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TABLEAUX APPROACH FOR CONTACT LOGICS INTERPRETED OVER INTERVALS

A b s t r a c t. Contact logics are modal logic that is developed for reasoning about region-based theories of space. We develope a tableaux approach for contact logics interpreted over intervals (CLIOI) on the reals. For obtaining sound and complete tableaux-based decision procedures, the main technical tool is the semantic tableaux approach. We use intensively the following concepts: tableaux methods, termination of tableaux methods, saturated tableaux, termination theorem, soundness theorem, truth lemma, and completeness theorem.

1. Introduction

In classical point-based theory, a point is defined as a primitive notation; it models an exact location in space and has no length, width, or thickness. Region-Based Theories of Space (RBTS), also known as pointless theory, arose as an alternative theory to classic point-based theory. RBTS is a powerful tool for modeling situations about image analysis or computer vision. Some of the well-known RBTS systems were purposed by Galton. His model consists of a nonempty set of regions and binary relation of adjacency between regions. If region a and b in W_G are in contact, a and b are called adjacent. The chessboard

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desk is a notable example of an adjacency space. Black and white cells are considered are regions, and two cells are adjacent relation, if and only if these two cells have an intersection point. The adjacency relation is symmetric and reflexive in this example.

After Galton's system, Balbiani et al. [3] and Vakarelov [12] present Kripke-type semantics for contact logics. In their setting, regions are defined as arbitrary subsets of points in a Kripke frame. The operations are join and complement, the relations are "contact"(C) and "part of"($\leq$). The contact relation denoted by aCb, and is defined as "a point x in a and a point y in b are such that $R(x, y)$". The ordinary inclusion relation is defined as the part-of relation denoted $a \leq b$. The language of contact logics and modal logic have a relation, CL is a linguistic restriction of the language of modal. Contact formulas aCb and $a \leq b$ correspond to modal formulas $\langle U\rangle(a \wedge \Diamond b)$ and $[U](a- > b)$ where $< U >$ and $[U]$ are the "somewhere" and "everywhere" modal connectives. As can be seen in [3, 11], the language of contact logics can be considered as a first-order language without quantifiers. As shown in [6, 7]. For instance, Balbiani and Kikot [1] defined Sahlqvist formulas concerning regions and obtained the Sahlqvist Correspondence Theorem in contact logics [1]. Some elementary properties like transitivity are not definable in contact logics. In modal logic, however, some properties that are definable in CL are not definable. Fitting introduced the tableaux approach for firstly classical logic [8]. After that, the tableaux approach was applied to intuitionistic logic and modal logic. The procedure combines proof theory and the semantic method. The semantic tableaux approach is a powerful tool for satisfiability problems in logic. If a formula is satisfiable, the tableaux approach will construct a model of the formula. The paper is structured as follows. In Section 2, we introduce the syntax and semantics of CLIOI. In Section 3, we give basic definitions and tableaux rules. In Section 4, we define interpretability conditions, and then prove termination and soundness theorems. The last section is devoted to saturation definition, truth lemma, and completeness for the tableaux system.

2. Syntax and semantics

The language of contact logics consists consist of a countably infinite set of Boolean variable BV and Boolean operations $(0, \cup, -)$. Members of BV are denoted by $p, q, etc.$ The set of Boolean terms BV_t are inductively defined as follows:

- $a ::= p \mid 0 \mid -a \mid (a \cup b)$.

The Boolean constructs 1 and $\cap$ are defined in a standard way. The abbreviations of Boolean constructs are: 1 for -0 and $a \cap b$ for $-(-a \cup -b)$. The set of formulas (BV_f) is inductively defined as follows:

- $\theta ::= a \equiv b \mid \perp \mid \neg\theta \mid (\theta \vee \chi)$.

For all $a, b \in \mathbb{R}$, if $a \leq b$ then let $[a, b] = \{c \in \mathbb{R} : \ a \leq c \leq b\}$ be the closed intervals determined by a and b. The closed interval $[a, b]$ is said to be regular when $a < b$. For all $a, b \in \mathbb{R}$, the sets $[a, +\infty) = \{c \in \mathbb{R} : \ a \leq c\}$ and $(-\infty, b] = \{c \in \mathbb{R} : \ c \leq b\}$ will also be called regular closed intervals. For all $S \subseteq \mathbb{R}$, let $Cl(S)$ and $Int(S)$ respectively denote the closure and the interior of S concerning the ordinary topology on $\mathbb{R}$. A subset S of $\mathbb{R}$ is said to be *regular closed* if $CL(Int(S)) = S$. An interpretation is a function associating a finite union $f(p)$ of regular closed intervals to each propositional variable p. Given an interpretation f, a function $\bar{f}$ is defined associating a finite union $\bar{f}(a)$ of regular closed intervals to each term a as follows:

- $\bar{f}(p) = f(p)$,
- $\bar{f}(0) = \emptyset$,
- $\bar{f}(-a) = Cl(\mathbb{R} \setminus \bar{f}(a))$,
- $\bar{f}(a \cup b) = \bar{f}(a) \cup \bar{f}(b)$.

Let us remark that for all terms a, $\bar{f}(a)$ is a finite union of regular closed intervals. The relation of satisfaction between interpretation and formulas is defined as follows:

- $f \models a \equiv b$ iff $\bar{f}(a) = \bar{f}(b)$,
- $f \not\models \bot$
- $f \models \neg\theta$ iff $f \not\models \theta$,
- $f \models \theta \vee \chi$ iff $f \models \theta$ or $f \models \chi$.

Formula θ is valid iff for all interpretations f, $f \models \theta$.

3. Tableau rules

In Section 3, we give the tableaux rules for contact logics interpreted over intervals. In semantics, satisfiability is defined between interpretations and formulas. Furthermore, the language of CL has two categories of expressions which are Boolean terms and formulas. As a result, tableaux nodes will be labeled by formulas or Boolean terms ($x \in a$, and $x \notin a$, where a is a Boolean term, and x is a variable). If a tableaux consists of one node labeled with θ, then it is called the root. The tableaux rules are given in formula rules (next page) and Boolean rules (Page 3). Rules are applied by extending the branches of constructed trees. Tableaux rules for terms are given in Figure 3.2. Tableaux rules for formulas are given in Figure 3.1.

Disjunction Rule

$$\frac{\theta \vee \chi}{\theta \mid \chi}$$

Conjunction Rule

$$\frac{\neg(\theta \vee \chi)}{\begin{array}{c}\neg\theta \\ \neg\chi\end{array}}$$

Negation Rule

$$\frac{\neg\neg\theta}{\theta}$$

T Equivalent Rule

$$\frac{a \equiv b}{\begin{array}{c|c} x \in a & x \notin a \\ x \in b & x \notin b \end{array}}$$

($x = x_0$ or x is old in the branch)

F Equivalent Rule

$$\frac{a \not\equiv b}{\begin{array}{c|c} x \in a & x \notin a \\ x \notin b & x \in b \end{array}}$$

(x is new in the branch)

Figure 3.1 Formula Rules

T Negation Rule

$$\frac{x \in -a}{x \notin a}$$

F Negation Rule

$$\frac{x \notin -a}{x \in a}$$

T Union Rule

$$\frac{x \in a \cup b}{x \in a \mid x \in b}$$

F Union Rule

$$\frac{x \notin a \cup b}{\begin{array}{c}x \notin a \\ x \notin b\end{array}}$$

Figure 3.2 Term Rules

A branch is closed if and only if one of the following holds:

i there is a node labeled $\perp$ in the branch;

ii there is a node labeled $x \in 0$ in the branch;

iii the branch contains two nodes respectively labeled $x \in a$, $x \notin a$.

where x is a variable and a is a Boolean term. If all branches in the tableaux are closed, a tableaux is called a closed tableaux. Let us apply the table rules to the contact formula $(a \cup b) \not\equiv -1 \wedge (a \cup b) \equiv -b$.

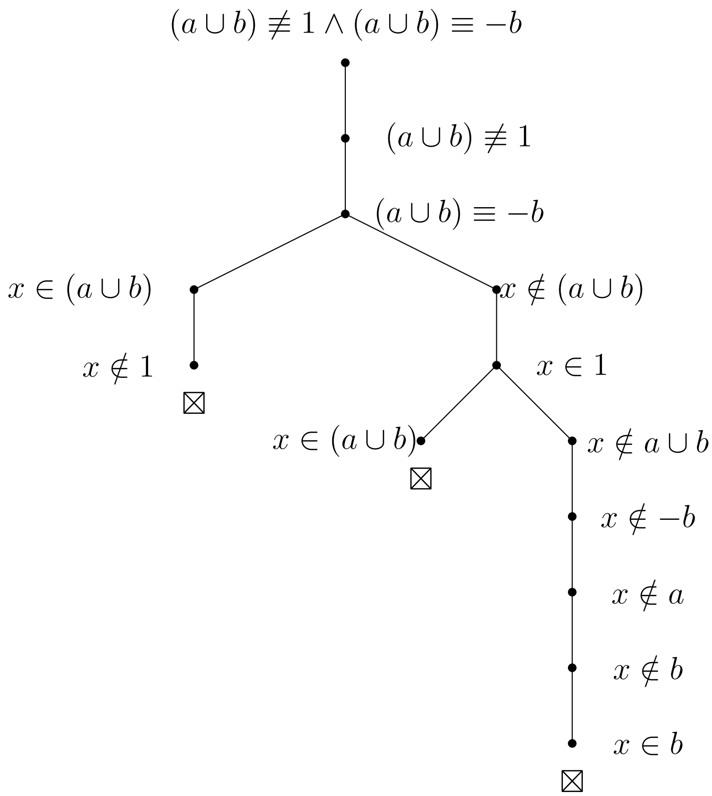

Figure 3.3 Closed tableaux

We will say a variable x occurs in a branch β iff there exists term a such that the expression $x : a$ is the label of some node in β. Given a formula θ, its initial tableaux is the labeled tree consisting of exactly one node (called root) labeled with θ. By abuse of notation, we will always consider, from the beginning of the computation starting with the initial tableaux of a given formula, that there is some occurring variable denoted x_0. Let $x_0, x_1, \ldots, x_n$ be a list of the variables occurring in β.

4. Soundness

In Theorem 4.5, we give the soundness of the tableaux system. Before the soundness theorem, we define the interpretability of a branch.

Let f be an interpretation. We say β is interpretable in f, if there exists $\bar{x}_0, \bar{x}_1, ..., \bar{x}_n$ in $\mathbb{R}$ such that the following conditions holds:

- for all nodes labeled "θ" that occur in β, $f \models \theta$,
- for all nodes labeled "$x_i \in a$" that occur in β, $\bar{x}_i \in \bar{f}(a)$,
- for all nodes labeled "$x_i \notin a$" that occur in β, $\bar{x}_i \notin \bar{f}(a)$.

We say that the variable x_i is regularly interpretable, if for all propositional variables p, $\bar{x}_i \in Int(f(p))$ or $\bar{x}_i \notin f(p)$.

Lemma 4.1. *Let x_i be a regularly interpreted variable, then for all terms a, $\bar{x}_i \in Int\bar{f}(a)$ or $\bar{x}_i \notin \bar{f}(a)$.*

Proof. Lemma will be proved by using mathematical induction on Boolean terms, and only the case of $a = -b$ is considered. It is easy to show the other cases. Suppose $\bar{x}_i \in Int\bar{f}(b)$ or $\bar{x}_i \notin \bar{f}(b)$, we demonstrate that $\bar{x}_i \in Int\bar{f}(-b)$ or $\bar{x}_i \notin \bar{f}(-b)$. By contradiction, suppose $\bar{x}_i \notin Int\bar{f}(-b)$ and $\bar{x}_i \in \bar{f}(-b)$. Since $\bar{x}_i \notin Int\bar{f}(-b)$, therefore $\bar{x}_i \in \mathbb{R} \setminus Int\bar{f}(-b)$. Hence, $\bar{x}_i \in Cl(\mathbb{R} \setminus \bar{f}(-b))$. Thus, $\bar{x}_i \in Cl(\mathbb{R} \setminus Cl(\mathbb{R} \setminus \bar{f}(b)))$. Consequently, $\bar{x}_i \in Cl(Int\bar{f}(b))$. Since $\bar{f}(b)$ is a finite union of regular closed intervals, therefore $Cl(Int\bar{f}(b)) = \bar{f}(b)$. As $\bar{x}_i \in Cl(Int\bar{f}(b))$, therefore $\bar{x}_i \in \bar{f}(b)$. Since $\bar{x}_i \in Int\bar{f}(b)$ or $\bar{x}_i \notin \bar{f}(b)$, therefore $\bar{x}_i \in Int\bar{f}(b)$. Since $\bar{x}_i \in \bar{f}(-b)$, therefore $\bar{x}_i \in Cl(\mathbb{R}\setminus\bar{f}(b))$. Hence, $\bar{x}_i \notin Int\bar{f}(b)$. It is a contradiction.

□

The finite sequence of reals $(\bar{x}_0, \bar{x}_1, ..., \bar{x}_n)$ is called interpretation of β in f. It is regular for the variable x_i, if x_i is regularly interpreted in it.

Now we can define interpretable tableaux. A tableaux is called interpretable in f, if one of the tableaux branches is interpretable in f. “From now on, when we write p^0 and p^1, we respectively mean $-p$ and$p$.

Lemma 4.2. *Let θ be a formula. If θ is satisfiable then the initial tableaux for θ is interpretable.*

Proof. Suppose θ is satisfiable. Let f be an interpretation such that $f \models \theta$. Let $p_1, ..., p_k$ be the propositional variables occuring in θ. Let $\epsilon_1, .., \epsilon_k \in \{0, 1\}$ be such that $\bar{f}(p_1^{\epsilon_1} \cap ... \cap p_k^{\epsilon_k})$ is a nonempty finite union of regular closed intervals. Let $\bar{x}_0$ be an element in the interior of this finite union. Obviously, $(\bar{x}_0)$ is a regular interpretation for the variable x_0 . □

Lemma 4.3. *A closed branch cannot be interpreted by a regular interpretation of its variables.*

Proof. Let β be a branch and f be an interpretation. Suppose β is interpretable in f and $x_0, x_1, ..., x_n$ occurs in β. There exists $\bar{x}_0, \bar{x}_1, ..., \bar{x}_n \in \mathbb{R}$ which satisfy compatibility conditions. Since β is closed, β has node labeled with $\perp$, "$x \in 0$" or "$x \in a$" and "$x \notin a$".

Case 1, β contains $\perp$. Therefore $f \models \perp$, it is a contradiction.

Case 2, β contains "$x \in 0$", since β interpretable in f, $\bar{x}_i \in \bar{f}(0)$ which means $\bar{x}_i \in \emptyset$. It is a contradiction.

Case 3, β contains "$x_i \in a$" and "$x_i \notin a$". Since β interpretable in M, $\bar{x}_i \in \bar{f}(a)$ and $\bar{x}_i \notin \bar{f}(a)$. It is a contradiction. □

Lemma 4.4. *If a tableaux t is regularly interpretable then all tableaux obtained from t by applying one of the tableaux rules are regularly interpretable.*

Proof. Let $x_0, ..., x_n$ be a list of the variables occurring in a branch β of t that is regularly interpretable in an interpretation of f. Let $\bar{x}_0, ..., \bar{x}_n$ in $\mathbb{R}$ be the associated elements. Suppose t' is an extension of t after applying one of the tableaux rules. We want to show that t' is regularly interpretable in f.

We firstly consider the case of the $\not\equiv$ rule. Suppose $a \not\equiv b$ occurs in t. We want to show that t' is regularly interpretable. This rule extend the current branch β in two new branches β' and β'' by adding (for β'), "$y' \in a$" and "$y' \notin b$" and adding (for β") "$y' \notin a$" and "$y' \in b$", y' being a new variable. We have to show that either branch β' can be regularly interpreted or branch β'' can be regularly interpreted. Since the branch β is regularly interpreted, it means that $f \models a \not\equiv b$. This implies that $\bar{f}(a) \neq \bar{f}(b)$. Since $\bar{f}(a)$ and $\bar{f}(b)$ are finite unions of closed intervals, this implies that there is a real number r which either belongs to $Int\bar{f}(a)$ and does not belong to $Int\bar{f}(b)$, or belongs to $\bar{f}(b)$ but does not belong to $\bar{f}(a)$. The former case corresponds to the branch β' whereas the latter case corresponds to the branch β''. In the branch β', we should extend the interpretation f by the interpretation f' which is like f on W and which is such that $f'(y') = r$. In the branch β'', it is the same: we should extend the interpretation f by the interpretation f'' which is like f on W and which is such that $f''(y') = r$. Then, clearly, in the former case, f' is a regular interpretation of the branch β' whereas in the latter case, f'' is a regular interpretation of the branch β''. We secondly consider the case of the $\equiv$ rule. Suppose $a \equiv b$ occurs in t. We want to show that t' is regularly interpretable. This rule extend the current branch β in two new branches β' and β'' by adding (for β'), $x' \in a$" and "$x' \in b$" and adding (for β) "$x' \notin a$" and "$x' \notin b$", x' being an old variable. We have to show that either branch β' can be regularly interpreted or branch β'' can be regularly interpreted. Since the branch β is regularly interpreted, it means that $f \models a \equiv b$. This implies that $\bar{f}(a) = \bar{f}(b)$. Since $\bar{f}(a)$ and $\bar{f}(b)$ are finite unions of closed intervals, this implies that there is a real number r which belongs to $Int\bar{f}(a)$ and $\bar{f}(b)$, or not belongs to $Int\bar{f}(b)$ and does not belong to $\bar{f}(a)$. The former case corresponds to the branch β' whereas the latter case corresponds to the branch β''. In the branch β', we should extend the interpretation f by the interpretation f' which is like f on W and which is such that $f'(x') = r$. In the branch β'', it is the same: we should extend the interpretation f by the interpretation f'' which is like f on W and which is such that $f''(x') = r$. Then, clearly, in the former case, f' is a regular interpretation of the branch β' whereas in the latter case, f'' is a regular interpretation of the branch β''.

□

Theorem 4.5. *If formula θ is satisfiable, then all tableaux computed from the initial tableaux for θ are regularly interpretable and, consequently, they are open.*

Proof. By Lemmas 4.3 and 4.4, since θ is satisfiable, then θ is regularly interpretable in some interpretation f. All tableaux obtained from θ are regularly interpretable and obviously, they are open. □

5. Completeness

In Theorem 5.4, the completeness theorem for the tableaux method is proved. In this respect, the concept of saturation is essential. A branch β in some tableaux is called saturated if the following conditions hold for all nodes $n \in \beta$:

- if n is labelled $\neg\neg\theta$, then β contains a node labelled θ,
- if n is labelled $\theta \vee \chi$, then β contains a node labelled θ or χ,
- if n is labelled $\neg(\theta \vee \chi)$, then β contains nodes labelled $\neg\theta$ and $\neg\chi$,
- if n is labelled $a \equiv b$ and β contains the variables x, then β contains nodes labelled $x \in a$ and $x \in b$, or β contains nodes labelled $x \notin a$ and $x \notin b$,
- if n is labelled $a \not\equiv b$, then for some variable x either β contains nodes label led$x \in a$ and $x \notin b$, or β contains nodes labelled $x \notin a$ and $x \in b$,
- if n is labelled $x \in -a$, then β contains a node labelled $x \notin a$,
- if n is labelled $x \notin -a$, then β contains a node labelled $x \in a$,
- if n is labelled $x \in a \cup b$, then β contains a node labelled $x \in a$ or $x \in b$,
- if n is labelled $x \notin a \cup b$, then β contains nodes labelled $x \notin a$ and $x \notin b$.

A tableaux is said to be saturated if all its branches are saturated.

Let β be a saturated open branch. Let $x_0, x_1, ..., x_n$ be a list of the variables occurring in β. Let $I_0, I_1, ..., I_n$ be the following finite unions of regular intervals (where $0 < \epsilon < \frac{1}{2}$):

· $I_0 = (-\infty, 1-\epsilon] \cup [1+\epsilon, 2-\epsilon] \cup ... \cup [n-1+\epsilon, n-\epsilon] \cup [n+\epsilon, +\infty)$,

· $I_1 = [1-\epsilon, 1+\epsilon]$,

. . .

· $I_n = [n-\epsilon, n+\epsilon]$.

Let $f : p \in BV \rightarrow f(p) \subseteq \mathbb{R}$ be the function defined as follows:

$$f(p) = \cup\{I_k : 0 \leq k \leq n \text{ and } "x_k \in p" \text{ occurs in } \beta\}.$$

Remark 5.1. For all p, $f(p)$ is a finite union of regular closed intervals.

Remark 5.2. Let $\bar{x}_k = k$ for each $k \in \mathbb{N}$ such that $k \leq n$. For all $k \in \mathbb{N}$, $k \leq n$, and for all $p \in BV$, $\bar{x}_k \in Int(f(p))$ or $\bar{x}_k \notin f(p)$.

Lemma 5.3 is important in the proof of the completeness theorem.

Lemma 5.3. *Let t be an open saturated tableaux, β be an open branch in t and f be an interpretation for β defined as above. Let p be a Boolean variable and $x_0, x_1, ..., x_n$ be variables which occur in β. For all $k \in \mathbb{N}$, $0 \leq k \leq n$, the following conditions are hold:*
(i) If $x_k \in a$ occurs in β, then $I_k \subseteq \bar{f}(a)$.
(ii) If $x_k \notin a$ occurs in β, then $Int(I_k) \cap \bar{f}(a) = \emptyset$.

Proof. The proof is done by induction on the term a. The case "p" follows by definition of f(p).

Let us consider $x_k \in -a$ occurs in β. We want to show that $I_k \subseteq \bar{f}(-a)$. Since $x_k \in -a$ occurs in β,then $x_k \notin a$ occurs in β. By induction hypothesis $Int(I_k) \cap \bar{f}(a) = \emptyset$. Therefore, $Int(I_k) \subseteq (\mathbb{R} \setminus \bar{f}(a)) \subseteq Cl(\mathbb{R} \setminus \bar{f}(a))$. By the definition of $\bar{f}$, $Int(I_k) \subseteq \bar{f}(-a)$. Consequently, $I_k \subseteq \bar{f}(-a)$.

Let us consider $x_k \in a \cup b$. We want to show that $I_k \subseteq \bar{f}(a \cup b)$. Since β is saturated, $x_k \in a$ occurs in β or $x_k \in b$ occurs in β. By induction hypothesis $I_k \subseteq \bar{f}(a)$ or $I_k \subseteq \bar{f}(b)$. Therefore, $I_k \subseteq \bar{f}(a) \cup \bar{f}(b)$. By the definition of $\bar{f}$, $I_k \subseteq \bar{f}(a \cup b)$.

Let us consider "$x_k \notin -a$" occurs in β. We want to show that $Int(I_k) \cap \bar{f}(-a) = \emptyset$. Obviously, "$x_k \in a$" occurs in β. By induction hypothesis, $I_k \subseteq \bar{f}(a)$. Therefore $Int(I_k) \cap Cl(\mathbb{R} \setminus \bar{f}(a)) = \emptyset$. So, $Int(I_k) \cap \bar{f}(-a) = \emptyset$.

Let us consider $x_k \notin a \cup b$. We want to satisfy that $Int(I_k) \cap \bar{f}(a \cup b) = \emptyset$. Since β is saturated, $x_k \notin a$ occurs in β and $x_k \notin b$ occurs in β. By the induction hypothesis, $Int(I_k) \cap \bar{f}(a) = \emptyset$ and $Int(I_k) \cap \bar{f}(b) = \emptyset$. By the definition of $\bar{f}$, $Int(I_k) \cap \bar{f}(a \cup b) = \emptyset$.

This completes the induction. □

Theorem 5.4. *Let θ be a formula. Starting the computations with the initial tableaux of θ, it is possible, after finitely many applications of the tableaux rules, to obtain a saturated tableaux.*

Proof. Let t_0 be the initial tableaux for θ. Notice that for all nodes labelled with a label of the form $a \not\equiv b$, it is not needed to apply the $\not\equiv$-rule more than once in order to obtain a saturated tableaux. Moreover, notice that for all labels of the form $a \not\equiv b$ occurring during the computations, $a \equiv b$ is a subformula of θ. Finally, notice that the symbols $x_1, x_2, \ldots$ occurring during the computations are only created by an application of the $\not\equiv$-rule. For all these reasons, the number of symbols $x_1, x_2, \ldots$ occurring during the computations will be at most equal to the number of subformulas of θ of the form $a \equiv b$. Hence, it is bounded. As a result, if immediately follows that starting the computations with t_0, it is possible, after finitely many applications of the tableaux rules, to obtain a saturated tableaux. □

Theorem 5.5. *If* $\models \theta$*, then there is a closed tableaux computed from* $\neg\theta$*.*

Proof. By contraposition. Suppose t is a saturated and open tableaux computed from $\neg\theta$ and β is an open branch in it. Since t is saturated, therefore β is saturated too. Let f be the interpretation for θ determined by β. By Lemma 5.3, we have $f \not\models \theta$, it contradicts the validity of θ.

□

A careful analysis of the tableau rules immediately leads us to the conclusion that the depth of a tableau computed from a given formula ϕ is linear in the number of symbols in ϕ. Since tableaux are finitely branching, for this reason, together with Theorems 4.5, 4.6 and 5.4 allow us to conclude that the problem of determining if a given formula is satisfiable in the class of all models is NP-complete. This complexity result has already been discussed in [4] where it was obtained by means of a more complicated argument based on the filtration method.

6. Conclusion

In this work, a general tableaux method for contact logic interpreted over intervals is described. Using the tableaux approach that we developed in the paper, it is easy to show that the satisfiability problem for the formulas of the language of CLIOI is NP-complete. Much remains to be done. We can extend the language with predicates such as $a < b$, $convex(a)$, $meets(a, b)$, etc. For instance, $a < b$ is true in a model if all real numbers in a's interpretation precede all real numbers in b's interpretation, $convex(a)$ is true in a model if a's interpretation consists of a regular closed interval, $meets(a, b)$ is true in a model if the intersection of a's interpretation with b's interpretation is a singleton.

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