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Algorithmic Computation of the Conley Index for Multivalued Maps with No Continuous Selector in Cubical Spaces.

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Abstract. In this note we present theory which facilitates the use of Conley index algorithms for cubical multivalued maps constructed from maximal dimensional cubes in the setting when cubes of arbitrary dimension are permitted.

Keywords: discrete multivalued dynamical system, cubical spaces, multivalued maps without continuous selector, Conley index, commutativity property.

1. Introduction

Multivalued maps reconstructed from sampled data often do not have a continuous selector. As a consequence, the algorithms constructing multivalued maps presented in [8] do not guarantee the acyclicity of values. This is because the union of a family of cubes may not be acyclic unless the cubes share a common point.

The modification of the algorithm which consists in replacing the union by, for example, its convex hull or other desirable acyclic set requires the use of multivalued map defined separately on cubes of each dimension. Unfortunately, the available algorithms computing maps induced in homology are based on some reductions which apply only to multivalued maps defined on cubes of maximal dimension.

In this paper we show how the problem of algorithmic computation of the Conley index with no continuous selector may be reduced to the analogous problem but with continuous selector. This, in particular, facilitates the use of the available algorithms and software. The proposed approach consists in mapping every cube of the original space into a cube of maximal dimension. This results in a new multivalued map which is admissible in the available software [10, CHomP]. What needs to be proved and what is proved in this paper is the fact that the outcome of the algorithm for the new map applies also to the original map. In consequence, we are able to use the existing software to compute index maps and Conley index of multivalued maps without a continuous selector. The theory needed to justify this is based on the commutativity property of Conley index for cubical multivalued maps.

The paper is organised as follows. Section 2. presents basic notions on cubical sets and maps. In Section 3. we define the mappings needed to link the original and modified multivalued map. We also introduce the notion of the extended boundary and study its properties. Finally, Section 4. provides the description of the dynamics of both systems and the proof that the Conley indices of these systems are related by the commutativity property.

2. Preliminaries

In this note we use the following notions. By a *cuboid* we mean a subset $B \subset \mathbb{R}^n$ of the form

$$J_1 \times \ldots \times J_n, \tag{1}$$

where each J_i satisfies $J_i = [a_i, b_i]$ or $J_i = \{a_i\}$ for some $a_i, b_i \in \mathbb{R}$, $a_i < b_i$. Such set J_i is called an *interval.* We say that an interval $J \subset \mathbb{R}$ is *elementary* if $J = [l, l+1]$ or $J = \{l\}$ for some $l \in \mathbb{Z}$. By an *elementary cube* or briefly a *cube* we mean a cuboid Q where each J_i is elementary.

The *dimension* of cuboid B is the number of intervals of positive length in (1). It is denoted by $\dim B$. In particular, if Q is a cube, then $\dim Q$ is equal to the number of intervals of length 1 in (1).

A set $X \subset \mathbb{R}^n$ is said to be *cubical*, if it can be written as a finite union of cubes. Note that every cubical set is compact. Let

$$\mathcal{K}^n := \{J_1 \times \ldots \times J_n \subset \mathbb{R}^n \mid J_i = [l, l+1] \text{ or } J_i = \{l\} \text{ for some } l \in \mathbb{Z}\}$$

denote the set of all cubes in $\mathbb{R}^n$ and

$$\mathcal{K} := \bigcup_{k=1}^{\infty} \mathcal{K}^k$$

be the set of all cubes. For a cubical set X we set

$$\mathcal{K}_d(X) := \{Q \in \mathcal{K} \mid Q \subset X,\ \dim Q = d\}$$

and

$$\mathcal{K}(X) := \{Q \in \mathcal{K} \mid Q \subset X\}.$$

In particular, if $X \subset \mathbb{R}^n$ and $n \in \mathbb{N}$ is fixed, then we write $\mathcal{K}_{\max}(X) := \mathcal{K}_n(X)$. The set $X \subset \mathbb{R}^n$ is a *full cubical set* if $X = \bigcup \mathcal{K}_{\max}(X)$.

Let X and Y be topological spaces. A map of the form $F\colon X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the power set of Y, is called a *multivalued map*. Such a multivalued map is denoted by $F\colon X \multimap Y$. The multivalued map F is said to be *upper semicontinuous* (*u.s.c.*), if for any closed set $B \subset Y$ the set

$$F^{-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$$

is closed in X, and it is said to be *lower semicontinuous* (*l.s.c*), if for any open set $U \subset Y$ the set $F^{-1}(U)$ is open in X. The set $F^{-1}(B)$ itself is called the *large counter image* of B. Let $A \subset X$. By the *image* of A under F we mean the set $F(A) := \bigcup\{F(x) \mid x \in A\}$.

With each cube $Q = J_1 \times \ldots \times J_k \in \mathcal{K}$ we associate a *cell*, that is the set of the form

$$\mathring{Q} := \mathring{J}_1 \times \ldots \times \mathring{J}_k,$$

where

$$\mathring{J}_i := \begin{cases} (l, l+1) & \text{if } J_i = [l, l+1], \\ \{l\} & \text{if } J_i = \{l\}. \end{cases}$$

Assume X and Y are cubical sets. A multivalued map $F\colon X \multimap Y$ is called *cubical*, if for every $x \in X$ the set $F(x)$ is cubical and for every $Q \in \mathcal{K}(X)$ the multivalued map $F|_{\mathring{Q}}$ is constant. More details on cubical sets and maps can be found in [5].

3. Proper Cubes and Extended Boundary

In this section we define the maps which allow to connect two cubical spaces and the multivalued maps defined on these cubical spaces.

Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\theta(x) := \begin{cases} x - \frac{k+1}{2} & \text{for } x \in [k, k+1] \text{ and } k \in 2\mathbb{Z}+1, \\ \frac{k}{2} & \text{for } x \in [k, k+1] \text{ and } k \in 2\mathbb{Z}. \end{cases} \tag{2}$$

The following proposition is straightforward.

Proposition 1. *Map θ is continuous and*

$$\theta([k, k+1]) = \begin{cases} [\frac{k-1}{2}, \frac{k+1}{2}] & \text{if } k \in 2\mathbb{Z}+1, \\ \{\frac{k}{2}\} & \text{if } k \in 2\mathbb{Z}. \quad \square \end{cases}$$

Fix $n \in \mathbb{N}$ and consider function $g : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g(x_1, \ldots, x_n) := (\theta(x_1), \ldots, \theta(x_n)). \tag{3}$$

It follows from Proposition 1 that g is continuous and maps every cube $Q \in \mathcal{K}^n$ onto cube $g(Q) \in \mathcal{K}^n$.

The following proposition shows the form of preimage of a singleton under g.

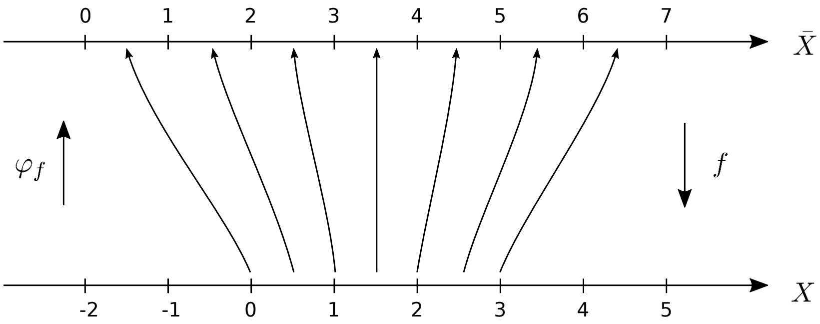

Figure 1. Maps $\varphi_f : X \multimap \bar{X}$ and $f : \bar{X} \to X$, where $X, \bar{X} \subset \mathbb{R}$.

Proposition 2. *For every $x \in \mathbb{R}^n$ the set $g^{-1}(x)$ is a cuboid.*

Proof. Note that for $y \in \mathbb{R}$ we have

$$\theta^{-1}(y) = \begin{cases} \{y+k+1\} & \text{for } y \in (k, k+1),\ k \in \mathbb{Z}, \\ [2y, 2y+1] & \text{for } y \in \mathbb{Z}. \end{cases}$$

Therefore, $g^{-1}(x) = J_1 \times \ldots \times J_n$, where each J_i is an interval. The conclusion follows. □

In the sequel, given a full cubical set $X \subset \mathbb{R}^n$, we consider it as a topological space with topology induced from $\mathbb{R}^n$ and we write $\bar{X} := g^{-1}(X)$. Then $f : \bar{X} \to X$ defined by $f(x) := g(x)$ for $x \in \bar{X}$ is a continuous surjection. Note also that f is closed, because every continuous map from compact space to Hausdorff space is closed.

Let $\varphi_f : X \multimap \bar{X}$ be the multivalued map defined by $\varphi_f(x) := f^{-1}(x)$ for $x \in X$ (cf. [4, Example 13.6]). One can easily verify that the image of a cubical subset of X under φ_f is a cubical subset of $\bar{X}$ (cf. Proposition 2). Therefore, X and $\bar{X}$ are compact ANRs.

The intuition behind the multivalued map φ_f is to extend all cubes from the space X to maximal dimensional cubes in the space $\bar{X}$. On the other hand, the map f enables us to relate cube in $\bar{X}$ to the corresponding cubes in X (cf. Figure 1).

Recall that a topological space is *contractible* if it has the homotopy type of a one-point space. A subset of $\mathbb{R}^n$ is *acyclic* if it has (co)homology of the one-point space. Clearly, contractibility implies acyclicity. Let A, B be compact ANRs. Recall that a compact set $C \neq \emptyset$ in A is said to be *cell-like* in A if C is contractible in every neighbourhood of C in A. A map $h : A \to B$ is called a *cell-like map* if h is a closed, continuous surjection and for each $y \in B$ the preimage $h^{-1}(y)$ is cell-like in A (cf. [11]). We will utilise these notions in the following lemma.

Lemma 3. *Let $F : X \multimap X$ be an upper semicontinuous cubical multivalued map with contractible values and let $\bar{X} = g^{-1}(X)$. Then*

$$\bar{F} := \varphi_f \circ F \circ f : \bar{X} \multimap \bar{X} \tag{4}$$

is an upper semicontinuous cubical multivalued map with contractible values.

Proof. Since f is closed, by [4, Proposition 14.6], the map φ_f is u.s.c. By [4, Proposition 14.10], the composition $\varphi_f \circ F \circ f$ is an u.s.c. multivalued map. One can easily verify that the image of a cubical subset of X under φ_f is a cubical subset of $\bar{X}$. Hence, by the definition of g, the sets $F(f(\bar{x}))$ and $\bar{F}(\bar{x})$ are cubical for any $\bar{x} \in \bar{X}$. Moreover, for each $u, v \in \mathring{Q}$ with $Q \in \mathcal{K}(\bar{X})$ we have $\bar{F}(u) = \bar{F}(v)$, because F is cubical. Hence $\bar{F}$ is cubical.

We still need to prove that $\bar{F}$ has contractible values. Since f is a single-valued map, it suffices to prove that $\varphi_f \circ F$ has contractible values. Since for each $x \in X$ sets $B := F(x)$ and $A := (\varphi_f \circ F)(x)$ are cubical, they are both compact ANRs. Note that the map $f|_A \colon A \to B$ is a cell-like map. Indeed, for every $y \in B$, the preimage $f|_A^{-1}(y)$ is a cuboid, that is it is nonempty compact set contractible in every its neighbourhood in A. By [6, Theorem of Section 4.2.], map $f|_A$ is homotopy equivalence. Thus, $(\varphi_f \circ F)(x)$ is contractible for each $x \in X$. □

Note that the surjectivity of f implies that $f \circ \varphi_f = \mathrm{id}_X$. For further reference we state the simple consequence of this as the following proposition.

Proposition 4. *For any $x \in \bar{X}$ the equality $f(\bar{F}(x)) = F(f(x))$ holds.*

Proof. Obviously, $f(\bar{F}(x)) = f(\varphi_f(F(f(x)))) = F(f(x))$. □

Now, let us consider a special class of multivalued mappings which allows computations of Conley index by the means of [8]. Let $X \subset \mathbb{R}^n$ be full cubical and $H \colon X \multimap X$ be u.s.c., cubical multivalued map with acyclic values such that for each $P \in \mathcal{K}(X)$

$$H(\mathring{P}) \supset \bigcup \{H(\mathring{Q}) \mid P \subset Q,\ Q \in \mathcal{K}_{\max}(X)\}.$$

A multivalued map H is called *admissible* if there exists a multivalued map $G \colon X \multimap X$ such that following conditions are satisfied:

(a) $G(\mathring{P}) = \bigcap \{H(\mathring{Q}) \mid P \subset Q,\ Q \in \mathcal{K}_{\max}(X)\}$ for each $P \in \mathcal{K}(X)$,

(b) values of G are acyclic.

In other words, for admissible map H there exists a lower semicontinuous multivalued map with acyclic values which is, in fact, its multivalued selector. For this multivalued selector there exists continuous selector. Thus, map H has a continuous selector. For details see [5, Section 6.2].

It turns out that the multivalued map $\bar{F} \colon \bar{X} \multimap \bar{X}$ given by (4) is admissible. The proof of the following proposition is postponed until the necessary definitions will be introduced.

Proposition 5. *$\bar{F} \colon \bar{X} \multimap \bar{X}$ is admissible.*

Let $n \in \mathbb{N}$ be fixed. We often use the set $I_n := \{1, \ldots, n\}$. We call the cube $P \in \mathcal{K}^n$ *proper* if $\dim f(P) = \dim P = n$. Note that for $C := \{P \in \mathcal{K}^n \mid \dim f(P) = \dim P = n\}$, the function f restricted to $\bigcup_{P \in C} \operatorname{int} P$ is an injection.

Now we show a few properties of proper cubes.

Lemma 6. *Assume that $P \in \mathcal{K}_{\max}(\bar{X})$ is a proper cube. Then*

$$f(\bar{X}) \setminus f(\operatorname{int} P) = f(\bar{X} \setminus \operatorname{int} P).$$

Proof. Since

$$f(\bar{X}) = f(\bar{X} \setminus \operatorname{int} P \cup \operatorname{int} P) = f(\bar{X} \setminus \operatorname{int} P) \cup f(\operatorname{int} P),$$

we have the inclusion $f(\bar{X}) \setminus f(\operatorname{int} P) \subset f(\bar{X} \setminus \operatorname{int} P)$.

In order to prove the inclusion in the opposite direction notice that $f(\bar{X} \setminus \operatorname{int} P) \subset f(\bar{X})$. Take a $y \in f(\bar{X} \setminus \operatorname{int} P)$ and suppose that $y \notin f(\bar{X}) \setminus f(\operatorname{int} P)$. This means that there are an $x' \in \bar{X} \setminus \operatorname{int} P$ and an $x \in \operatorname{int} P$ such that $f(x') = f(x) = y$. Since $\bar{X}$ is full cubical and $P \in \mathcal{K}_{\max}(\bar{X})$, one can easily check that

$$\bar{X} \setminus \operatorname{int} P = \bigcup \mathcal{K}_{\max}(\bar{X} \setminus \operatorname{int} P).$$

Hence, $x' \in Q$ for some $Q \in \mathcal{K}_{\max}(\bar{X} \setminus \operatorname{int} P)$.

Observe that for any $u, u' \in \mathbb{R}$ such that $u \neq u'$

$$\theta(u) = \theta(u') \Rightarrow \exists p \in \mathbb{Z} \text{ such that } u, u' \in [2p, 2p+1]. \tag{5}$$

Since $x \neq x'$, there is a $j \in I_n$ such that $x_j \neq x'_j$. Since $f(x) = f(x')$, we have $\theta(x_j) = \theta(x'_j)$. Therefore, we get from (5) that there exists a $p \in \mathbb{Z}$ such that $x_j \in [2p, 2p+1]$. But, $x \in \operatorname{int} P$ and P is proper. This means that

$$\operatorname{int} P = (2k_1 - 1, 2k_1) \times \ldots \times (2k_n - 1, 2k_n)$$

and $x_j \in (2k_j - 1, 2k_j)$, which contradicts $x_j \in [2p, 2p+1]$. □

Lemma 7. *Assume that $P, Q \in \mathcal{K}_{\max}(\bar{X})$ are cubes such that P is a proper cube and $P \neq Q$. If $P \cap Q \neq \emptyset$ then $f(Q) \subset \operatorname{bd} f(P)$.*

Proof. Let $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then $P \cap Q \neq \emptyset$ implies that $Q = [a_1 + u_1, b_1 + u_1] \times \ldots \times [a_n + u_n, b_n + u_n]$ for some $u_i \in \{-1, 0, 1\}$ and $i \in I_n$. There exists at least one $i \in I_n$ such that $u_i \neq 0$, because $P \neq Q$. Fix such an i. Since P is proper, each a_i is odd. In particular

$$f(P) = [\frac{a_1 - 1}{2}, \frac{a_1 + 1}{2}] \times \ldots \times [\frac{a_n - 1}{2}, \frac{a_n + 1}{2}]. \tag{6}$$

Therefore, $u_i \neq 0$ implies $a_i + u_i \in 2\mathbb{Z}$ and $f_i(x_i) = \frac{a_i + u_i}{2}$ for each $x = (x_1, \ldots, x_n) \in Q$. Since $f_i(x_i) \in \operatorname{bd}([\frac{a_i - 1}{2}, \frac{a_i + 1}{2}])$, we get from (6) that $f(Q) \subset \operatorname{bd} f(P)$. □

Let $Q \in \mathcal{K}^n$ and $\dim Q = n$. By $I'(Q) \subset I_n$ we mean the set of all the indices for which the components of f restricted to Q are constant. Let $I''(Q) := I_n \setminus I'(Q)$. Note that for each $i \in I''(Q)$ the ith component of f restricted to Q is injective. Moreover, Q is proper if and only if $I'(Q) = \emptyset$. One can also easily verify that $\dim f(Q) = k$ if and only if $\operatorname{card} I''(Q) = k$.

Now we get back to the proof of Proposition 5.

Proof. Let $G \colon \bar{X} \multimap \bar{X}$ be a multivalued map defined as in point (a) of definition of admissibility. Such a map exists and it is lower semicontinuous cubical map (cf. [5, Section 6.2]).

We need to prove that values of G are acyclic. Let $x \in \bar{X}$ and $I_s(x) := \{i \in I_n \mid x_i \in \mathbb{Z}\}$. Since for $I_s(x) = \emptyset$ we have $G(x) = \bar{F}(\mathring{Q})$ for some $Q \in \mathcal{K}_{\max}(\bar{X})$ and the conclusion is clear, consider the case $I_s(x) \neq \emptyset$. Let $G(x) = \bar{F}(\mathring{Q}_1) \cap \ldots \cap \bar{F}(\mathring{Q}_k)$. Each $Q_i = J_1^i \times \ldots \times J_n^i \in \mathcal{K}_{\max}(\bar{X})$ and $J_j^i = [a_j^i, b_j^i]$, where either $a_j^i = x_j$ or $b_j^i = x_j$ for every $j \in I_s(x)$. Together with the assumption that X is full cubical this means that $k = 2^{\operatorname{card} I_s(x)}$ and all cubes Q_i differ only at intervals indexed by $I_s(x)$.

Note that if for some $j \in I_s(x)$ the index $j \in I'(Q_i)$ for some Q_i, then for $Q_l = J_1^i \times \ldots \times J_j^l \times \ldots \times J_n^i$ we have $j \in I''(Q_l) = I_n \setminus I'(Q_l)$. In that way one can find Q_m where $m \in \{1, \ldots, k\}$, such that for each $j \in I_s(x)$ the restriction $f_j|_{J_j^m}$ is not constant. It is easy to see that for every $i = 1, \ldots, k$ we have $I'(Q_i) \setminus I_s(x) = I'(Q_m) \setminus I_s(x)$. Hence, $I'(Q_m) \subset I'(Q_i)$ for each $i = 1, \ldots, k$. Therefore, $f(Q_m) \supset f(Q_i)$ for $i = 1, \ldots, k$. Since F is u.s.c, we have $F(f(\mathring{Q}_m)) \subset F(f(\mathring{Q}_i))$. One can also easily check that for any $Q \in \mathcal{K}_{\max}(\bar{X})$ the equality $f(\mathring{Q}) = f(\mathring{Q})$ holds. Combining above facts we get $\varphi_f(F(f(\mathring{Q}_m))) \subset \varphi_f(F(f(\mathring{Q}_i)))$, that is $\bar{F}(\mathring{Q}_m) \subset \bar{F}(\mathring{Q}_i)$ for each $i = 1, \ldots, k$.

Since $G(x) = \bar{F}(\mathring{Q}_1) \cap \ldots \cap \bar{F}(\mathring{Q}_k) = \bar{F}(\mathring{Q}_m)$ for some Q_m and $\bar{F}$ is cubical, the contractibility of $\bar{F}$ implies acyclicity of $G(x)$. □

Lemma 8. *Assume that the cube $Q \in \mathcal{K}^n$, $\dim Q = n$, and Q is not proper, that is $\dim f(Q) = k$ for some $0 \leq k < n$, and the cube $P \in \mathcal{K}^n$ is proper and $P \cap Q \neq \emptyset$. Then $\dim P \cap Q = k$ and $\dim f(P \cap Q) = k$.*

Proof. The cube P is proper, hence for each $i \in I_n$ the component f_i of the function f restricted to P is injective. Thus, f restricted to $P \cap Q$ is one-to-one.

Let $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and $Q = [a_1 + u_1, b_1 + u_1] \times \ldots \times [a_n + u_n, b_n + u_n]$, where $u_i \in \{-1, 1\}$ for $i \in I'(Q)$. We have $u_i = 0$ for $i \in I''(Q)$. It is easy to see that $P \cap Q = [c_1, d_1] \times \ldots \times [c_n, d_n]$, where $d_i - c_i = 0$ for $i \in I'(Q)$ and $d_i - c_i = 1$ for $i \in I''(Q)$. It follows that $\dim P \cap Q = k$ and $\dim f(P \cap Q) = k$. □

Lemma 9. *Let $\bar{A} \subset \bar{X}$ be a full cubical set. For each $y \in \operatorname{bd} f(\bar{A})$*

(a) $f^{-1}(y)$ is a cuboid and $\dim f^{-1}(y) > 0$,

(b) there exists a unique cube $Q \in \mathcal{K}_{\max}(\bar{X})$ such that Q is not proper and $f^{-1}(y) \subset Q$. Moreover, $Q \cap P \neq \emptyset$ for a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$,

(c) $\dim f^{-1}(y) = n - \dim f(Q)$ provided $f^{-1}(y) \subset Q$ for a cube $Q \in \mathcal{K}_{\max}(\bar{X})$ which is not proper.

Proof. Let $y = (y_1, \ldots, y_n) \in \operatorname{bd} f(\bar{A})$. In order to prove (a) notice that since the set $f(\bar{A})$ is cubical, there exists an $i \in I_n$ such that $y_i \in \mathbb{Z}$. This means that there is a nonempty subset $I'_y \subset I_n$ of all these indices such that $f_i^{-1}(y_i) = [a_i, b_i]$ with $a_i \neq b_i$. By the definition of f (cf. Proposition 2), we have $f^{-1}(y) = [a_1, b_1] \times \ldots \times [a_n, b_n]$ with

$$b_i - a_i = \begin{cases} 1, & \text{if } i \in I'_y \\ 0, & \text{otherwise.} \end{cases}$$

Since $I'_y \neq \emptyset$, we get $\dim f^{-1}(y) > 0$.

Now we prove (b). Since X is full cubical and $\bar{X} = g^{-1}(X)$, also $\bar{X}$ is full cubical. Therefore, for any $y \in \operatorname{bd} f(\bar{A})$ there exists a $Q \in \mathcal{K}_{\max}(\bar{X})$ such that $f^{-1}(y) \subset Q$. Moreover, Q is not proper because $I'_y \neq \emptyset$. This proves existence.

To prove uniqueness suppose to the contrary that there are cubes $Q, Q' \in \mathcal{K}_{\max}(\bar{X})$ which are not proper, $Q \neq Q'$ and $f^{-1}(y) \subset Q \cap Q'$. We claim that $\dim f(Q) =$

$\dim f(Q')$. Indeed, if $\dim f(Q) = k_1$, $\dim f(Q') = k_2$ and $k_1 \neq k_2$ then assume that $k_1 < k_2$ and take $i_0 \in I'(Q') \setminus I'(Q)$ such that $b_{i_0} - a_{i_0} = 1$, where $f^{-1}(y) = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Since $f^{-1}(y) \subset Q$ and, in consequence, $y \in f(Q)$, i_0th component of f restricted to Q is constant. Thus, $i_0 \in I'(Q)$, which contradicts the choice of i_0. Since there are no two cubes with nonempty intersection such that the restriction of f to these cubes is constant on coordinates in I'_y, there exists an $i \in I'_y$ such that either $f|_Q$ or $f|_{Q'}$ on the ith coordinate is injective. This means $\operatorname{card} f(f^{-1}(y)) > 1$, a contradiction.

We still need to prove the existence of a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ such that $P \cap Q \neq \emptyset$. For this end take an $R \in \mathcal{K}_{\max}(\operatorname{cl}(f(\bar{X}) \setminus f(\bar{A})))$ such that $y \in R$. By definition of f, there exists a proper cube $P \in \mathcal{K}_{\max}(\bar{X})$ such that $f(P) = R$. To show that $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ suppose to the contrary that $P \subset \bar{A}$. Since $R \subset \operatorname{cl}(f(\bar{X}) \setminus f(\bar{A}))$ and $R \subset f(\bar{A})$, we have $R \subset \operatorname{bd} f(\bar{A})$ and $\emptyset \neq \operatorname{int} R \subset \operatorname{int} \operatorname{bd} f(\bar{A}) = \emptyset$, a contradiction.

To show (c), let $f^{-1}(y) = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset Q = J_1 \times \ldots \times J_n$ for some Q which is not proper and $b_i - a_i = 1$ for $i \in I'_y$ and $b_i - a_i = 0$ for $i \in I_n \setminus I'_y$. We have $\dim f^{-1}(y) = \operatorname{card} I'_y$. Since $f^{-1}(y) \subset Q$, we have $I'_y \subset I'(Q)$. Now we prove that $I'(Q) \subset I'_y$. Let $i \in I'(Q)$. This means that $f_i|_{J_i}$ is constant and for every $z_i \in J_i$ there exist $c_i, d_i \in \mathbb{R}$ such that $d_i - c_i = 1$ and $f_i^{-1}(z_i) = [c_i, d_i]$. Thus, $i \in I'_y$ and $I'(Q) = I'_y$. Moreover,

$$\begin{aligned} \dim f(Q) &= \operatorname{card} I''(Q) = \operatorname{card} I_n \setminus I'(Q) = \operatorname{card} I_n \setminus I'_y \\ &= \operatorname{card} I_n - \operatorname{card} I'_y = n - \dim f^{-1}(y), \end{aligned}$$

which completes the proof of (c). □

Let $\bar{A} \subset \bar{X}$ be a full cubical set. By the *extended boundary of* $\bar{A}$ we mean the set

$$\begin{aligned} \operatorname{Bd} \bar{A} := \operatorname{bd} \bar{A} \cup \bigcup \{ & Q \in \mathcal{K}_{\max}(\bar{A}) \mid \dim f(Q) < n, \\ & \exists P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A})), \dim f(P) = n, P \cap Q \neq \emptyset \}. \end{aligned} \tag{7}$$

The main idea behind extended boundary of a set $\bar{A}$ in addition to being a boundary is to gather all not proper cubes from $\bar{A}$ such that their intersection with $\operatorname{bd} \bar{A}$ is nonempty and their image under f is contained in $\operatorname{bd} f(\bar{A})$. In general, this is not true that the value of any element in $\operatorname{Bd} \bar{A}$ is in $\operatorname{bd} f(\bar{A})$.

Now, we show a few properties of the extended boundary.

Lemma 10. *Let $\bar{A} \subset \bar{X}$ be a full cubical set. Then $f^{-1}(\operatorname{bd} f(\bar{A})) \cap \bar{A} \subset \operatorname{Bd} \bar{A}$.*

Proof. Let $x \in f^{-1}(\operatorname{bd} f(\bar{A})) \cap \bar{A}$. Set $y := f(x)$ and $B := f^{-1}(y)$. Then $y \in \operatorname{bd} f(\bar{A})$. Since by Lemma 9 (a) $\dim f^{-1}(y) > 0$, we have $\operatorname{card} B > 1$. Let $I'_y \subset I_n$ be the set of all indices such that $f_i^{-1}(y_i) = [a_i, b_i]$ with $a_i \neq b_i$. We have $I'_y \neq \emptyset$. Assume $\operatorname{card} I'_y = n - k$. By Proposition 2, B is a cuboid and, by Lemma 9, $n - k > 0$. By the same lemma there exists a unique $Q \in \mathcal{K}_{\max}(\bar{X})$ such that $B \subset Q$, Q is not proper, $f|_Q$ is constant on coordinates in I'_y and injective on coordinates in $I_n \setminus I'_y$ and there exists a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ such that $Q \cap P \neq \emptyset$. This means that $\dim f(Q) = n - (n - k) = k$.

In order to prove that $x \in \operatorname{Bd} \bar{A}$ we will consider cases $Q \not\subset \bar{A}$ and $Q \subset \bar{A}$. Consider the case $Q \not\subset \bar{A}$. Then $Q \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ and for any $P \in \mathcal{K}_{\max}(\bar{A})$ we have $P \cap Q \subset Q \subset \operatorname{cl}(\bar{X} \setminus \bar{A})$ and $P \cap Q \subset P \subset \bar{A}$. Thus $P \cap Q \subset \operatorname{bd} \bar{A}$ and

$$Q \cap \bar{A} = Q \cap \bigcup_{P \in \mathcal{K}_{\max}(\bar{A})} P \subset \operatorname{bd} \bar{A} \subset \operatorname{Bd} \bar{A},$$

that is $B \cap \bar{A} \subset \operatorname{Bd} \bar{A}$. It follows that $x \in \operatorname{Bd} \bar{A}$. Consider the case $Q \subset \bar{A}$. We have proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ such that $Q \cap P \neq \emptyset$. This means that $Q \subset \operatorname{Bd} \bar{A}$. Thus $B \subset \operatorname{Bd} \bar{A}$. It follows that $x \in \operatorname{Bd} \bar{A}$. □

Proposition 11. *Let $\bar{A} \subset \bar{X}$ be a full cubical set. If $x \in \bar{A}$ and $f(x) \in \operatorname{bd} f(\bar{A})$ then there exists an $x' \in \operatorname{bd} \bar{A}$ such that $f(x) = f(x')$.*

Proof. Let $x \in \bar{A} \setminus \operatorname{bd} \bar{A}$ and $f(x) \in \operatorname{bd} f(\bar{A})$. Put $B := f^{-1}(f(x))$. Clearly, $x \in B \subset f^{-1}(\operatorname{bd} f(\bar{A}))$. By Lemma 9 there exists a $Q \in \mathcal{K}_{\max}(\bar{X})$ which is not proper, satisfies $\dim f(Q) = k$ for some $k < n$ and such that $B \subset Q$. Since $\bar{A}$ is full cubical and $x \in \operatorname{int} \bar{A}$, we have $Q \subset \bar{A}$.

The set $\operatorname{bd} f(\bar{A})$ is cubical and by definition of f^{-1} set $f^{-1}(\operatorname{bd} f(\bar{A}))$ is full cubical. Since $B \subset f^{-1}(\operatorname{bd} f(\bar{A}))$, we have $Q \subset f^{-1}(\operatorname{bd} f(\bar{A}))$. By Lemma 10 we get $Q \subset \operatorname{Bd} \bar{A}$. Moreover, we can find a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{A}))$ such that $Q \cap P \neq \emptyset$. It follows that $P \cap Q \subset \operatorname{bd} \bar{A}$.

Now assume that $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and $Q = [a_1 + u_1, b_1 + u_1] \times \ldots \times [a_n + u_n, b_n + u_n]$, where $u_i \in \{-1, 0, 1\}$. Clearly, $u_i \in \{-1, 1\}$ for $i \in I'(Q)$ and $u_i = 0$ for $i \in I''(Q)$. Then $P \cap Q = J_1 \times \ldots \times J_n$, where

$$J_i = \begin{cases} \{a_i\} & \text{if } u_i = -1, \\ \{b_i\} & \text{if } u_i = 1, \\ [a_i, b_i] & \text{if } u_i = 0. \end{cases}$$

Since by Proposition 2 B is a cuboid, we have $B = J_1' \times \ldots \times J_n'$, where

$$J_i' = \begin{cases} [a_i + u_i, b_i + u_i] & \text{if } u_i \in \{-1, 1\}, \\ \{e_i\} & \text{if } u_i = 0 \text{ for some } e_i \in (a_i, b_i). \end{cases}$$

Since $a_i = b_i - 1$, the intersection $B \cap P \cap Q \neq \emptyset$ and $B \cap P \cap Q \subset \operatorname{bd} \bar{A}$. Hence, taking an $x' \in B \cap P \cap Q$ we obtain $f(x') = f(x)$ and $x' \in \operatorname{bd} \bar{A}$. □

Lemma 12. *Let $\bar{N} \subset \bar{X}$ be a full cubical set. For all $Q \in \mathcal{K}_{\max}(\bar{N})$ such that $Q \subset \operatorname{Bd} \bar{N}$ the inclusion $f(Q) \subset \operatorname{bd} f(\bar{N})$ holds.*

Proof. For any $Q \subset \operatorname{Bd} \bar{N}$ there exists a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{N}))$ such that $P \cap Q \neq \emptyset$. By Lemma 7, the inclusion $f(Q) \subset \operatorname{bd} f(P)$ holds. Clearly, $Q \subset \operatorname{Bd} \bar{N} \subset \bar{N}$ and $f(Q) \subset f(\bar{N})$. Suppose that $f(Q) \cap \operatorname{int} f(\bar{N}) \neq \emptyset$. We have $f(Q) \cap \operatorname{int} f(\bar{N}) \subset \operatorname{bd} f(P) \cap \operatorname{int} f(\bar{N})$. Since P is a proper cube, we also have $f(\operatorname{int} P) = \operatorname{int} f(P)$ and $\operatorname{cl} \operatorname{int} f(P) = f(P)$. By Lemma 6,

$$\emptyset = (f(\bar{N}) \setminus f(\operatorname{int} P)) \cap f(\operatorname{int} P) = f(\bar{N} \setminus \operatorname{int} P) \cap f(\operatorname{int} P).$$

Hence, $f(\bar{N} \setminus \operatorname{int} P) = f(\bar{N})$ and

$$\operatorname{int} f(\bar{N}) \cap f(\operatorname{int} P) \subset f(\bar{N} \setminus \operatorname{int} P) \cap f(\operatorname{int} P) = \emptyset.$$

Therefore, $f(\operatorname{int} P) \cap \operatorname{int} f(\bar{N}) = \operatorname{int} f(P) \cap \operatorname{int} f(\bar{N}) = \emptyset$. By [7, Corollary 1.1.2], the intersection $f(P) \cap \operatorname{int} f(\bar{N}) = \emptyset$ and $\operatorname{bd} f(P) \cap \operatorname{int} f(\bar{N}) \subset f(P) \cap \operatorname{int} f(\bar{N}) = \emptyset$, a contradiction. Hence, $f(Q) \cap \operatorname{int} f(\bar{N}) = \emptyset$. □

4. Dynamics of Maps F and $\bar{F}$

In this section we give the description of invariant sets and isolating neighbourhoods of dynamical systems generated by maps $\bar{F}$ and F. Let X and $\bar{X}$ be spaces as in the previous section. An upper semicontinuous mapping $F\colon X \times \mathbb{Z} \multimap X$ with compact values is called a *discrete multivalued dynamical system* if

(i) for all $x \in X$, $F(x, 0) = \{x\}$,

(ii) for all $n, m \in \mathbb{Z}$ with $nm \geq 0$ and all $x \in X$

$$F(F(x, n), m) = F(x, n + m),$$

(iii) for all $x, y \in X$, $y \in F(x, -1) \iff x \in F(y, 1)$.

Given a discrete multivalued dynamical system $F\colon X \times \mathbb{Z} \multimap X$ and $n \in \mathbb{Z}$ set $F^n\colon X \ni x \to F(x, n) \subset X$. Since for every $n \in \mathbb{Z}$ the map F^n coincides with the composition of n copies of either $F^1\colon X \multimap X$ or $(F^1)^{-1}\colon X \multimap X$, it is justified to say that the multivalued map $F := F^1\colon X \multimap X$ *generates* the discrete multivalued dynamical system $F\colon X \times \mathbb{Z} \multimap X$. This allows us to consider multivalued maps as discrete multivalued dynamical systems.

Now we recall basic notions of multivalued dynamics. Let $F\colon X \multimap X$ be a multivalued map generating a discrete multivalued dynamical system. A mapping $\sigma\colon \mathbb{Z} \to X$ is called a *solution for F through $x \in X$* if $\sigma(0) = x$ and $\sigma(n + 1) \in F(\sigma(n))$ for all $n, n + 1 \in \mathbb{Z}$. Given $N \subset X$, by an *invariant part of N with respect to F* we mean the set

$$\operatorname{Inv}(N, F) := \{x \in N \mid \exists \sigma\colon \mathbb{Z} \to N \text{ a solution for } F \text{ through } x\}.$$

A compact set $N \subset X$ is called an *isolating neighbourhood* if $\operatorname{Inv}(N, F) \subset \operatorname{int} N$. Details concerning these notions may be found in [3].

The following proposition states the relation between invariant sets of the multivalued maps F and $\bar{F}$ given by (4).

Proposition 13. *Assume that $\bar{N} \subset \bar{X}$ is a full cubical set. Then $f(\operatorname{Inv}(\bar{N}, \bar{F})) = \operatorname{Inv}(f(\bar{N}), F)$.*

Proof. By Proposition 4, for any $x \in \bar{X}$ the equality $f(\bar{F}(x)) = F(f(x))$ holds. Let $\bar{\sigma} : \mathbb{Z} \to \bar{N}$ be a solution for $\bar{F}$ through $x \in \operatorname{Inv}(\bar{N}, \bar{F})$. Consider $\sigma(n) := f(\bar{\sigma}(n))$. For every $n \in \mathbb{Z}$ we have

$$f(\bar{\sigma}(n + 1)) \in f(\bar{F}(\bar{\sigma}(n))) = F(f(\bar{\sigma}(n))) = F(\sigma(n)).$$

Thus, $\sigma\colon \mathbb{Z} \to f(\bar{N})$ is a solution for F through $f(x)$ in $f(\bar{N})$.

To prove the opposite inclusion, take a $y \in \operatorname{Inv}(f(\bar{N}), F)$. Let $\tau\colon \mathbb{Z} \to f(\bar{N})$ be a solution for F through y. Since f is surjective, for every $n \in \mathbb{Z}$ we have $\tau(n) = f(u_n)$ for a $u_n \in \bar{N}$. In particular, $y = f(u_0)$. Hence,

$$u_{n+1} \in \varphi_f(\tau(n+1)) \subset \varphi_f(F(\tau(n))) = \bar{F}(u_n).$$

Therefore, $\bar{\tau}\colon \mathbb{Z} \to \bar{N}$ defined by $\bar{\tau}(n) := u_n$ is a solution for $\bar{F}$ through u_0 in $\bar{N}$. It follows that $u_0 \in \operatorname{Inv}(\bar{N}, \bar{F})$ and $y = f(u_0) \in f(\operatorname{Inv}(\bar{N}, \bar{F}))$. □

Proposition 14. *Let $\bar{N} \subset \bar{X}$ be a full cubical set and let $y \in \operatorname{Inv}(f(\bar{N}), F)$. Then $\varphi_f(y) \subset \operatorname{Inv}(\bar{N}, \bar{F})$.*

Proof. Let $y = f(x) \in \operatorname{Inv}(f(\bar{N}), F)$ for some $x \in \bar{N}$. This means that there is a solution $\sigma\colon \mathbb{Z} \to f(\bar{N})$ for F through y in $f(\bar{N})$. Thus, for each $n \in \mathbb{Z}$ we have $\sigma(n) = f(x_n)$ for some $x_n \in \bar{N}$. Clearly,

$$x_{n+1} \in \varphi_f(\sigma(n+1)) \subset \varphi_f(F(\sigma(n))) = \varphi_f(F(f(x_n))) = \bar{F}(x_n).$$

Therefore, $\bar{\sigma} : \mathbb{Z} \to \bar{N}$ such that $\bar{\sigma}(n) := x_n$ is a solution for $\bar{F}$ through x in $\bar{N}$.

Now let $x' \in \varphi_f(y)$. Define $\bar{\sigma}'\colon \mathbb{Z} \to \bar{N}$ by

$$\bar{\sigma}'(n) := \begin{cases} \bar{\sigma}(n), & \text{if } n \neq 0 \\ x', & \text{otherwise.} \end{cases}$$

Clearly, $\bar{\sigma}'(1) \in \bar{F}(\bar{\sigma}'(0))$, because

$$\bar{\sigma}'(1) = \bar{\sigma}(1) \in \bar{F}(\bar{\sigma}(0)) = \varphi_f(F(f(x))) = \varphi_f(F(f(x'))) = \bar{F}(\bar{\sigma}'(0)),$$

and $\bar{\sigma}'(0) \in \bar{F}(\bar{\sigma}'(-1))$, because by Proposition 4

$$f(\bar{\sigma}'(0)) = y = f(\bar{\sigma}(0)) \in f(\bar{F}(\bar{\sigma}(-1))) = F(f(\bar{\sigma}(-1))) = F(f(\bar{\sigma}'(-1)))$$

and

$$\bar{\sigma}'(0) \in \varphi_f(f(\bar{\sigma}'(0))) \subset \varphi_f(F(f(\bar{\sigma}'(-1)))) = \bar{F}(\bar{\sigma}'(-1)).$$

Hence, $\bar{\sigma}'$ is a solution for $\bar{F}$ through x' in $\bar{N}$. This shows that $x' \in \operatorname{Inv}(\bar{N}, \bar{F})$ and proves that $\varphi_f(y) \subset \operatorname{Inv}(\bar{N}, \bar{F})$. □

Proposition 15. *Let $\bar{N} \subset \bar{X}$ be a full cubical set. If $\bar{N}$ is an isolating neighbourhood for $\bar{F}$, then $f(\bar{N})$ is an isolating neighbourhood for F.*

Proof. Suppose by contrary that $f(\bar{N})$ is not an isolating neighbourhood for F. Then there exists a $y \in \operatorname{Inv}(f(\bar{N}), F) \cap \operatorname{bd} f(\bar{N})$. By Proposition 13 there exists an $x \in \operatorname{Inv}(\bar{N}, \bar{F})$ such that $y = f(x)$. By Proposition 11 there exists an $x' \in \operatorname{bd} \bar{N}$ such that $f(x) = f(x') = y$. We get from Proposition 14 that $x' \in \operatorname{Inv}(\bar{N}, \bar{F})$. Since $x' \in \operatorname{bd} \bar{N}$, this contradicts the assumption that $\bar{N}$ is an isolating neighbourhood for $\bar{F}$. □

Proposition 16. *Let $\bar{N} \subset \bar{X}$ be a full cubical set. If $\bar{N}$ is an isolating neighbourhood for $\bar{F}$, then $\operatorname{Inv}(\bar{N}, \bar{F}) \subset \bar{N} \setminus \operatorname{Bd} \bar{N}$.*

Proof. Suppose that the conclusion is not true. Then, there exists an $x \in \operatorname{Inv}(\bar{N}, \bar{F}) \cap \operatorname{Bd} \bar{N}$. We cannot have $x \in \operatorname{bd} \bar{N}$, because $\bar{N}$ is an isolating neighbourhood for $\bar{F}$. Hence, $x \in \operatorname{Bd} \bar{N} \setminus \operatorname{bd} \bar{N}$. This means that there exists a $Q \in \mathcal{K}_{\max}(\bar{N})$ with $x \in Q$ and such that there is a proper cube $P \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{N}))$ satisfying $P \cap Q \neq \emptyset$. By Propositions 13 and 15 we have $f(x) \in \operatorname{Inv}(f(\bar{N}), F) \subset \operatorname{int} f(\bar{N})$, and by Lemma 12 obviously $f(x) \in \operatorname{bd} f(\bar{N})$, a contradiction. □

In order to investigate Conley index we need one more notion. A pair $P = (P_1, P_2)$ of compact sets such that $P_2 \subset P_1 \subset N$ is called a *weak index pair in isolating neighbourhood* N if

(a) $F(P_i) \cap N \subset P_i$ for $i = 1, 2$,

(b) $\operatorname{bd}_F(P_1) := P_1 \cap \operatorname{cl}(F(P_1) \setminus P_1) \subset P_2$,

(c) $\operatorname{Inv}(N, F) \subset \operatorname{int}(P_1 \setminus P_2)$,

(d) $P_1 \setminus P_2 \subset \operatorname{int} N$.

Theorem 17. *Let $\bar{N} \subset \bar{X}$ be a full cubical set. Assume that $\bar{N}$ is an isolating neighbourhood for $\bar{F}$ and $P = (P_1, P_2)$ is a weak index pair for $\bar{F}$ in $\bar{N}$ consisting of full cubical sets. Then $f(P) = (f(P_1), f(P_2))$ is a weak index pair for F in $f(\bar{N})$.*

Proof. For the sake of clarity let $Q := (Q_1, Q_2)$ denote the pair $f(P) = (f(P_1), f(P_2))$. To prove property (a) of weak index pair, take a $y \in F(Q_i) \cap f(\bar{N})$ for $i = 1, 2$. Since $y \in f(\bar{N})$, there is an $x \in \bar{N}$ such that $y = f(x)$. Clearly, $y \in F(Q_i) = F(f(P_i))$ and $x \in \varphi_f(y) \subset \varphi_f(F(f(P_i))) = \bar{F}(P_i)$. By property (a) of weak index pair P for $\bar{F}$, we have $x \in P_i$ and $y = f(x) \in f(P_i) = Q_i$.

Now we prove property (d). Suppose by the contrary that there exists a $y \in Q_1$ such that $y \notin Q_2$ and $y \in \operatorname{bd} f(\bar{N})$. Take an $x \in P_1$ such that $y = f(x)$. We have $x \notin P_2$ and since P is a weak index pair we obtain $x \in P_1 \setminus P_2 \subset \operatorname{int} \bar{N}$. Since $x \in \varphi_f(y)$ and $x \in \operatorname{int} \bar{N}$, we get from Lemma 10 that $x \in \operatorname{Bd} \bar{N} \cap P_1$. Thus, there is an $S \in \mathcal{K}_{\max}(\bar{N})$ such that $x \in S \subset \operatorname{Bd} \bar{N} \cap P_1$ and $S \not\subset P_2$ because P_1 and P_2 are full cubical. It follows that there exists a proper cube $R \in \mathcal{K}_{\max}(\operatorname{cl}(\bar{X} \setminus \bar{N}))$ satisfying $R \cap S \neq \emptyset$. Therefore, $R \cap S \subset \operatorname{bd} \bar{N}$ and $R \cap S \subset S \subset P_1 \setminus P_2$ which contradicts $P_1 \setminus P_2 \subset \operatorname{int} \bar{N}$.

In order to prove property (b) assume by contrary that there exists a $y \in Q_1 \cap \operatorname{cl}(F(Q_1) \setminus Q_1) \setminus Q_2$. Then $y \in Q_1 \setminus Q_2$ and $y \in \operatorname{cl}(F(Q_1) \setminus Q_1)$. Take the sequence $\{y_n\}_{n \in \mathbb{N}} \subset F(Q_1) \setminus Q_1$ such that $y_n \to y$. By the just proved property (d) we have $y \in \operatorname{int} f(\bar{N})$. Thus, for sufficiently large $n \in \mathbb{N}$ we have $y_n \in F(Q_1) \cap \operatorname{int} f(\bar{N}) \subset F(Q_1) \cap f(\bar{N}) \subset Q_1$, a contradiction.

To show property (c) we will first prove that $\operatorname{Inv}(f(\bar{N}), F) \cap Q_2 = \emptyset$. Assume the contrary. Then there exists a $y \in \operatorname{Inv}(f(\bar{N}), F) \cap Q_2$. Since $Q_2 = f(P_2)$ we may take an $x \in P_2$ such that $y = f(x)$. Since $y \in \operatorname{Inv}(f(\bar{N}), F)$, by Proposition 14, we have $x \in \operatorname{Inv}(\bar{N}, \bar{F})$. This proves that $P_2 \cap \operatorname{Inv}(\bar{N}, \bar{F}) \neq \emptyset$ which contradicts the assumption that P is a weak index pair for $\bar{F}$ in $\bar{N}$ and proves that $\operatorname{Inv}(f(\bar{N}), F) \cap Q_2 = \emptyset$.

Next, we will prove that $\operatorname{Inv}(f(\bar{N}), F) \subset \operatorname{int} Q_1$. Assume the contrary. Since $\operatorname{Inv}(f(\bar{N}), F) = f(\operatorname{Inv}(\bar{N}, \bar{F})) \subset f(\operatorname{int}(P_1 \setminus P_2)) \subset f(P_1) = Q_1$, there is a $y \in \operatorname{Inv}(f(\bar{N}), F) \cap \operatorname{bd} Q_1$. Since $y \in \operatorname{bd} Q_1 \subset Q_1$, we can take an $x \in P_1$ such that $y = f(x)$. Since $y \in \operatorname{Inv}(f(\bar{N}), F)$, by Proposition 14, we have $x \in \operatorname{Inv}(\bar{N}, \bar{F})$. It follows that $x \in \varphi_f(y) \subset \varphi_f(\operatorname{bd} f(P_1))$ and $x \in P_1$. Hence, by Lemma 10, we get

$x \in \operatorname{Bd} P_1$. But, also $x \in \operatorname{Inv}(\bar{N}, \bar{F}) \subset \operatorname{Inv}(P_1, \bar{F})$. Therefore, $\operatorname{Bd} P_1 \cap \operatorname{Inv}(P_1, \bar{F}) \neq \emptyset$. But, by Proposition 16 we get $\operatorname{Inv}(P_1, \bar{F}) \subset P_1 \setminus \operatorname{Bd} P_1$, a contradiction. □

Assume F has acyclic values. Given a weak index pair $P = (P_1, P_2)$ in an isolating neighbourhood $N \subset X$ let

$$T(P) := T_N(P) := (P_1 \cup (X \setminus \operatorname{int} N), P_2 \cup (X \setminus \operatorname{int} N))$$

and let $F_P \colon P \multimap T(P)$ be a restriction of F to the weak index pair P. Note that F_P is a well defined map of pairs. Consider the inclusion $i_P \colon P \to T(P)$. The map i_P induces an isomorphism in Alexander-Spanier cohomology. Thus, we can consider an endomorphism $H^*(F_P) \circ (H^*(i_P))^{-1} \colon H^*(P) \to H^*(P)$. It is called the *index map* associated with the weak index pair P. Applying the Leray functor to $(H^*(P), H^*(F_P) \circ (H^*(i_P))^{-1})$ we obtain a graded module over $\mathbb{Z}$ equipped with the endomorphism. It is called the *cohomological Conley index* of $\operatorname{Inv}(N, F)$ and it is denoted by $C(\operatorname{Inv}(N, F), F)$. For more details regarding the definition of the cohomological Conley index see [3].

Now, we prove that the Conley index of an isolated invariant set $\bar{S}$ in $\bar{X}$ is related by the commutativity property to the index of $f(\bar{S})$ in X with f the continuous map introduced in Section 3..

In the case of $f \colon \bar{X} \to X$ given by (3) the statement of [1, Theorem 7.1] remains valid despite the fact that f is not an injection. More precisely, we have the following theorem.

Theorem 18. *Let $X \subset \mathbb{R}^n$ be a full cubical set and let $F \colon X \multimap X$ be discrete multivalued dynamical system with cubical and contractible values. Assume $\bar{X}$, $f \colon \bar{X} \to X$ and $\bar{F} \colon \bar{X} \multimap \bar{X}$ are defined as in Section 3.. Then, the map $\bar{F}$ is admissible, the map $\psi := \varphi_f F$ is upper semicontinuous with compact acyclic values and diagram*

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\ \bar{F}\ } & \bar{X} \\ {\scriptstyle f}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\ F\ ]{} & X \end{array} \tag{8}$$

commutes. Moreover, if a full cubical set $\bar{N} \subset \bar{X}$ is an isolating neighbourhood with respect to $\bar{F}$ which admits a weak index pair in $\bar{N}$ consisting of a pair of full cubical sets, then $f(\operatorname{Inv}(\bar{N}, \bar{F}))$ is an isolated invariant set with respect to F and $C(\operatorname{Inv}(\bar{N}, \bar{F}), \bar{F}) = C(f(\operatorname{Inv}(\bar{N}, \bar{F})), F)$.

Before the proof consider an example which shows the importance of the assumption concerning existence of a weak index pair consisting of full cubical sets. Naturally, for any isolating neighbourhood there exists a weak index pair [3, Theorem 4.12]. However, it does not need to be a pair of full cubical sets.

Let $\bar{X} = [0, 11] \subset \mathbb{R}$ and define $\bar{F} \colon \bar{X} \multimap \bar{X}$ by

$$\bar{F}(x) := \begin{cases} [0,1] & \text{if } x \in [0,4), \\ [0,7] & \text{if } x \in [4,5], \\ [4,7] & \text{if } x \in (5,6), \\ [4,11] & \text{if } x \in [6,7], \\ [10,11] & \text{if } x \in (7,11]. \end{cases}$$

Consider an isolating neighbourhood $\bar{N} = [3, 8]$. Clearly, $\bar{N}$ is full cubical and $\operatorname{Inv} \bar{N} = [4, 7]$. There exists a weak index pair (e.g. $P_1 = [3, 8]$ and $P_2 = \{3, 8\}$), but there is no weak index pair consisting of full cubical sets.

Proof. By Proposition 5, the map $\bar{F}$ is admissible.

Since both maps φ_f and F are upper semicontinuous with compact values, by [4, Proposition 14.10], their composition is upper semicontinuous with compact values. For each $x \in X$ the value $F(x)$ is cubical and contractible. By the argument similar to the one in the proof of Lemma 3, the map $\varphi_f \circ F$ has cubical and contractible values, which implies acyclicity of ψ. Thus, ψ is upper semicontinuous with compact acyclic values.

To prove commutativity of the diagram (8) notice that $\psi \circ f = \varphi_f \circ F \circ f = \bar{F}$ and $f \circ \psi = f \circ \varphi_f \circ F = \operatorname{id}_X \circ F = F$.

By Proposition 13, the set $f(\operatorname{Inv}(\bar{N}, \bar{F}))$ is an invariant set. By Proposition 15, the set $f(\bar{N})$ is an isolating neighbourhood of $f(\operatorname{Inv}(\bar{N}, \bar{F}))$.

Let $P = (P_1, P_2)$ be a weak index pair in $\bar{N}$ consisting of full cubical sets. By Theorem 17, the pair $(Q_1, Q_2) = (f(P_1), f(P_2))$ is a weak index pair for F in $f(\bar{N})$. By [1, Lemma 3.3(i)], we have $F(P) \subset T_{\bar{N}}(P)$. Therefore, we have a well defined map of pairs $\bar{F}_{P, T_{\bar{N}}(P)} \colon P \multimap T_{\bar{N}}(P)$. Analogously, we define $F_{Q, T_{f(\bar{N})}(Q)} \colon Q \multimap T_{f(\bar{N})}(Q)$. We have the following commutative diagram

$$
\begin{array}{ccccc}
(P_1, P_2) & \xrightarrow{\bar{F}} & (T_{\bar{N},1}(P), T_{\bar{N},2}(P)) & \xleftarrow{\bar{j}} & (P_1, P_2) \\
\downarrow f & \psi \nearrow & \downarrow f & & \downarrow f \\
(Q_1, Q_2) & \xrightarrow[F]{} & (T_{f(\bar{N}),1}(f(P)), T_{f(\bar{N}),2}(f(P))) & \xleftarrow[j]{} & (Q_1, Q_2)
\end{array}
$$

where $\bar{j}$ and j denote inclusions. These inclusions induce isomorphisms in cohomology (cf. [1, Lemma 3.3(ii)]). Therefore, the index maps $I_P := H^*(\bar{F}_{P, T_{\bar{N}}(P)}) \circ H^*(\bar{j})^{-1}$, $I_Q := H^*(F_{Q, T_{f(\bar{N})}(Q)}) \circ H^*(j)^{-1}$ and $I_{QP} := H^*(\psi_{Q, T_{\bar{N}}(P)}) \circ H^*(\bar{j})^{-1}$ are well defined and the diagram

$$
\begin{array}{ccc}
H^*(P) & \xleftarrow{I_P} & H^*(P) \\
f^* \uparrow & \swarrow I_{QP} & \uparrow f^* \\
H^*(Q) & \xleftarrow[I_Q]{} & H^*(Q)
\end{array}
$$

commutes. Since $(H^*(P), I_P)$ and $(H^*(Q), I_Q)$ are linked in the sense of [9], the Leray reductions of these graded cohomology modules with endomorphisms are isomorphic. □

5. Conclusions

Theory developed in this note allows to use available algorithms to compute components necessary to investigate Conley index for discrete multivalued dynamical system. The multivalued map generating the system does not need to satisfy the requirement of existence of continuous selector.

Since in applications the multivalued map is usually defined only on maximal dimensional cubes, we still need to ensure correct values for lower dimensional cubes. This can be done by choosing type of value for lower dimensional cubes. By default the available software takes as a value for lower dimensional cube the union of values of maximal dimensional cubes which are cofaces of lower dimensional one. However, the union of contractible values is not necessary contractible. Therefore, we need to extend the union to a contractible set. An example of type of values is convex hull of the union, star-shaped set or other contractible set containing the default union of values. Note that none of the above-mentioned types of values guarantee the existence of continuous selector of the multivalued map (cf. examples in [2]).

These considerations enable us briefly summarise steps for real computations. As an input for described in this paper procedure we need to deliver an u.s.c. cubical multivalued map and a policy of determining values on lower dimensional cubes. In order to proceed with computations, one should make use of algorithms of finding an isolating neighbourhood and a weak index pair (cf. [12, 5, 1]) for obtained admissible multivalued map. Finally, (co)homological computations are done by algorithms implemented in [10].

An extensive example of using methods developed in this paper as well as an output from available software can be found in [2].

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7. References

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