also known as a collective set theory or mereology; a full formalization followed much later, in the years 1927–1931. This work is valuable because it removes the problem of Russell’s antinomy, but retains the original intuition of the term set introduced by Georg Cantor.

The paradox of set theory (i.e. the antinomy of irreflexive classes) is conceptually the simplest theory within the set theory. It was discovered by Russell (in 1901) while he studied the work of G. Frege *Grundgesetze der Arithmetik* (1893). The antinomy of irreflexive classes constitutes that there is no set composed of only those sets, which are not elements of themselves:

$$Z = \{X : X \notin X\}.$$

If we assume that R is such a set, and ask whether R is its own element, we obtain that:

$$R \in R \rightarrow R \notin R.$$

Hence, if we assume that $R \in R$, pursuant to the rules of inference, we obtain that:

$$R \notin R,$$

and vice versa. Thus, we get two contradictory expressions. Leśniewski suspected that the essence of the paradox lies in the mistaken notion of the term “set”, hence the desire to consolidate properly the foundations of mathematics was born, and the mereology was created.

Leśniewski uses only one primitive term: part. This term is reserved only for proper parts, i.e. those which are different from the whole. The relation of being a part is irreflexive and transitive. In contrast to the relation of ingrediens (i.e. improper part) which partially orders the set of objects on which it is determined, i.e. it is reflexive, antisymmetric, and transitive. For Leśniewski being an ingrediens of a set is equivalent to being an element of this set. The essence of the concepts of “element” and “set” is explained by Borkowski very well:

⁴ The complete collection of Leśniewski’s works may be found in [10].

“Terms “set”, “element of a set” are used in double meaning. One meaning is that the term “set” signifies objects made of parts, collectives, that is conglomerates of various types. Elements of such a set are understood as its parts, and the term “part” is understood in its common meaning, e.g. a leg of a table constitutes part of the table. A pile of stones in this meaning is a set of those stones. Particular stones and various parts of those stones, that is molecules or atoms are equally elements of this set. According to this meaning the set of given stones is identical with e.g. a set of all atoms constructing those stones. Elements of a set understood in this manner, e.g. a set of all tables, are not only particular tables, but e.g. legs of tables and other parts of tables. (...). The second meaning of the term “set” and “element of a set” is used e.g. when we talk about a set of European countries and we recognize particular European countries such as e.g. Poland, France, Italy etc. to be elements of the set, and we do not use various parts of those countries as elements of this set. In this meaning e.g. the Tatra Mountains or the Małopolska Upland are not elements of the set of European countries, despite the fact that they are parts of certain European countries. In this meaning we often use those terms when we talk e.g. about a set of Polish cities and we recognize particular cities e.g. Wrocław, Warsaw etc. as elements of this set and we do not recognize particular streets, squares and other parts of those cities (...) as elements of this set. In this meaning we cannot identify the notion of an element of a set with the common notion of part [1]”.

Mereology, hence, is the theory in which the whole can be seen in different ways⁵: let us consider, for example, a chessboard, and a class of its own fields. Each field of the chessboard is its element (because it is its part); it is an element of the class of its fields (which has 64 elements), but the same chessboard can be considered as a class of eight-fields rectangular bands (hence it has 8 elements). This perspective is totally unacceptable in classical set theory, where the set is uniquely determined by its elements.

To define the concept of a whole Leśniewski uses the terms “class” and “mereological sum”. A mereological class is a concrete object which consists of all its parts, while the sum of any non-empty set of objects is a concrete set of such parts that overlap with some elements of that set. Thus, one can immediately see the difference between the classical set theory and the theory of collective sets: the latter requires that components are connected to each other; we take into account the relationship between the individual elements, in the former this relationship is not taken into consideration. Of course it causes some problems for the theory itself, but also opens new possibilities for the understanding of the concept of continuum, where the whole cannot be considered only as a sum of its parts. In this perspective, perhaps a new definition of mereological continuum might be offered in future.

⁵ A detailed description of the essence of mereology presented in the language of the first order logic is presented in [7] and [8].

4. Śleszyński's notes

In one of Śleszyński's manuscript dating from the period 1919–1921, stored in the archive of the University of Warsaw [9], we find a detailed analysis of the first two works of Leśniewski [2, 3] dedicated to the foundations of set theory⁶. Let us begin with some basic concepts given by Leśniewski:

DEFINITIONS (p. 22):

- I. „I apply the term: ‘ingrediens of the object P ’ to denote the object P itself or each of its part”.
- II. The term: ‘set of objects m ’ I apply to denote any object P , which fulfills the following condition:
 - if J is an ingrediens of the object P , then some ingrediens of an object J is an ingrediens of some m , which is an ingrediens of the object P .
- III. The term: ‘class of objects m ’ – I apply to denote each object P , which fulfills the following conditions:
 - Each m is an ingrediens of the object P ,
 - If J is an ingrediens of the object P , then some ingrediens of the object J is an ingrediens of some m .
- IV. „I apply the term: ‘element of the object P ’, to denote any object $P1$, if at some denotation of the term x , the following conditions are fulfilled:
 - P is class of objects x ,
 - $P1$ is x .

AXIOMS (p. 20):

- I. If an object P is part of the object $P1$, then the object $P1$ is not part of the object P .
- II. If an object P is part of an object $P1$, the object $P1$ is part of an object $P2$, then the object P is part of the object $P2$.
- III. If an object is m , then some object is a class of m .
- IV. If P is a class of objects m , and $P1$ is a class of objects m , then P is $P1$.

The first definition given by Leśniewski defines the notion of a proper part, and the second and third define the concept of a collection (a set and a class). The concept of a class (a set) given above differs from the classical understanding, that of Cantor: in mereological collections (classes), as we mentioned before, the relation between the elements of sets (classes) is taken into consideration, while in Cantorian concept it is ignored. The fourth definition of Leśniewski defines the concept of an element of a class. While the first two axioms determine the properties of the relation of being a part, which is irreflexive and transitive, Axioms 3 and 4 are related to the existence and uniqueness of classes.

We find these definitions, axioms and some statements in the manuscript of Śleszyński, with accompanying notes, in the first twenty-five pages.

⁶ A PhD thesis written by P. Borowik under the supervision of J. Woleński was dedicated to the same topic, but the presented analysis was made independently.

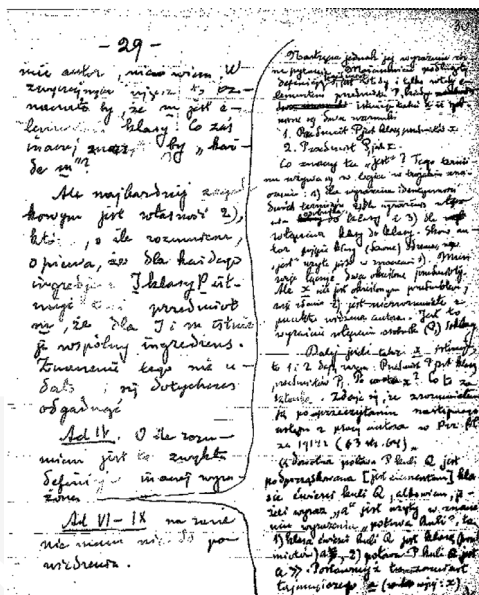


Fig. 1. Page 29 of Śleszyński's Manuscript [9]

Regarding the first two axioms of Leśniewski (AI, AII), Śleszyński formalizes them using the language from *Principia Mathematica* as follows:

Let apb mean that a is a part of an object b . Then the Axioms I and II can be written as follows:

$$(I) \quad apb \cdot \supset \sim (bpa)$$

$$(II) \quad apb \cdot bpc \cdot \supset \cdot apc$$

Here we find nothing special. Śleszyński compares Leśniewski's axioms to the axiomatization of line made by Vailati, where the predicate "p" means that a single point precedes the other. Hence, the system of axioms (I), (II) can be replaced by the system (Ia) and (II):

$$(Ia) \quad \sim (apa)$$

$$(II) \quad apb \cdot bpc \cdot \supset \cdot apc$$

Then Śleszyński refers to Leśniewski's proofs, and – for example – the proof (Ia) can be done in the following way:

$$(1) \quad (I) [b \setminus a] \rightarrow apa \cdot \supset \cdot \sim (apa)$$

$$(1).L \rightarrow \sim (apa)^7$$

⁷ Here "·" means conjunction; the sign: "· \supset ·" – implication; "(1).L \rightarrow " the result following from the conjunction of the assumption (1) and the logical rules signed by L; "(I) [b \setminus a]" – we substitute "b" for "a" in assumption (I).

Instead, if we begin with Axioms (Ia) and (II), then the proof would be as follows (applying of course the classic rules of inference, which Śleszyński symbolically denotes by the letter: *L*):

- (1) (II) $[c \setminus a] \rightarrow apb \cdot bpa \cdot \supset \cdot apa$
- (2) (Ia) $\rightarrow \sim (apa)$
- (3) (1).(2).*L* $\rightarrow \sim (apb \cdot bpa)$
- (4) (3).*L* $\rightarrow apb \cdot \supset \cdot \sim (bpa)$, which proves (I).

We must add here that Leśniewski deliberately does not use the language of classical logic, because he believes that it is imprecise; he distinguishes between the concept of existence and that of being: every object that exists – by Leśniewski – is, but not everyone which is, exists.

Now, let us take into consideration the definition of ingrediens (and being an element) (DF.I and DF.IV). Śleszyński compares this notion to the concept of inclusion of one class in another class, and distinguishes three different meanings of the word “is” (jest):

- (1) It expresses the identity of two objects.
- (2) It expresses the inclusion of a single object into a class.
- (3) It expresses the inclusion of one class into another class.

He writes:

“Since the author (Leśniewski) distinguishes the concept of class, so “is” has the meaning as described in point (1). It must therefore combine two determined objects. But x is not a determined object, so the sentence (number 2 in DF. IV) is incomprehensible from the point of view of the author. It expresses the inclusion of a single object ($P1$) into a class.

Furthermore, if such x exists, then by the 1st and the 2nd condition it follows that: an object P is a class of objects $P1$. Hence, what about this x ? What’s the trick? It seems that I have understood it after reading the paragraph [given below] from the work of the author published in the Review of Philosophy (Przegląd Filozoficzny) in 1914, vol. 3, p. 64.

<Any half P of a ball Q is subordinated [= is an element] of a class of quarters of the ball Q , since if the term “ a ” is used in a sense of being “half of a ball”, then:

- (1) A class of quarters of the ball Q is a class of objects “ a ”,
- (2) A half of the ball Q is “ a ”>.

Let’s put here instead of the mysterious “ a ” (previously x), what it really means. Then, the condition (2) [in DF. IV] is eliminated since it is obvious, while the condition (1) would be as follows:

A class of quarters of the ball Q is a class of halves of the ball Q , and the author does not hesitate to declare it explicitly in the article in the Review of Philosophy (p. 64): <a class of halves of the ball Q is also a class of quarters of the ball Q >. The same expression I cannot find in this work, hence the comprehension becomes very difficult”.

In fact, the Definition III underlines that a class (a collection) can be viewed in different ways (not all objects have to be **m** if a class is defined as a collection of objects **m**): the ball can be seen as a class of its halves and as a class of its quarters. Moreover, the Definition II is redundant in comparison with Definition III.

Also subjected to the criticism is the term “objects **m**”. If there was only one item **m**, then **m** means the name of the object – by Śleszyński. Conversely, if the object is not an individual, but there are many **m**'s, what are these **m**'s? (p. 28). Addressing this issue, Śleszyński states that: “**m** is the common name for the elements of a class”(p. 28) and could not even suspect that Leśniewski uses this term unconsciously.

Next, Śleszyński investigates the correctness of Leśniewski's proofs applying the language of *Principia Mathematica* (p. 48) but he does not make any essential changes. He notes the use of three terms: ingrediens, element and sub-multitude which are synonyms. It seems that Śleszyński did not fully understand the concept of a class applied by Leśniewski, which totally differs from the classical concept. Hence he criticizes Leśniewski, but he does not reject his ideas (p. 55). Additionally, analyzing Leśniewski's proofs of theorems and performing them again in a classical way, he greatly appreciates Leśniewski's accuracy and precision (p. 73).

At the end of the manuscript, we also find the analysis of Leśniewski's paper entitled: *Is the class of classes which are not subordinated to themselves, subordinated to itself?*, where a formulation of the antinomy of irreflexive classes is given. Śleszyński criticizes Leśniewski's use of the term “subordinated” (p. 80) and does not understand why he uses it. At this point we find the example of a class of halves of a ball and a class of quarters of a ball given by Śleszyński, who concludes that the term “class” used by Leśniewski does not denote “a collection of objects, in which there is no internal relationship between objects [intuitively that's how we interpret the collections in the sense of G. Cantor]” (p. 84).

On the page 94 of Śleszyński's comments we find a beautiful summary of the whole theory of Leśniewski written in the language of PM. The first part consists of two axioms (for the relation of being a part and its property: irreflexivity and transitivity), followed by 19 claims. The second part consists of the definition of ingrediens and 5 claims. The third part contains the definitions of multitude and class together with 2 axioms and 4 theorems (p. 95). The fourth part (the most important contribution of Śleszyński) contains the definition of an element and 17 claims (p. 96), in which it is shown that the term ‘element’ is synonymous with the term ‘ingrediens’; that every object is its own element, and that the ratio epsilon (Peano's “is”) is transitive, and “part of the multitude of objects **m** may not be any of these objects” (p. 99) – which is an absolute novelty in relation to the classical set theory. This shows the impossibility of construction of the antinomy of Russell within mereology.

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