REPORTS ON MATHEMATICAL LOGIC
55 (2020), 3–39
doi:10.4467/20842589RM.20.001.12433

Roberto GIUNTINI, Claudia MUREŞAN,
Francesco PAOLI

PBZ*-LATTICES: STRUCTURE THEORY AND SUBVARIETIES

A b s t r a c t. We investigate the structure theory of the variety of *PBZ*-lattices* and some of its proper subvarieties. These lattices with additional structure originate in the foundations of quantum mechanics and can be viewed as a common generalisation of orthomodular lattices and Kleene algebras expanded by an extra unary operation. We lay down the basics of the theories of ideals and of central elements in PBZ*-lattices, we prove some structure theorems, and we explore some connections with the theories of subtractive and binary discriminator varieties.

Received 31 January 2019
Keywords and phrases: orthomodular lattice, PBZ-lattice, Brouwer-Zadeh lattice, Kleene lattice, unsharp quantum theory, subtractive variety.*
AMS Subject Classification: primary 08B15; secondary 06B10, 08B26, 03G25, 03G12.

1. Introduction

The papers [14] and [15] contain the beginnings of an algebraic investigation of a variety of lattices with additional structure, the variety $\mathbb{PBZL}^*$ of *PBZ*-lattices*. The key motivation for the introduction of this class of algebras comes from the foundations of quantum mechanics. Consider the structure

$$\mathbf{E}(\mathbf{H}) = \left(\mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', ^\sim, \mathbb{O}, \mathbb{I}\right),$$

where:

- $\mathcal{E}(\mathbf{H})$ is the set of all effects of a given complex separable Hilbert space $\mathbf{H}$, i.e., positive linear operators of $\mathbf{H}$ that are bounded by the identity operator $\mathbb{I}$;
- $\wedge_s$ and $\vee_s$ are the meet and the join, respectively, of the *spectral ordering* $\leq_s$ so defined for all $E, F \in \mathcal{E}(\mathbf{H})$:

  $$E \leq_s F \text{ iff } \forall \lambda \in \mathbb{R} : \ M^F(\lambda) \leq M^E(\lambda),$$

  where for any effect E, M^E is the unique spectral family [19, Ch. 7] such that $E = \int_{-\infty}^{\infty} \lambda \, dM^E(\lambda)$ (the integral is here meant in the sense of norm-converging Riemann-Stieltjes sums [23, Ch. 1]);
- $\mathbb{O}$ and $\mathbb{I}$ are the null and identity operators, respectively;
- $E' = \mathbb{I} - E$ and $E^\sim = P_{\ker(E)}$ (the projection onto the kernel of E).

The operations in $\mathbf{E}(\mathbf{H})$ are well-defined. The spectral ordering is indeed a lattice ordering [21, 18] that coincides with the usual ordering of effects induced via the trace functional when both orderings are restricted to the set of projection operators of the same Hilbert space.

A PBZ*-lattice can be viewed as an abstraction from this concrete physical model, much in the same way as an orthomodular lattice can be viewed as an abstraction from a certain structure of projection operators in a complex separable Hilbert space. The faithfulness of PBZ*-lattices to the physical model whence they stem is further underscored by the fact that they reproduce at an abstract level the "collapse" of several notions of *sharp physical property* that can be observed in $\mathbf{E}(\mathbf{H})$.

Further motivation for the study of $\mathbb{PBZL}^*$ comes from its emerging relationships with many related algebraic structures (orthomodular lattices, Kleene algebras, Stone algebras). In particular, PBZ*-lattices can be seen as a common generalisation of orthomodular lattices and of Kleene algebras with an additional unary operation.

This paper is devoted to laying down the basics of the structure theory of the variety $\mathbb{PBZL}^*$ and of some of its subvarieties; let us briefly summarise its contents. In Section 2 we dispatch a number of preliminaries in order to keep the paper reasonably self-contained, including a short résumé of the results in [14] and [15]. In Section 3, we study decompositions of PBZ*-lattices. As it happens for orthomodular lattices, and more generally for members of all Church varieties [22], direct decompositions in a PBZ*-lattice **L** are induced by certain members of L (the so-called *central elements*) that form a Boolean algebra and that can be conveniently described. In particular, we show that the central elements in a PBZ*-lattice **L** are those elements that "commute" with any $a \in L$, and that this "commuting" relation generalises the analogous relation of decisive importance in the context of orthomodular lattices. In Section 4, we introduce the notion of a p-ideal (ideal closed under perspectivity), mimicking the corresponding definition available for orthomodular lattices. Although in the general case p-ideals lack many of the strong properties one would expect from a reasonable notion of an ideal, as soon as we zoom in on the subvariety $\mathbb{SDM}$ satisfying the strong De Morgan law $(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$, we can show that such ideals coincide with the $\mathbb{SDM}$-ideals in the sense of Ursini (whence also with 0-classes of congruences, since $\mathbb{PBZL}^*$ and all its subvarieties are 0-subtractive). We also prove that the 0-assertional logic of $\mathbb{SDM}$ is strongly algebraisable and we characterise its equivalent variety semantics. Finally, we observe that the variety $V(\mathbb{AOL})$ generated by antiortholattices — that is, PBZ*-lattices with no nontrivial sharp element — is a binary discriminator variety and we further simplify the description of ideals in that case. In the concluding Section 5, after streamlining the known equational basis for $V(\mathbb{AOL})$, we axiomatise the varietal join of orthomodular lattices and the variety generated by antiortholattices in the lattice of subvarieties of $\mathbb{PBZL}^*$.

2. Preliminaries

2.1 Universal Algebra and Lattice Theory

For basic information on universal algebra, the reader is referred to [6, 17].

Throughout this paper, all algebras will be nonempty; by a *trivial algebra* we will mean a one–element algebra, and a *trivial variety* will be a variety consisting solely of trivial algebras. If $\mathbf{A}$ is an algebra, then A will be the universe of $\mathbf{A}$; in some cases, such as those of congruence lattices, lattices will be designated by their set reducts. If $\mathbb{V}$ is a variety of algebras of similarity type ν and $\mathbf{A}$ or a reduct of $\mathbf{A}$ is a member of $\mathbb{V}$, then $(\mathrm{Con}_{\mathbb{V}}(\mathbf{A}), \cap, \vee, \Delta_A, \nabla_A)$ will be the bounded lattice of the congruences of $\mathbf{A}$ with respect to ν; when $\mathbb{V}$ is the variety of lattices, $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ will be denoted by $\mathrm{Con}(\mathbf{A})$. With $\mathbb{V}$ assumed implicit, the congruence of $\mathbf{A}$ generated by an $S \subseteq A \times A$ will be denoted by $Cg(S)$; for all $a, b \in A$, the principal congruence $Cg(\{(a, b)\})$ will be denoted by $Cg(a, b)$.

For any lattice $\mathbf{L}$ and any $x, y \in L$, the principal filter (resp. ideal) of $\mathbf{L}$ generated by x will be denoted by $[x)$ (resp. $(x]$), and, if $x \leq y$, then $[x, y] = [x) \cap (y]$ will be the interval of $\mathbf{L}$ bounded by x and y. The dual of any (bounded) lattice $\mathbf{M}$ will be denoted by $\mathbf{M}^d$. If $\mathbf{A}$ is an algebra with a bounded lattice reduct, then such a reduct will be indicated by $\mathbf{A}_l$. In this case, a congruence θ of $\mathbf{A}$ (or any of its reducts) is said to be *pseudo-identical* iff $0^{\mathbf{A}}/\theta = \{0^{\mathbf{A}}\}$ and $1^{\mathbf{A}}/\theta = \{1^{\mathbf{A}}\}$.

2.2 PBZ*-lattices

We recap in this section some definitions and results on PBZ*-lattices (the latter mostly from [14] and [15], except when explicitly noted) that will be needed in the following.

Definition 2.1. A *bounded involution lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, {}', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with partial order $\leq$ and the following conditions are satisfied for all $a, b \in L$:

- $a'' = a$;
- $a \leq b$ implies $b' \leq a'$.

Note that, for any bounded involution lattice $\mathbf{L}$, the involution $' : L \to L$ is a dual lattice isomorphism of $\mathbf{L}_l$.

Definition 2.2. A bounded involution lattice $\mathbf{L} = (L, \wedge, \vee, ', 0, 1)$ is a *pseudo-Kleene algebra* in case it satisfies any of the following two equivalent conditions:

(i) for all $a, b \in L$, if $a \leq a'$ and $b \leq b'$, then $a \leq b'$;

(ii) for all $a, b \in L$, $a \wedge a' \leq b \vee b'$.

The class of bounded involution lattices is a variety, here denoted by $\mathbb{BI}$. The involution of a pseudo-Kleene algebra is called *Kleene complement*. The variety of pseudo-Kleene algebras, for which see e.g. [10], is denoted by $\mathbb{PKA}$. Distributive pseudo-Kleene algebras are variously called *Kleene lattices* or *Kleene algebras* in the literature. Observe that in [14], embracing the terminological usage from [12, p. 12], pseudo-Kleene algebras were referred to as "Kleene lattices". In [15], however, the authors switched to the less ambiguous "pseudo-Kleene algebras".

In unsharp quantum logic, there are several competing purely algebraic characterisations of sharp effects [12, Ch. 7]. A quantum effect or property is usually called *sharp* if it satisfies the noncontradiction principle:

Definition 2.3. Let $\mathbf{L}$ be a bounded involution lattice.

(i) An element $a \in L$ is said to be *Kleene-sharp* iff $a \wedge a' = 0$. $S_K(\mathbf{L})$ denotes the class of Kleene-sharp elements of $\mathbf{L}$.

(ii) $\mathbf{L}$ is an *ortholattice* iff $S_K(\mathbf{L}) = L$.

(iii) $\mathbf{L}$ is an *orthomodular lattice* iff $\mathbf{L}$ is an ortholattice and, for all $a, b \in L$, if $a \leq b$, then $b = (b \wedge a') \vee a$.

The variety of ortholattices is denoted by $\mathbb{OL}$. Among ortholattices, orthomodular lattices play a crucial role in the standard (sharp) approach to quantum logic. The class of orthomodular lattices is actually a variety, hereafter denoted by $\mathbb{OML}$.

It is well-known that an ortholattice $\mathbf{L}$ is orthomodular if and only if, for all $a, b \in L$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$. In the wider setting of bounded involution lattices, the previous condition does not imply the stronger condition of orthomodularity above. We will call this weaker condition *paraorthomodularity.*

Definition 2.4. An algebra $\mathbf{L}$ with a bounded involution lattice reduct is said to be *paraorthomodular* iff, for all $a, b \in L$:

$$\text{if } a \leq b \text{ and } a' \wedge b = 0, \text{ then } a = b.$$

It turns out that the class of paraorthomodular pseudo-Kleene algebras is a proper quasivariety, whence we cannot help ourselves to the strong universal algebraic properties that characterise varieties. It is then natural to wonder whether there exists an *expansion* of the language of bounded involution lattices where the paraorthomodular condition can be equationally recovered. The appropriate language expansion is provided by including an additional unary operation and moving to the type $(2, 2, 1, 1, 0, 0)$, familiar in unsharp quantum logic from the investigation of *Brouwer-Zadeh lattices* (see [9] or [12, Ch. 4.2]).

Definition 2.5. A *Brouwer-Zadeh lattice* (or *BZ-lattice*) is an algebra

$$\mathbf{L} = \big(L, \wedge, \vee, ', {}^{\sim}, 0, 1\big)$$

of type $(2, 2, 1, 1, 0, 0)$, such that:

(i) $(L, \wedge, \vee, ', 0, 1)$ is a pseudo-Kleene algebra;

(ii) for all $a, b \in L$, the following conditions are satisfied:

(1) $a \wedge a^{\sim} = 0$;	(2) $a \leq a^{\sim\sim}$;
(3) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$;	(4) $a^{\sim\prime} = a^{\sim\sim}$.

The operation $^{\sim}$ is called the *Brouwer complement* of the BZ–lattice. The class of all BZ-lattices is a variety, denoted by $\mathbb{BZL}$; $\mathbb{OL}$ can be identified with the subvariety of $\mathbb{BZL}$ whose relative equational basis w.r.t. $\mathbb{BZL}$ is given by the equation $x^{\sim} = x'$. In any BZ-lattice, we set $\Diamond x = x^{\sim\sim}$ and $\Box x = x'^{\sim}$. The following arithmetical lemma, the proof of which is variously scattered in the above-mentioned literature and elsewhere [7, 8], will be used without being referenced in what follows.

Lemma 2.6. *Let* **L** *be a BZ-lattice. For all* $a, b \in L$*, the following conditions hold:*

(i)	$a^{\sim\sim\sim} = a^{\sim}$;	***(vi)***	$\Box(a \wedge b) = \Box a \wedge \Box b$;
(ii)	$a^{\sim} \leq a'$;	***(vii)***	$\Diamond(a \vee b) = \Diamond a \vee \Diamond b$;
(iii)	$(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$;	***(viii)***	$\Diamond(a \wedge b) \leq \Diamond a \wedge \Diamond b$;
(iv)	$a^{\sim} \vee b^{\sim} \leq (a \wedge b)^{\sim}$;	***(ix)***	*if* $a' \leq a$*, then* $a^{\sim} = 0$.
(v)	$(\Box(a'))' = \Diamond a$;		

We remarked above that Kleene-sharpness is not the unique purely algebraic characterisation of a sharp quantum property. Two noteworthy alternatives now become available in our expanded language of BZ-lattices.

Definition 2.7. Let **L** be a BZ-lattice.

(i) An element $a \in L$ is said to be $\Diamond$*-sharp* iff $a = \Diamond a$; the class of all $\Diamond$-sharp elements of **L** will be denoted by $S_{\Diamond}(\mathbf{L})$.

(ii) An element $a \in L$ is said to be *Brouwer-sharp* iff $a \vee a^{\sim} = 1$; the class of all Brouwer-sharp elements of **L** will be denoted by $S_B(\mathbf{L})$.

It is easy to derive from the previous lemma that, in any BZ-lattice **L**, $S_{\Diamond}(\mathbf{L}) = \{a^{\sim} : a \in L\} = \{a \in L : a' = a^{\sim}\}$. For any BZ-lattice **L**, we have that $S_{\Diamond}(\mathbf{L}) \subseteq S_B(\mathbf{L}) \subseteq S_K(\mathbf{L})$. However, in any BZ-lattice of effects of a Hilbert space (under the meet and join operation induced by the spectral ordering) these three classes coincide. Consequently, it makes sense to investigate whether there is a class of BZ-lattices for which this collapse result can be recovered at a purely abstract level. The next definition and theorem answer this question in the affirmative.

Definition 2.8. A *BZ*-lattice* is a BZ-lattice **L** that satisfies, for all $a \in L$, the condition

$$(*) \qquad (a \wedge a')^{\sim} \leq a^{\sim} \vee \Box a.$$

Theorem 2.9. *Let* **L** *be a paraorthomodular BZ*-lattice. Then,*

$$S_{\Diamond}(\mathbf{L}) = S_B(\mathbf{L}) = S_K(\mathbf{L}).$$

As pleasing as this result may be, the class of paraorthomodular BZ*-lattices still suffers from a major shortcoming: the paraorthomodularity condition is quasiequational. However, the next result shows that it can be replaced by an equation, so that paraorthomodular BZ*-lattices form a variety, which we will denote by $\mathbb{PBZL}^*$ and whose members will be called, in brief, *PBZ*-lattices.*

Theorem 2.10. *Let* **L** *be a BZ*-lattice. The following conditions are equivalent:*

(1) **L** *is paraorthomodular;*

(2) **L** *satisfies the following* $\Diamond$-orthomodularity *condition for all* $a, b \in L$*:*

$$(a^{\sim} \vee (\Diamond a \wedge \Diamond b)) \wedge \Diamond a \leq \Diamond b.$$

Every bounded lattice can be embedded as a sublattice into a PBZ*-lattice [14, Lm. 5.3]. Consequently, $\mathbb{PBZL}^*$ satisfies no nontrivial identity in the language of lattices.

The naturalness of the concept of a PBZ*-lattice is further reinforced by the circumstance that BZ-lattices of effects of a Hilbert space, under the spectral ordering, qualify as instances of PBZ*-lattices:

Theorem 2.11. *Let* **H** *be a complex separable Hilbert space. The algebra*

$$\mathbf{E}(\mathbf{H}) = \left\langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', ^{\sim}, \mathbb{O}, \mathbb{I} \right\rangle,$$

(see the introduction for the notation) is a PBZ-lattice. Moreover,*

$$S_K(\mathbf{E}(\mathbf{H})) = S_{\Diamond}(\mathbf{E}(\mathbf{H})) = S_B(\mathbf{E}(\mathbf{H}))$$

is an orthomodular subuniverse of $\mathbf{E}(\mathbf{H})$ *consisting of all the projection operators of* **H**.

A natural question is whether the class of all PBZ*-lattices of the form $\mathbf{E}(\mathbf{H})$, for some complex separable Hilbert space **H**, *generates* the variety $\mathbb{PBZL}^*$. The answer to this question is known to be negative. In fact, there are identities (e.g. $x \approx (x \vee x^{\sim}) \wedge \Diamond x$) that hold in the class of all PBZ*-lattices of effects of some Hilbert space but fail in $\mathbb{PBZL}^*$. The (probably very difficult) problem of axiomatising the proper subvariety of $\mathbb{PBZL}^*$ that is so generated is open at the time of writing.

All orthomodular lattices become, of course, PBZ*-lattices when endowed with a Brouwer complement that equals their Kleene complement.

In every PBZ*-lattice $\mathbf{L}$, $S_K(\mathbf{L})$ is always the universe of the largest orthomodular subalgebra $\mathbf{S}_K(\mathbf{L})$ of $\mathbf{L}$, so that $\mathbf{L}$ is orthomodular iff it satisfies $x^\sim \approx x'$. Further examples of PBZ*-lattices are given by those algebras in this class that are "as far apart as possible" from orthomodular lattices. In any orthomodular lattice $\mathbf{L}$, $S_K(\mathbf{L}) = L$; on the other hand, by definition, a PBZ*-lattice $\mathbf{L}$ is an *antiortholattice* iff $S_K(\mathbf{L}) = \{0, 1\}$. We denote by $\mathbb{AOL}$ the class of antiortholattices.

Lemma 2.12.

(i) *A PBZ*-lattice* $\mathbf{L}$ *belongs to* $\mathbb{AOL}$ *iff* $0^\sim = 1$ *and, for all* $a \in L \setminus \{0\}$, $a^\sim = 0$.

(ii) *Every* $\mathbf{L} \in \mathbb{AOL}$ *is directly indecomposable.*

(iii) $\mathbb{AOL}$ *is a proper universal class.*

The Brouwer complement of Lemma 2.12.(i) is called *trivial*.

For all $n \geq 1$, the n-element Kleene chain with universe $D_n = \{0, d_1, d_2, \ldots, d_{n-2}, 1\}$, with $0 < d_1 < d_2 < \ldots < d_{n-2} < 1$, is an antiortholattice $\mathbf{D}_n$ under the trivial Brouwer complement. To avoid notational overloading, the reduct $(\mathbf{D}_n)_l$ will simply be denoted by $\mathbf{D}_n$, as well. Note that every finite chain is self–dual both as a bounded lattice and as a Kleene algebra, so the notation $\mathbf{D}_n^d$ is superfluous in these cases; the same can be stated about direct products of finite chains, in particular about Boolean algebras.

The following easy results are observed (sometimes implicitly) in the literature on BZ-lattices, in particular in [14] and in [15]:

Lemma 2.13. *(i)* *Any pseudo-Kleene algebra, endowed with the trivial Brouwer complement, becomes a BZ-lattice.*

(ii) *Any paraorthomodular pseudo-Kleene algebra which, endowed with the trivial Brouwer complement, satisfies condition* $(*)$, *becomes an antiortholattice.*

(iii) *Any pseudo-Kleene algebra in which* 0 *is meet–irreducible is paraorthomodular and satisfies condition* $(*)$ *when endowed with the trivial Brouwer complement, whence it becomes an antiortholattice.*

We will repeatedly have the occasion to consider the following identities in the language of BZ–lattices:

SDM (the *Strong de Morgan law*) $(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$;

WSDM (*weak SDM*) $(x \wedge y^{\sim})^{\sim} \approx x^{\sim} \vee \Diamond y$;

DIST $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;

J2 $x \approx (x \wedge (y \wedge y')^{\sim}) \vee (x \wedge \Diamond (y \wedge y'))$;

SK $x \wedge \Diamond y \leq \Box x \vee y$.

Clearly, SDM implies WSDM. Observe that $\mathbb{OML}$ satisfies SDM, J2 and SK. Trivially, $\mathbb{AOL}$ satisfies WSDM and J2, whence $\mathbb{OML} \vee \mathrm{V}(\mathbb{AOL})$ satisfies these two identities.

We list some useful properties of the variety $\mathrm{V}(\mathbb{AOL})$ generated by antiortholattices, including an axiomatisation relative to $\mathbb{PBZL}^*$.

Lemma 2.14. *Let* $\mathbf{L} \in \mathrm{V}(\mathbb{AOL})$. *Then each of the distributive identities*

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)\,;$$
$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$$

holds if any one of the variables x, y, *or* z *is evaluated in* $S_{\Diamond}(\mathbf{L}) = S_K(\mathbf{L})$.

Theorem 2.15.

(i) An equational basis for $\mathrm{V}(\mathbb{AOL})$ *relative to* $\mathbb{PBZL}^*$ *is given by the identities*

(AOL1) $(x^{\sim} \vee y^{\sim}) \wedge (\Diamond x \vee z^{\sim}) \approx ((x^{\sim} \vee y) \wedge (\Diamond x \vee z))^{\sim}$;

(AOL2) $x \approx (x \wedge y^{\sim}) \vee (x \wedge \Diamond y)$;

(AOL3) $x \approx (x \vee y^{\sim}) \wedge (x \vee \Diamond y)$.

(ii) Every subdirectly irreducible member of $\mathrm{V}(\mathbb{AOL})$ *is an antiortholattice.*

Clearly $\mathrm{V}(\mathbb{AOL}) \cap \mathbb{OML}$ is the variety $\mathbb{BA}$ of Boolean algebras.

The lattice $\mathbf{L}_{\mathbb{PBZL}^*}$ of subvarieties of $\mathbb{PBZL}^*$ has $\mathbb{BA}$ as a unique atom. It is well-known that $\mathbb{BA}$ has a single orthomodular cover [5, Cor. 3.6]: the variety $\mathrm{V}(\mathbf{MO}_2)$, generated by the simple modular ortholattice with 4 atoms. Moreover:

Theorem 2.16. *There is a single non-orthomodular cover of* $\mathbb{BA}$ *in* $\mathbf{L}_{\mathbb{PBZL}^*}$*, the variety* $\mathrm{V}(\mathbf{D}_3)$ *generated by the* 3*-element antiortholattice chain, whose equational basis relative to* $\mathrm{V}(\mathbb{AOL})$ *is given by the identity SK.*

Two other notable subvarieties of $\mathrm{V}(\mathbb{AOL})$ are the variety $\mathbb{DIST}$, whose equational basis relative to $\mathrm{V}(\mathbb{AOL})$ (or, equivalently, relative to $\mathbb{PBZL}^*$) is given by the distribution identity DIST, and the variety $\mathbb{SAOL}$, whose equational basis relative to $\mathrm{V}(\mathbb{AOL})$ is given by the Strong De Morgan identity SDM. We have that:

Theorem 2.17. $\mathrm{V}(\mathbf{D}_5) = \mathbb{DIST} \cap \mathbb{SAOL}$*.*

A more circumscribed study of $\mathbb{DIST}$ and its subvarieties, also focussing on the relationship with known classes of algebras (including Kleene-Stone algebras, Łukasiewicz algebras, Heyting-Wajsberg algebras) is currently ongoing [20, 16].

2.3 Subtractive Varieties

Subtractive varieties were introduced by Ursini [24] to enucleate the common features of pointed varieties with a good ideal theory, like groups, rings or Boolean algebras. They were further investigated in [1, 2, 3, 25].

Definition 2.18. Let $\mathbb{V}$ a variety of type ν, and let 0 be a nullary term (or equationally definable constant) of type ν. $\mathbb{V}$ is called 0-*subtractive* if there exists a binary term s, also of type ν, s.t. $\mathbb{V}$ satisfies the identities $s(x,x) \approx 0$ and $s(x,0) \approx x$. A variety of type ν which is 0-subtractive w.r.t. at least one constant 0 of type ν is called *subtractive* tout court.

It is not hard to see that subtractivity is a congruence property: namely, a variety $\mathbb{V}$ is 0-subtractive exactly when in each $\mathbf{A} \in \mathbb{V}$ congruences permute at 0 (meaning that for all θ, φ in $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$, $0^{\mathbf{A}}/(\theta \circ \varphi) = 0^{\mathbf{A}}/(\varphi \circ \theta)$).

To investigate ideals in this context, first and foremost, we need a workable general notion of ideal encompassing all the intended examples mentioned above (normal subgroups of groups, two-sided ideals of rings, ideals or filters of Boolean algebras). Ursini's candidate for playing this role is defined below.

Definition 2.19. (i) If $\mathbb{K}$ is a class of similar algebras whose type ν is as in Definition 2.18, then a term $p(\overrightarrow{x}, \overrightarrow{y})$ of type ν is a $\mathbb{K}$-*ideal term* in $\overrightarrow{x}$ iff $\mathbb{K} \vDash p(0, ..., 0, \overrightarrow{y}) \approx 0$.

(ii) A nonempty subset J of the universe of an $\mathbf{A} \in \mathbb{K}$ is a $\mathbb{K}$-*ideal* of $\mathbf{A}$ (w.r.t. 0) iff for any $\mathbb{K}$-ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{x}$ we have that $p^{\mathbf{A}}\left(\overrightarrow{a}, \overrightarrow{b}\right) \in J$ whenever $\overrightarrow{a} \in J$ and $\overrightarrow{b} \in A$.

We will denote by $\mathcal{I}_{\mathbb{K}}(\mathbf{A})$ the set (or the lattice) of all $\mathbb{K}$-ideals of $\mathbf{A}$, dropping the subscript whenever $\mathbb{K}$ can be contextually identified; observe that $\{0\}, A \in \mathcal{I}_{\mathbb{K}}(\mathbf{A})$. The main reason that backs our previous claim to the effect that subtractive varieties have a good ideal theory is given by the following result. Let $\mathbb{V}$ be a variety of type ν. Recall that an algebra $\mathbf{A}$ from $\mathbb{V}$ is said to be 0-*regular* iff the map sending a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ to its 0-class $0^{\mathbf{A}}/\theta$ is injective. The variety $\mathbb{V}$ is said to be 0-*regular* iff every $\mathbf{A} \in \mathbb{V}$ is 0-regular; this happens exactly when there exists a finite family of binary ν-terms (called *Fichtner terms*) $\{d_i(x,y)\}_{i \leq n}$ such that $\mathbb{V} \vDash d_1(x,y) \approx 0 \& ... \& d_n(x,y) \approx 0 \Leftrightarrow x \approx y$.

Theorem 2.20. *(i) Subtractive varieties have normal ideals. That is, if* $\mathbb{V}$ *is a* 0*-subtractive variety and* $\mathbf{A} \in \mathbb{V}$*, then*

$$\mathcal{I}_{\mathbb{V}}(\mathbf{A}) = \left\{ I \subseteq A : I = 0^{\mathbf{A}}/\theta \text{ for some } \theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) \right\}.$$

(ii) If $\mathbb{V}$ *is a* 0*-subtractive and* 0*-regular variety, then, for every* $\mathbf{A} \in \mathbb{V}$, $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ *is isomorphic to* $\mathcal{I}_{\mathbb{V}}(\mathbf{A})$.

Actually, the situation described by the previous theorem can be made more precise as follows. Let $\mathbf{A}$ be an algebra in a 0-subtractive variety $\mathbb{V}$. Then the following maps are well-defined: $\cdot^{\delta}, \cdot^{\varepsilon} : \mathcal{I}_{\mathbb{V}}(\mathbf{A}) \to \mathrm{Con}_{\mathbb{V}}(\mathbf{A})$, for all $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$,

$$\begin{aligned} I^{\delta} &= \bigwedge \{\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0^{\mathbf{A}}/\theta = I\}, \\ I^{\varepsilon} &= \bigvee \{\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0^{\mathbf{A}}/\theta = I\}. \end{aligned}$$

Henceforth, all unnecessary superscripts will be dropped for the sake of conciseness. Note that, for all $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$, we have $0/I^{\delta} = 0/I^{\varepsilon} = I$, so that $I^{\delta} = \min\{\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0/\theta = I\}$ and $I^{\varepsilon} = \max\{\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0/\theta = I\}$. Moreover, the map $I \mapsto [I^{\delta}, I^{\varepsilon}]$ is a lattice isomorphism from $\mathcal{I}_{\mathbb{V}}(\mathbf{A})$

to the following lattice of intervals of $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$: $\{[\min(C_\theta), \max(C_\theta)] : \theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A})\}$, where, for all $\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A})$, $C_\theta = \{\alpha \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0/\alpha = 0/\theta\} = \{\alpha \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : 0/\theta \in A/\alpha\}$, which is a complete sublattice of $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$. Clearly, if $\mathbb{V}$ is in addition 0-regular, then $C_\theta = \{\theta\}$ for all $\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A})$, hence all intervals of the form $\left[I^\delta, I^\varepsilon\right]$ for some $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$ are trivial, and Theorem 2.20.(ii) follows as a special case.

Definition 2.21. Let $\mathbf{A}$ be an algebra in a 0-subtractive variety $\mathbb{V}$. $\mathbf{A}$ is said to be *reduced* iff $\{0\}^\varepsilon = \Delta_A$.

The class of all reduced algebras in $\mathbb{V}$ will be denoted by $\mathbb{V}_\varepsilon$. Clearly, for all $\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A})$, we have $\mathbf{A}/\theta \in \mathbb{V}_\varepsilon$ iff $\theta = I^\varepsilon$ for some $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$.

For a 0-subtractive variety, a property that generalises 0-regularity is finite congruentiality. Roughly put, a 0-subtractive variety is finitely congruential if it has a family of terms that do "part of the job" usually dispatched by the Fichtner terms for 0-regularity.

Definition 2.22. A variety $\mathbb{V}$, whose type ν is as in Definition 2.18, is *finitely congruential* iff there exists a finite set $\{d_i(x,y)\}_{i\leq n}$ of binary ν-terms s.t., whenever $\mathbf{A} \in \mathbb{V}$ and $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$, we have:

$$I^\varepsilon = \{(a,b) : d_i^{\mathbf{A}}(a,b) \in I \text{ for all } i \leq n\}.$$

The next results establish a significant connection between the theory of subtractive varieties and *abstract algebraic logic* (for information on this research area, the reader is referred to [13]). It turns out that, for a 0-subtractive variety $\mathbb{V}$, the properties of the 0-assertional logic of $\mathbb{V}$ yield relevant information on the properties of the class of reduced algebras in $\mathbb{V}$, and conversely.

Theorem 2.23. *If $\mathbf{A}$ is a member of a 0-subtractive variety $\mathbb{V}$, then $\mathcal{I}_{\mathbb{V}}(\mathbf{A})$ is the class of all deductive filters on $\mathbf{A}$ of the 0-assertional logic of $\mathbb{V}$, and for all $I \in \mathcal{I}_{\mathbb{V}}(\mathbf{A})$, $I^\varepsilon = \Omega^{\mathbf{A}}(I)$.*

Theorem 2.24. *[2, Thm. 3.12] For a 0-subtractive variety $\mathbb{V}$ the following are equivalent:*

(i) The 0-assertional logic of $\mathbb{V}$ is equivalential.

(ii) $\mathbb{V}_\varepsilon$ is closed under subalgebras and direct products.

Theorem 2.25. *[2, Thm. 3.16] For a* 0*-subtractive variety* $\mathbb{V}$ *the following are equivalent:*

(i) The 0*-assertional logic of* $\mathbb{V}$ *is strongly algebraisable with* $\mathbb{V}_\varepsilon$ *as an equivalent algebraic semantics.*

(ii) $\mathbb{V}_\varepsilon$ *is a variety.*

Important examples of subtractive varieties are *binary discriminator varieties*. Recall that a *discriminator variety* [26] is a variety $\mathbb{V}$ of given type ν for which there exists a ternary ν-term $t(x, y, z)$ that realises the ternary discriminator function

$$t(a, b, c) = \begin{cases} c \text{ if } a = b, \\ a, \text{ otherwise} \end{cases}$$

on any subdirectly irreducible member of $\mathbb{V}$ (equivalently, on any member of some class $\mathbb{K}$ such that $\mathbb{V} = \mathrm{V}(\mathbb{K})$). The introduction of *binary discriminator varieties* by Chajda, Halaš, and Rosenberg [11] was aimed at singling out an appropriate weakening of the ternary discriminator that can vouchsafe some of the strong properties of discriminator varieties (like congruence distributivity or congruence permutability) only "locally", i.e. at 0. Binary discriminator varieties are paramount among subtractive varieties with *equationally definable principal ideals* [1]; they were thoroughly studied in the unpublished [4].

Definition 2.26. [11] Let A be a nonempty set and fix $0 \in A$. The 0-*binary discriminator* on A is the binary function b_0^A on A defined by:

$$b_0^A(a, c) = \begin{cases} a \text{ if } c = 0, \\ 0 \text{ otherwise.} \end{cases}$$

An algebra $\mathbf{A}$ with a term definable element 0 is said to be a 0-*binary discriminator algebra* in case the 0-binary discriminator b_0^A on A is a term operation on $\mathbf{A}$. A variety $\mathrm{V}(\mathbb{K})$ is a 0-*binary discriminator variety* if it is generated by a class $\mathbb{K}$ of 0-binary discriminator algebras such that the property is witnessed by the same terms for all members of $\mathbb{K}$.

3. Central Elements

One of the most distinctive and far-reaching chapters in the theory of orthomodular lattices is the study of the commuting relation and of central elements (see e.g. [5, § 2]). Given an orthomodular lattice $\mathbf{L}$ and $a, b \in L$, a is said to *commute* with b in case $(a \wedge b) \vee (a' \wedge b) = b$. Such a relation is reflexive and symmetric. An element $a \in L$ is said to be *central* in $\mathbf{L}$ in case it commutes with all elements of L. The next celebrated result is one of the most useful tools for practicioners of the field:

Theorem 3.1 (Foulis-Holland)**.** *[5, Prop. 2.8] If $\mathbf{L}$ is an orthomodular lattice and $a, b, c \in L$ are such that a commutes both with b and with c, then the set $\{a, b, c\}$ generates a distributive sublattice of $\mathbf{L}_l$.*

Although these investigations were carried out in the special context of orthomodular lattices, the notion of a central element is deeply rooted in universal algebra. Recall that, if $\mathbf{A}$ is an algebra in a double-pointed variety $\mathbb{V}$ with constants $0, 1$, an element $e \in A$ is *central* in $\mathbf{A}$ in case the congruences $Cg\,(e, 0)$ and $Cg\,(e, 1)$ are complementary factor congruences of $\mathbf{A}$ [22]. By $C(\mathbf{A})$ we denote the *centre* of A, i.e. the set of central elements of the algebra $\mathbf{A}$. In particular, if $\mathbf{A}$ is a *Church algebra* [22], namely, if there is an "if-then-else" term operation $q^{\mathbf{A}}$ on $\mathbf{A}$ s.t., for all $a, b \in A$, $q^{\mathbf{A}}\left(1^{\mathbf{A}}, a, b\right) = a$ and $q^{\mathbf{A}}\left(0^{\mathbf{A}}, a, b\right) = b$, then, by defining

$$x \wedge y = q(x, y, 0),\ x \vee y = q(x, 1, y) \text{ and } x' = q(x, 0, 1),$$

we get:

Theorem 3.2. *[22, Thm. 3.7] The algebra $c\,[\mathbf{A}] = (C(\mathbf{A}), \wedge, \vee, ', 0, 1)$ is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of $\mathbf{A}$.*

In Church algebras, central elements can be equationally characterised [22, Prop. 3.6]. When studying a specific variety $\mathbb{V}$, however, more informative descriptions of central elements in members of $\mathbb{V}$ can sometimes be found either in terms of certain properties of intrinsic interest, or by appropriately streamlining the above-mentioned equational characterisation. As regards members of $\mathbb{OML}$, for example, it can be shown that the two definitions of central element we have given above are equivalent. For $\mathbb{PBZL}^*$, on the other hand, a more economical equational description of central elements was provided in [14, Lm. 5.9]. We reproduce this lemma for the reader‘s convenience:

Lemma 3.3. *Let* $\mathbf{L} \in \mathbb{PBZL}^*$. *Then* $e \in L$ *is central in* $\mathbf{L}$ *iff it satisfies the following conditions for all* $a, b \in L$*:*

C1 $a = (e \wedge a) \vee (e' \wedge a)$;

C2 $(a \vee b) \wedge e = (a \wedge e) \vee (b \wedge e)$;

C3 $(a \vee b) \wedge e' = (a \wedge e') \vee (b \wedge e')$;

C4 $((e \vee b) \wedge (e' \vee a))^{\sim} = (e \vee b^{\sim}) \wedge (e' \vee a^{\sim})$.

The aim of the next subsection is to improve on this result, giving a description of the centre in a generic PBZ*-lattice that resembles as closely as possible the one we have in orthomodular lattices.

3.1 Central Elements in PBZ*-lattices

Throughout the rest of this subsection, unless mentioned otherwise, $\mathbf{L}$ will be an arbitrary PBZ*-lattice. Observe that $C(\mathbf{L}) \subseteq S_K(\mathbf{L})$, by C1 for $a = 1$. Since $e' = e^{\sim}$ for all $e \in S_K(\mathbf{L})$, it follows that, for all $e \in L$: $e \in C(\mathbf{L})$ iff $e^{\sim} \in C(\mathbf{L})$. In [14, Thm. 5.4] it is also proved that central elements in any member $\mathbf{L}$ of $\mathrm{V}(\mathbb{AOL})$ are exactly the members of $S_K(\mathbf{L})$. The problem of finding a manageable description of central elements in PBZ*-lattices, that relates in a perspicuous way to the notions of a centre and of a commutator in orthomodular lattice, was however left open. We now set about filling this gap. We show that the property of being central, in a generic PBZ*-lattice, is actually two-sided. In order to belong to $C(\mathbf{L})$, an $e \in L$ should not only "commute" with any element of L, in a sense of commuting that appropriately generalises the corresponding (reflexive and symmetric) relation on orthomodular lattices; but it should also be such that, for any $a \in L$, $(e \wedge a)^{\sim} = e^{\sim} \vee a^{\sim}$ and $(e' \wedge a)^{\sim} = \Box e \vee a^{\sim}$. This latter component is sort of "hidden" in the case of $\mathbb{OML}$, where the Strong De Morgan identity is satisfied across the board.

We define the following binary relations on L:

$C_{\mathbf{L}} = \{(a,b) \in L^2 : (a \wedge b) \vee (a' \wedge b) = b\}$;
$C_{\mathrm{SDM},\mathbf{L}} = C_{\mathbf{L}} \cap \{(a,b) \in L^2 : (a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim} \text{ and } (a' \wedge b)^{\sim} = \Box a \vee b^{\sim}\}$.

Both relations are clearly reflexive. Also, for all $a \in L$, $(1,a) \in C_{\mathrm{SDM},\mathbf{L}} \subseteq C_{\mathbf{L}}$, while the following four conditions are mutually equivalent: i) $(a,1) \in C_{\mathrm{SDM},\mathbf{L}}$; ii) $(a,1) \in C_{\mathbf{L}}$; iii) $a \vee a' = 1$; iv) $a \in S_K(\mathbf{L})$. Hence, $C_{\mathrm{SDM},\mathbf{L}}$ is symmetric exactly when $C_{\mathbf{L}}$ is symmetric, which in turn obtains if and only if $\mathbf{L}$ is orthomodular — in which case, apparently, the two relations coincide.

The next definition provides a formal clothing to the above informal remarks about the property of being central in PBZ*-lattices. Our goal is to show that it exactly captures, for this variety, the universal algebraic property of centrality, as encoded by Lemma 3.3.

Definition 3.4. Let $\mathbf{L} \in \mathbb{PBZL}^*$. An element $a \in L$ is said to be *PBZ*-central* iff $(a,b) \in C_{\mathrm{SDM},\mathbf{L}}$ for all $b \in L$.

We will denote by $C_{pbz}(\mathbf{L})$ the set of all PBZ*-central elements of $\mathbf{L}$:

$$C_{pbz}(\mathbf{L}) = \{a \in L : (\forall b \in L)\,((a,b) \in C_{\mathrm{SDM},\mathbf{L}})\}.$$

We also consider the following subset of L:

$$C_p(\mathbf{L}) = \{a \in L : (\forall b \in L)\,((a,b) \in C_{\mathbf{L}})\}.$$

In virtue of the above,

$$\begin{aligned} &C_{pbz}(\mathbf{L}) \\ &= C_p(\mathbf{L}) \cap \{a \in L : (\forall b \in L)\,((a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}, (a' \wedge b)^{\sim} = \Box a \vee b^{\sim})\} \\ &= C_p(\mathbf{L}) \cap \{a \in S_K(\mathbf{L}) : (\forall b \in L)\,((a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}, (a' \wedge b)^{\sim} = \Box a \vee b^{\sim})\} \\ &\subseteq C_p(\mathbf{L}) \subseteq S_K(\mathbf{L}). \end{aligned}$$

Lemma 3.5. *Let* $\mathbf{L} \in \mathbb{PBZL}^*$. *Then:*

(i) $C_p(\mathbf{L}) = \{a \in S_K(\mathbf{L}) : (\forall b \in L)\,((a \wedge b) \vee (a^{\sim} \wedge b) = b)\} = \{a \in S_K(\mathbf{L}) : (\forall b \in L)\,((a \vee b) \wedge (a^{\sim} \vee b) = b)\}$;

(ii) for all $a \in S_K(\mathbf{L})$, $a \in C_p(\mathbf{L})$ *iff* $a^{\sim} \in C_p(\mathbf{L})$; *furthermore,* $a \in C_{pbz}(\mathbf{L})$ *iff* $a^{\sim} \in C_{pbz}(\mathbf{L})$;

(iii) if $\mathbf{L}$ *satisfies WSDM, then* $C_{pbz}(\mathbf{L}) = C_p(\mathbf{L})$*.*

Proof. Let $a \in S_K(\mathbf{L})$, arbitrary.

(i) The fact that $a' = a^\sim$ gives us the first equality. To obtain the second equality, notice that, for all $b \in L$, we have: $(a, b) \in C_{\mathbf{L}}$ iff $(a \wedge b) \vee (a' \wedge b) = b$ iff $(a' \vee b') \wedge (a \vee b') = b'$ iff $(a \vee b') \wedge (a^\sim \vee b') = b'$, hence: $(a, b) \in C_{\mathbf{L}}$ for all $b \in L$ iff $(a \vee b) \wedge (a^\sim \vee b) = b$ for all $b \in L$.

(ii) From (i), along with the fact that $a = \Diamond a$.

(iii) From the fact that $C_p(\mathbf{L}) \subseteq S_K(\mathbf{L})$. □

Note from this proof that $C_{\mathbf{L}}$ and $C_{\mathrm{SDM},\mathbf{L}}$ preserve the Kleene complement and that, under WSDM, the relation $C_{\mathrm{SDM},\mathbf{L}} \cap (S_K(\mathbf{L}) \times L) = C_{\mathbf{L}} \cap (S_K(\mathbf{L}) \times L)$ preserves the Brouwer complement. Some useful properties of members of $C_p(\mathbf{L})$ follow in the next two lemmas.

Lemma 3.6. *Let* $\mathbf{L} \in \mathbb{PBZL}^*$ *and let* $a \in L$ *be such that* $a^\sim \in C_p(\mathbf{L})$*. Then, for all* $b, c \in L$*:*

(i) $a^\sim \vee b = a^\sim \vee (\Diamond a \wedge b)$ *and* $a^\sim \wedge b = a^\sim \wedge (\Diamond a \vee b)$*;*

(ii) $b \vee (c \wedge \Diamond a) = (b \vee c \vee a^\sim) \wedge (b \vee \Diamond a)$ *and* $b \wedge (c \vee \Diamond a) = (b \wedge c \wedge a^\sim) \vee (b \wedge \Diamond a)$*;*

(iii) $a^\sim \wedge (b \vee c) = a^\sim \wedge (b \vee (a^\sim \wedge c))$ *and* $a^\sim \vee (b \wedge c) = a^\sim \vee (b \wedge (a^\sim \vee c))$*.*

Proof. By adapting the corresponding proofs in [14, Lm. 5.10]. □

Lemma 3.7. *Let* $\mathbf{L} \in \mathbb{PBZL}^*$*, and let* $a \in L$ *be such that* $a^\sim \in C_p(\mathbf{L})$*. Then the following hold for all* $b, c \in L$*:*

(i) $b \vee (c \wedge a^\sim) = (b \vee c) \wedge (b \vee a^\sim)$ *and* $b \wedge (c \vee a^\sim) = (b \wedge c) \vee (b \wedge a^\sim)$*;*

(ii) $a^\sim \wedge (b \vee c) = (a^\sim \wedge b) \vee (a^\sim \wedge c)$ *and* $a^\sim \vee (b \wedge c) = (a^\sim \vee b) \wedge (a^\sim \vee c)$*.*

Proof. We prove the first equalities in each pair. The lattice duals are derived similarly.

(i) Let $x = b \vee (c \wedge a^\sim)$. Then:

$$
\begin{aligned}
x &= x \wedge (x \vee (c \wedge (x \vee \Diamond a))) && \text{Absorption} \\
&= (x \vee a^\sim) \wedge (x \vee \Diamond a) \wedge (x \vee (c \wedge (x \vee \Diamond a))) && \text{Lemma 3.5} \\
&= (x \vee a^\sim) \wedge (x \vee (c \wedge (x \vee \Diamond a))) && \text{Lattice prop.} \\
&= (b \vee a^\sim) \wedge (x \vee (c \wedge (x \vee \Diamond a))) && \text{Absorption} \\
&= (b \vee a^\sim) \wedge (x \vee c) && \\
&= (b \vee c) \wedge (b \vee a^\sim) && \text{Absorption}
\end{aligned}
$$

The penultimate equality is obtained by observing that, according to Lemma 3.6.(i), since $\Diamond(a^{\sim}) = a^{\sim}$, we have: $c \leq b \vee c \vee \Diamond a = b \vee \Diamond a \vee (a^{\sim} \wedge c) = x \vee \Diamond a$.

(ii)

$$\begin{array}{lll} a^{\sim} \wedge (b \vee c) & = a^{\sim} \wedge (b \vee (a^{\sim} \wedge c)) & \text{Lemma 3.6.(iii)} \\ & = ((a^{\sim} \wedge c) \vee a^{\sim}) \wedge (b \vee (a^{\sim} \wedge c)) & \text{Absorption} \\ & = (a^{\sim} \wedge b) \vee (a^{\sim} \wedge c) & \text{(i)} \end{array}$$

□

Theorem 3.8. *Let* $\mathbf{L} \in \mathbb{PBZL}^*$*. Then* $C_{pbz}(\mathbf{L}) = C(\mathbf{L})$*.*

Proof. Let $e \in L$.

Assume that $e^{\sim} \in C_{pbz}(\mathbf{L}) \subseteq C_p(\mathbf{L})$. We verify all the conditions C1 through C4 in Lemma 3.3. C1 holds because $(e^{\sim}, a) \in C_{\mathrm{SDM},\mathbf{L}} \subseteq C_{\mathbf{L}}$ for all $a \in L$. C2 holds by Lemma 3.7.(ii). C3 holds by Lemma 3.7.(ii) and Lemma 3.5, which ensures us that $e^{\sim\prime} = \Diamond e \in C_p(\mathbf{L})$. From the latter fact, the equality $e^{\sim} \vee \Diamond e = 1$ and Lemma 3.7.(i), we obtain:

$$\begin{aligned} a \wedge b &\leq (a \vee b) \wedge (a \vee \Diamond e) \wedge (b \vee e^{\sim}) \\ &= (a \vee b) \wedge (e^{\sim} \vee b) \wedge (\Diamond e \vee a) \wedge (e^{\sim} \vee \Diamond e) \\ &= (b \vee (a \wedge e^{\sim})) \wedge (\Diamond e \vee (a \wedge e^{\sim})) \\ &= (a \wedge e^{\sim}) \vee (b \wedge \Diamond e), \end{aligned}$$

whence, also using the equality $e^{\sim} \wedge \Diamond e = 0$ and Lemma 3.5.(ii),

$$\begin{array}{lll} ((e^{\sim} \vee b) \wedge (\Diamond e \vee a))^{\sim} & = ((e^{\sim} \wedge a) \vee (\Diamond e \wedge b) \vee (b \wedge a))^{\sim} & \text{Lemma 3.7} \\ & = ((e^{\sim} \wedge a) \vee (\Diamond e \wedge b))^{\sim} & \\ & = (e^{\sim} \wedge a)^{\sim} \wedge (\Diamond e \wedge b)^{\sim} & \\ & = (e^{\sim} \vee b^{\sim}) \wedge (\Diamond e \vee a^{\sim}) & e^{\sim}, \Diamond e \in C_{pbz}(\mathbf{L}) \end{array}$$

Conversely, if $e^{\sim}$ is central in $\mathbf{L}$, then by C1 we have that for all $a \in L$, $a = (a \wedge e^{\sim}) \vee (a \wedge \Diamond e)$, whereby $e^{\sim} \in C_p(\mathbf{L})$. Now, recall that if $e^{\sim}$ is central, then so is $\Diamond e$, whereby C4 holds for both elements — i.e., for all $a, b \in L$:

(i) $((e^{\sim} \vee b) \wedge (\Diamond e \vee a))^{\sim} = (e^{\sim} \vee b^{\sim}) \wedge (\Diamond e \vee a^{\sim})$;

(ii) $((\Diamond e \vee b) \wedge (e^{\sim} \vee a))^{\sim} = (\Diamond e \vee b^{\sim}) \wedge (e^{\sim} \vee a^{\sim})$.

Letting $b = e^{\sim}$ in (i) and using Lemma 3.7.(ii), we have that $(e^{\sim} \wedge a)^{\sim} = ((e^{\sim} \vee e^{\sim}) \wedge (\Diamond e \vee a))^{\sim} = \Diamond e \vee a^{\sim}$. Similarly, letting $b = \Diamond e$ in (ii), we obtain $(\Diamond e \wedge a)^{\sim} = e^{\sim} \vee a^{\sim}$. □

3.2 Central Elements in the Variety Generated by Antiortholattices

In any member $\mathbf{L}$ of $\mathrm{V}(\mathbb{AOL})$ central elements are exactly the sharp elements: $C(\mathbf{L}) = S_K(\mathbf{L})$. Therefore, for any $a \in \mathbf{L}$, $\mathbf{L}$ is decomposable as $\mathbf{L}/Cg(a^{\sim}, 0) \times \mathbf{L}/Cg(a^{\sim}, 1)$, or equivalently as $\mathbf{L}/Cg(a^{\sim}, 0) \times \mathbf{L}/Cg(\Diamond a, 0)$. However, the mentioned results in [22] do not provide us with a uniform recipe to obtain an explicit description of the factors in this decomposition. This situation marks a sharp contrast with the case of orthomodular lattices, where the following result is available:

Theorem 3.9. *[5, Lm. 2.7] Let $\mathbf{L} \in \mathbb{OML}$ and let $e \in C(\mathbf{L})$. Then $\mathbf{L} \simeq \mathbf{L}_1 \times \mathbf{L}_2$, where $\mathbf{L}_1, \mathbf{L}_2$ are algebras whose universes are the intervals $[0, e]$ and $[0, e']$, respectively.*

In this subsection, we similarly characterise the factors in these decompositions in terms of algebras on intervals in $\mathbf{L}$.

Lemma 3.10. *Let $\mathbf{L} \in \mathrm{V}(\mathbb{AOL})$ and let $a \in L$. Define:*

$$\mathbf{L}_1 = \left([0, a^{\sim}], \wedge, \vee, '^1, {}^{\sim 1}, 0, a^{\sim}\right);$$
$$\mathbf{L}_2 = \left([0, \Diamond a], \wedge, \vee, '^2, {}^{\sim 2}, 0, \Diamond a\right),$$

where for all b in $[0, a^{\sim}]$, $b'^1 = b' \wedge a^{\sim}$, $b^{\sim 1} = b^{\sim} \wedge a^{\sim}$, while for all c in $[0, \Diamond a]$, $c'^2 = c' \wedge \Diamond a$, $c^{\sim 2} = c^{\sim} \wedge \Diamond a$. Then the algebras $\mathbf{L}_1$ and $\mathbf{L}_2$ are in $\mathrm{V}(\mathbb{AOL})$.

Proof. Clearly, $\mathbf{L}_1$ and $\mathbf{L}_2$ are bounded lattices. We now verify the remaining properties for $\mathbf{L}_1$; by replacing a by $a^{\sim}$, we obtain our claim for $\mathbf{L}_2$.

($\mathbf{L}_1$ is a pseudo-Kleene algebra). Let $b, c \in [0, a^{\sim}]$. Then, using Lemma 2.14.(iii), $b'^1{}'^1 = (b' \wedge a^{\sim})' \wedge a^{\sim} = (b \vee \Diamond a) \wedge a^{\sim} = b \wedge a^{\sim} = b$. Moreover, $(b \wedge c)'^1 = (b \wedge c)' \wedge a^{\sim} = (b' \vee c') \wedge a^{\sim} = (b' \wedge a^{\sim}) \vee (c' \wedge a^{\sim}) = b'^1 \vee c'^1$. Finally, since $b \wedge b' \leq c \vee c'$ in $\mathbf{L}$, we use Lemma 2.14 and obtain

$$b \wedge b'^1 = b \wedge b' \wedge a^{\sim} \leq \left(c \vee c'\right) \wedge (c \vee a^{\sim}) = c \vee \left(c' \wedge a^{\sim}\right) = c \vee c'^1.$$

($\mathbf{L}_1$ is paraorthomodular). Let $b, c \in [0, a^{\sim}]$. Suppose that $b \leq c$ and that $b'^1 \wedge c = b' \wedge a^{\sim} \wedge c = 0$. Since $c \leq a^{\sim}$, $b' \wedge a^{\sim} \wedge c = b' \wedge c$, whence paraorthomodularity of $\mathbf{L}$ yields the desired result.

($\mathbf{L}_1$ is a BZ-lattice). Let $b, c \in [0, a^{\sim}]$. Clearly, $b \wedge b^{\sim 1} = 0$. Further, $b^{\sim 1 \sim 1} = (b^{\sim} \wedge a^{\sim})^{\sim} \wedge a^{\sim}$. Since $b^{\sim}$ and $a^{\sim}$ are sharp elements, $(b^{\sim} \wedge a^{\sim})^{\sim} = (b^{\sim} \wedge a^{\sim})' = \Diamond b \vee \Diamond a$, hence $b^{\sim 1 \sim 1} = (b^{\sim} \wedge a^{\sim})^{\sim} \wedge a^{\sim} = (b^{\sim} \wedge a^{\sim})' \wedge a^{\sim} = b^{\sim 1 '1}$. By Lemma 2.14 $b^{\sim 1 \sim 1} = (\Diamond b \vee \Diamond a) \wedge a^{\sim} = \Diamond b \wedge a^{\sim} \geq b$. Finally, it is easily seen that if $b \leq c$, then $c^{\sim 1} \leq b^{\sim 1}$.

($\mathbf{L}_1$ is a BZ*-lattice). Let $b \in [0, a^{\sim}]$. Then, by Lemma 2.14 and WSDM, we have that:

$$\begin{aligned}
\left(b \wedge b'^1\right)^{\sim 1} &= (b \wedge b' \wedge a^{\sim})^{\sim} \wedge a^{\sim} \\
&= (b \wedge b')^{\sim} \wedge a^{\sim} \\
&= (b^{\sim} \vee \Box b) \wedge a^{\sim} \\
&= (b^{\sim} \wedge a^{\sim}) \vee (\Box b \wedge a^{\sim}) \\
&= (b^{\sim} \wedge a^{\sim}) \vee ((b' \wedge a^{\sim})^{\sim} \wedge a^{\sim}) \\
&= b^{\sim 1} \vee b'^{1 \sim 1}.
\end{aligned}$$

($\mathbf{L}_1 \in \mathrm{V}(\mathbb{AOL})$). By way of example, we check that the reformulation of AOL2 in terms of the new operations $^{\sim 1}$ and $'^1$ is satisfied in any antiortholattice $\mathbf{M}$. Thus, for $a, b, c \in \mathbf{M}$, consider the element

$$t^{\mathbf{M}}(a, b, c) = (b \wedge c^{\sim} \wedge a^{\sim}) \vee (b \wedge (c^{\sim} \wedge a^{\sim})^{\sim} \wedge a^{\sim}).$$

If $c > 0$, then $t^{\mathbf{M}}(a, b, c) = 0 \vee (b \wedge 1 \wedge a^{\sim}) = b \wedge a^{\sim}$. If $c = 0$, then

$$t^{\mathbf{M}}(a, b, c) = (b \wedge a^{\sim}) \vee (b \wedge \Diamond a \wedge a^{\sim}) = (b \wedge a^{\sim}) \vee 0 = b \wedge a^{\sim}.$$

Since $\mathbf{L} \in \mathrm{V}(\mathbb{AOL})$, it follows that for all $a, b, c \in L$, $t^{\mathbf{L}}(a, b, c) = b \wedge a^{\sim}$, which equals b whenever $b \leq a^{\sim}$, hence $\mathbf{L}_1$ satisfies AOL2. □

Theorem 3.11. *Let $\mathbf{L} \in \mathrm{V}(\mathbb{AOL})$ and let $a \in L$. Then $\mathbf{L} \simeq \mathbf{L}_1 \times \mathbf{L}_2$, where $\mathbf{L}_1, \mathbf{L}_2$ are defined as in Lemma 3.10.*

Proof. Let $\varphi : L \to L_1 \times L_2$ be defined, for any $b \in L$, by $\varphi(b) = (b \wedge a^{\sim}, b \wedge \Diamond a)$. We first show that φ is a bijection. If $\varphi(b) = \varphi(c)$ for some $b, c \in L$, then $b \wedge a^{\sim} = c \wedge a^{\sim}$ and $b \wedge \Diamond a = c \wedge \Diamond a$. Thus, by AOL2,

$$b = (b \wedge a^{\sim}) \vee (b \wedge \Diamond a) = (c \wedge a^{\sim}) \vee (c \wedge \Diamond a) = c.$$

Now, let $(x,y)\in L_1\times L_2$. Then $x\leq a^{\sim}$ and $y\leq \Diamond a$, whence $y\wedge a^{\sim}\leq \Diamond a\wedge a^{\sim}=0$. Let us compute $\varphi(x\vee y)$. Using Lemma 2.14, we obtain

$$\begin{aligned}(x\vee y)\wedge a^{\sim} &= (x\wedge a^{\sim})\vee(y\wedge a^{\sim})\\ &= x\vee 0=x.\end{aligned}$$

Similarly, $(x\vee y)\wedge\Diamond a=y$ and thus φ is onto.

Next, we show that φ preserves meets and the unary operations (observe that this is sufficient in virtue of Lemma 3.10). For meets,

$$\begin{aligned}\varphi\left(b\wedge^{\mathbf{L}}c\right) &= (b\wedge c\wedge a^{\sim}, b\wedge c\wedge\Diamond a)\\ &= (b\wedge a^{\sim}, b\wedge\Diamond a)\wedge^{\mathbf{L}_1\times\mathbf{L}_2}(c\wedge a^{\sim}, c\wedge\Diamond a)\\ &= \varphi(b)\wedge^{\mathbf{L}_1\times\mathbf{L}_2}\varphi(c).\end{aligned}$$

With regards to Kleene complements, resorting again to Lemma 2.14,

$$\begin{aligned}\varphi\left(b^{\prime\mathbf{L}}\right) &= (b'\wedge a^{\sim}, b'\wedge\Diamond a)\\ &= ((b'\vee\Diamond a)\wedge a^{\sim}, (b'\vee a^{\sim})\wedge\Diamond a)\\ &= \left((b\wedge a^{\sim})'\wedge a^{\sim}, (b\wedge\Diamond a)'\wedge\Diamond a\right)\\ &= \left((b\wedge a^{\sim})^{\prime 1}, (b\wedge\Diamond a)^{\prime 2}\right)\\ &= \varphi(b)^{\prime\mathbf{L}_1\times\mathbf{L}_2}.\end{aligned}$$

For Brouwer complements the computation is similar. It is safely left to the reader, who is warned that the WSDM identity $(x\wedge y^{\sim})^{\sim}\approx x^{\sim}\vee\Diamond y$ will be needed somewhere down the line. □

4. Ideal Theory

It is well-known from the theory of orthomodular lattices that $\mathbb{OML}$-ideals admit a manageable characterization in terms of lattice ideals *closed under perspectivity*, for short *p-ideals*: in other words, in terms of lattice ideals I of an orthomodular lattice $\mathbf{L}$ such that, whenever $a\in I$, then also $b\Cap a=b\wedge(b'\vee a)\in I$ for all $b\in L$ [5, Prop. 4.7]. The aim of this section is to generalise this idea within the expanded language of $\mathbb{PBZL}^*$. Unfortunately, in the general case these p-ideals do not even coincide with 0-classes of congruences — and, a fortiori, no isomorphism result between the lattices of p-ideals and of congruences can be attained. The situation improves if we restrict ourselves to the subvariety $\mathbb{SDM}$ of $\mathbb{PBZL}^*$, axiomatised relative to $\mathbb{PBZL}^*$ by the Strong De Morgan identity. In fact, in any $\mathbf{L} \in \mathbb{SDM}$ p-ideals coincide with Ursini $\mathbb{SDM}$-ideals, hence with 0-classes of congruences, given the fact that $\mathbb{PBZL}^*$ (and thus, all the more so, $\mathbb{SDM}$) is a 0-subtractive variety.

4.1 Ideals in PBZ*-lattices

We start by defining the notion of a p-ideal for generic PBZ*-lattices. Hereafter, whenever $\mathbf{L}$ is a PBZ*-lattice and $a, b \in L$, let $a \Cap b = a \wedge (a^{\sim} \vee b)$.

Definition 4.1. Let $\mathbf{L}$ be a PBZ*-lattice. $I \subseteq L$ is a *p-ideal* iff it is a lattice ideal of $\mathbf{L}$ s.t. if $a \in I$, then $\Diamond b \Cap \Diamond a \in I$ for all $b \in L$.

Lemma 4.2. *Let* $\mathbf{L}$ *be a PBZ*-lattice, let* I *be a p-ideal of* $\mathbf{L}$*, and let* $a, b \in L$*. Then: (i) if* $a \in I$*, then* $\Diamond a \in I$*; (ii)* $\Diamond a \Cap \Diamond b \in I$ *iff* $\Diamond b \Cap \Diamond a \in I$*.*

Proof. (i) Let $b = 1$ in Definition 4.1.
(ii) Suppose that $\Diamond b \wedge (b^{\sim} \vee \Diamond a) \in I$. Then, since I is a p-ideal,

$$\Diamond a \wedge (a^{\sim} \vee \Diamond (\Diamond b \wedge (b^{\sim} \vee \Diamond a))) = \Diamond a \wedge (a^{\sim} \vee (\Diamond b \wedge (b^{\sim} \vee \Diamond a))) \in I.$$

By Theorem 3.1 applied to $\mathbf{S}_K(\mathbf{L})$, however,

$$\Diamond a \wedge (a^{\sim} \vee (\Diamond b \wedge (b^{\sim} \vee \Diamond a))) = \Diamond a \wedge (a^{\sim} \vee \Diamond b),$$

whence our conclusion. □

We now aim at defining a customary notion of equivalence between elements in a PBZ*-lattice modulo a given p-ideal. Note that $\{0\}$ is a p-ideal; thus,

Definition 4.3. Let $\mathbf{L}$ be a PBZ*-lattice, and let I be a p-ideal of $\mathbf{L}$. The elements $a, b \in L$ are said to be *I-modally equivalent* iff

$$(\Diamond a)^{\sim} \Cap \Diamond b, (\Diamond b)^{\sim} \Cap \Diamond a, (\Box a)^{\sim} \Cap \Box b, (\Box b)^{\sim} \Cap \Box a \in I.$$

The elements $a, b \in L$ are said to be *modally equivalent* iff they are $\{0\}$-modally equivalent, namely iff $\Diamond a = \Diamond b$ and $\Box a = \Box b$.

Definition 4.4. Let $\mathbf{L}$ be a PBZ*-lattice, and let I be a p-ideal of $\mathbf{L}$. We define the following binary relation on L:

$$\rho(I) = \{(a,b) \in L : (\Diamond a)^{\sim} \Cap \Diamond b, (\Diamond b)^{\sim} \Cap \Diamond a, (\Box a)^{\sim} \Cap \Box b, (\Box b)^{\sim} \Cap \Box a \in I\}.$$

Thus, $(a,b) \in \rho(I)$ iff a and b are I-modally equivalent, and, in particular, $(a,b) \in \rho(\{0\})$ iff a and b are modally equivalent. The relation $\rho(I)$ also admits a less cumbersome description:

Theorem 4.5. *Let $\mathbf{L}$ be a PBZ*-lattice, and let I be a p-ideal of $\mathbf{L}$. For $a, b \in L$ the following conditions are equivalent:*

(i) $(\Diamond a \vee \Diamond b) \wedge (a^{\sim} \vee b^{\sim}), ((\Box a)^{\sim} \vee (\Box b)^{\sim}) \wedge (\Box a \vee \Box b) \in I$*;*

(ii) there exist $s, t \in I$ such that $\Diamond a \vee s = \Diamond b \vee s$ and $\Box a \vee t = \Box b \vee t$;

(iii) $(a,b) \in \rho(I)$*.*

Proof. (i) implies (ii).

Let $s = (\Diamond a \vee \Diamond b) \wedge (a^{\sim} \vee b^{\sim})$ and $t = ((\Box a)^{\sim} \vee (\Box b)^{\sim}) \wedge (\Box a \vee \Box b)$. Then:

$$\begin{aligned} \Diamond a \vee s &= \Diamond a \vee ((\Diamond a \vee \Diamond b) \wedge (a^{\sim} \vee b^{\sim})) \\ &= \Diamond a \vee \Diamond b && \text{(Thm. 3.1 in } \mathbf{S}_K(\mathbf{L})\text{)} \\ &= \Diamond b \vee s. \end{aligned}$$

Similarly, $\Box a \vee t = \Box b \vee t$.

(ii) implies (iii).

Suppose that there exist $s, t \in I$ such that $\Diamond a \vee s = \Diamond b \vee s$ and $\Box a \vee t = \Box b \vee t$. First, we prove $(\Diamond b)^{\sim} \Cap \Diamond a \in I$.

$$\begin{aligned} (\Diamond b)^{\sim} \Cap \Diamond a &= b^{\sim} \wedge (\Diamond b \vee \Diamond a) \\ &\leq b^{\sim} \wedge (\Diamond s \vee \Diamond b \vee \Diamond a) \\ &= b^{\sim} \wedge (\Diamond (s \vee \Diamond a) \vee \Diamond b) \\ &= b^{\sim} \wedge (\Diamond (s \vee \Diamond b) \vee \Diamond b) \\ &= b^{\sim} \wedge (\Diamond b \vee \Diamond s) \\ &= (\Diamond b)^{\sim} \Cap \Diamond s. \end{aligned}$$

Since I is a p-ideal of $\mathbf{L}$ and $s \in I$, we have that $(\Diamond b)^{\sim} \Cap \Diamond s \in I$ and therefore $(\Diamond b)^{\sim} \Cap \Diamond a \in I$. The remaining conditions are proved similarly and thus $(a,b) \in \rho(I)$.

(iii) implies (i).

Suppose $(a,b) \in \rho(I)$; we prove that $(\Diamond a \vee \Diamond b) \wedge (a^{\sim} \vee b^{\sim}) \in I$. By assumption,

$$(a^{\sim} \wedge (\Diamond a \vee \Diamond b)) \vee (b^{\sim} \wedge (\Diamond b \vee \Diamond a)) = ((\Diamond a)^{\sim} \Cap \Diamond b) \vee ((\Diamond b)^{\sim} \Cap \Diamond a) \in I.$$

By Theorem 3.1 applied to $\mathbf{S}_K(\mathbf{L})$, however,

$$(a^{\sim} \wedge (\Diamond a \vee \Diamond b)) \vee (b^{\sim} \wedge (\Diamond b \vee \Diamond a)) = (\Diamond a \vee \Diamond b) \wedge (a^{\sim} \vee b^{\sim}),$$

whence our claim follows. A similar proof establishes the other claim. □

Theorem 4.6. *Let* $\mathbf{L}$ *be a PBZ*-lattice, and let* I *be a p-ideal of* $\mathbf{L}$*. Then* $\rho(I)$ *is an equivalence relation on* L *that preserves the operations* $'$ *and* $^{\sim}$*.*

Proof. Since, for all $a \in L$, $(\Diamond a)^{\sim} \Cap \Diamond a, (\Box a)^{\sim} \Cap \Box a = 0 \in I$, $\rho(I)$ is reflexive. Symmetry is trivial. For transitivity, suppose $(a,b), (b,c) \in \rho(I)$. By Theorem 4.5, there exist:

- $s_1, t_1 \in I$ such that $\Diamond a \vee s_1 = \Diamond b \vee s_1$ and $\Box a \vee t_1 = \Box b \vee t_1$;
- $s_2, t_2 \in I$ such that $\Diamond b \vee s_2 = \Diamond c \vee s_2$ and $\Box b \vee t_2 = \Box c \vee t_2$.

Thus $s_1 \vee s_2 \in I$ and $\Diamond a \vee s_1 \vee s_2 = \Diamond b \vee s_1 \vee s_2 = \Diamond c \vee s_1 \vee s_2$, and similarly for the other condition, whence by Theorem 4.5 again, $(a,c) \in \rho(I)$. The unary operations are clearly preserved. □

Although $\rho(I)$ is always an equivalence relation, it need not always be a congruence, as the next example shows.

Example 4.7. Consider the distributive antiortholattice whose lattice reduct is the ordinal sum of $\mathbf{D}_2^2$ with itself, with atoms a, b and the fixpoint $c = c'$. Observe that $(a,c) \in \rho(\{0\})$, because $\Diamond a = \Diamond c = 1$ and $\Box a = \Box c = 0$. However, $\Diamond(a \wedge b) = \Diamond 0 = 0$ and $\Diamond(c \wedge b) = \Diamond b = 1$, whence $\rho(\{0\})$ does not preserve meets.

Our next goal is to tweak the notion of p-ideal in such a way that its associated equivalence is necessarily a congruence.

Definition 4.8. Let $\mathbf{L}$ be a PBZ*-lattice, and let I be a p-ideal of $\mathbf{L}$. I is a *weak De Morgan ideal* iff for all $a, b \in L$, whenever $(a,b) \in \rho(I)$, then for all $c \in L$ it is the case that $\Diamond(a \wedge c)^{\sim} \Cap \Diamond(b \wedge c) \in I$.

Lemma 4.9. *Let* **L** *be a PBZ*-lattice, and let I be a p-ideal of* **L**. *The following conditions are equivalent:*

(i) $\rho(I)$ is a congruence;

(ii) I is a weak De Morgan ideal.

Proof. (i) implies (ii). Suppose that $\rho(I)$ is a congruence and let $(a,b) \in \rho(I)$, $c \in L$. Then $(\Diamond(a \wedge c), \Diamond(b \wedge c)) \in \rho(I)$. It follows that $(\Diamond(a \wedge c)^{\sim} \Cap \Diamond(b \wedge c), 0) \in \rho(I)$, which implies $\Diamond(a \wedge c)^{\sim} \Cap \Diamond(b \wedge c) \in I$.

(ii) implies (i). Let I be a weak De Morgan ideal. By Theorem 4.6, to attain our conclusion it will suffice to show that $\rho(I)$ preserves meets. Thus, let $(a,b) \in \rho(I)$ and $c \in L$. In virtue of our assumption, $\Diamond(a \wedge c)^{\sim} \Cap \Diamond(b \wedge c) \in I$ and, taking into account the symmetry of $\rho(I)$, $\Diamond(b \wedge c)^{\sim} \Cap \Diamond(a \wedge c) \in I$. It remains to show that

$$(\Box(a \wedge c))^{\sim} \Cap \Box(b \wedge c), (\Box(b \wedge c))^{\sim} \Cap \Box(a \wedge c) \in I.$$

However, since $(a,b) \in \rho(I)$ and $\rho(I)$ preserves the unary operations, $(\Box a, \Box b) \in \rho(I)$. Given that I is a weak De Morgan ideal, thus,

$$\begin{aligned}(\Box(a \wedge c))^{\sim} \Cap \Box(b \wedge c) &= \Diamond(a' \vee c') \Cap (b' \vee c')^{\sim} \\ &= (\Box a \wedge \Box c)^{\sim} \Cap \Diamond(\Box b \wedge \Box c) \\ &= \Diamond(\Box a \wedge \Box c)^{\sim} \Cap \Diamond(\Box b \wedge \Box c) \in I.\end{aligned}$$

Similarly, $(\Box(b \wedge c))^{\sim} \Cap \Box(a \wedge c) \in I$. □

4.2 Ideals in the Strong De Morgan Subvariety

The subvariety of $\mathbb{PBZL}^*$ that is axiomatised relative to $\mathbb{PBZL}^*$ by the Strong De Morgan law SDM, here labelled $\mathbb{SDM}$, includes $\mathbb{OML}$ and stands out for its smooth theory of ideals. In fact, we have that:

Lemma 4.10. *Let* $\mathbf{L} \in \mathbb{SDM}$*, and let I be a p-ideal of* **L**. *Then I is a weak De Morgan ideal and therefore $\rho(I)$ is a congruence.*

Proof. If $(a,b) \in \rho(I)$, then $(a^{\sim}, b^{\sim}) \in \rho(I)$ by Theorem 4.6. Thus, Theorem 4.5 guarantees that there is $s \in I$ such that $a^{\sim} \vee s = \Diamond a^{\sim} \vee s = \Diamond b^{\sim} \vee s = b^{\sim} \vee s$. Then, for an arbitrary $c \in L$, $a^{\sim} \vee c^{\sim} \vee s = b^{\sim} \vee$

$c^{\sim} \vee s$. Applying SDM, we have that $\Box\left(\left(a \wedge c\right)^{\sim}\right) \vee s = \left(a \wedge c\right)^{\sim} \vee s = \left(b \wedge c\right)^{\sim} \vee s = \Box\left(\left(b \wedge c\right)^{\sim}\right) \vee s$. A further recourse to Theorem 4.5 yields $\left(\left(a \wedge c\right)^{\sim}, \left(b \wedge c\right)^{\sim}\right) \in \rho\left(I\right)$, whence $\left(\Diamond\left(a \wedge c\right), \Diamond\left(b \wedge c\right)\right) \in \rho\left(I\right)$, which implies, in particular, that I is weak De Morgan. Lemma 4.9 takes care of the remaining claim. $\Box$

We are now in a position to prove that within the boundaries of this subvariety, p-ideals coincide with ideals in the sense of Ursini.

Theorem 4.11. *If* $\mathbf{L} \in \mathbb{SDM}$, *then the class of p-ideals of* $\mathbf{L}$ *coincides with* $\mathcal{I}_{\mathbb{SDM}}\left(\mathbf{L}\right)$.

Proof. Let $I \in \mathcal{I}_{\mathbb{SDM}}\left(\mathbf{L}\right)$, whence by Theorem 2.20.(i) $I = 0/\theta$ for some $\theta \in \mathrm{Con}_{\mathbb{BZL}}\left(\mathbf{L}\right)$. Clearly, I is a lattice ideal of $\mathbf{L}$. Furthermore, if $a \in I$, then $(\Diamond a, 0) \in \theta$ and then $\Diamond a \in I$. What remains to show is that, for an arbitrary $b \in L$, $\Diamond b \Cap \Diamond a \in I$. Since $\Diamond a \in I = 0/\theta$, $(\Diamond b \Cap \Diamond a, 0) = (\Diamond b \Cap \Diamond a, \Diamond b \Cap 0) \in \theta$, which means $\Diamond b \Cap \Diamond a \in I$. Conversely, it will be enough to prove that if I is a p-ideal of $\mathbf{L}$, then $I = 0/\rho\left(I\right)$. However, by Theorem 4.5,

$$\begin{aligned} 0/\rho\left(I\right) &= \left\{a \in L : (a, 0) \in \rho\left(I\right)\right\} \\ &= \left\{a \in L : \Diamond a \leq s, \Box a \leq t \text{ for some } s, t \in I\right\}. \end{aligned}$$

If $a \in I$, then choose $s = t = \Diamond a \in I$ to obtain $a \in 0/\rho\left(I\right)$. If $a \in 0/\rho\left(I\right)$, then there is $s \in I$ such that $a \leq \Diamond a \leq s$, whence $a \in I$. $\Box$

Observe that, by Lemma 4.10 and Theorem 4.11, whenever $\mathbf{L} \in \mathbb{SDM}$, all members of $\mathcal{I}_{\mathbb{SDM}}\left(\mathbf{L}\right)$ are weak De Morgan ideals.

Theorem 4.12. *Let* $\mathbf{L} \in \mathbb{SDM}$, *and let* $I \in \mathcal{I}_{\mathbb{SDM}}\left(\mathbf{L}\right)$. *Then* $\rho\left(I\right) = I^{\varepsilon}$.

Proof. By the proof of Theorem 4.11 $0/\rho\left(I\right) = I$, whence $\rho\left(I\right) \subseteq I^{\varepsilon}$. For the converse inequality, suppose $(a, b) \in I^{\varepsilon}$. Since I^{ε} is a congruence,

$$\left(\left(\Diamond a \vee \Diamond b\right) \wedge \left(a^{\sim} \vee b^{\sim}\right), 0\right), \left(\left(\Box a \vee \Box b\right) \wedge \left(\left(\Box a\right)^{\sim} \vee \left(\Box b\right)^{\sim}\right), 0\right) \in I^{\varepsilon}.$$

So $\left(\Diamond a \vee \Diamond b\right) \wedge \left(a^{\sim} \vee b^{\sim}\right) \in 0/I^{\varepsilon} = I$ and $\left(\Box a \vee \Box b\right) \wedge \left(\left(\Box a\right)^{\sim} \vee \left(\Box b\right)^{\sim}\right) \in 0/I^{\varepsilon} = I$. By Theorem 4.5, this means that $(a, b) \in \rho\left(I\right)$. $\Box$

Corollary 4.13. $\mathbb{SDM}$ *is finitely congruential.*

Proof. We have to find a finite set of terms $\{d_i(x,y)\}_{i\leq n}$ that witnesses finite congruentiality according to Definition 2.22. Thus, let

$$d_1(x,y) = (\Diamond x)^{\sim} \Cap \Diamond y, \quad d_2(x,y) = (\Diamond y)^{\sim} \Cap \Diamond x,$$
$$d_3(x,y) = (\Box x)^{\sim} \Cap \Box y, \quad d_4(x,y) = (\Box y)^{\sim} \Cap \Box x.$$

If $\mathbf{L} \in \mathbb{SDM}$ and $I \in \mathcal{I}_{\mathbb{SDM}}(\mathbf{L})$, then by Theorems 4.12 and 4.5 $\rho(I) = I^{\varepsilon}$. As a result, $(a,b) \in I^{\varepsilon} = \rho(I)$ iff $d_i^{\mathbf{A}}(a,b) \in I$, for all $i \leq 4$. □

Theorem 4.14. *The* 0*-assertional logic of* $\mathbb{PBZL}^*$ *is not equivalential.*

Proof. Consider again the antiortholattice of Example 4.7. Being simple, this antiortholattice belongs to $\mathbb{PBZL}^*_{\varepsilon}$. Moreover, the set $\{0, a, a', 1\}$ is a subuniverse of such, isomorphic to $\mathbf{D}_4$, and its middle congruence, that collapses only a and a', is a nonzero pseudo-identical congruence. Our claim follows then from Theorem 2.24. □

Lemma 4.15. *Let* $\mathbf{L} \in \mathbb{SDM}$*. The following are equivalent:*

(i) $\rho(\{0\}) = \Delta_{\mathbf{L}}$.

(ii) $\mathbf{L}$ *satisfies the quasi-identity* $\Box x \leq \Box y \,\&\, \Diamond x \leq \Diamond y \Rightarrow x \leq y$.

(iii) $\mathbf{L}$ *satisfies the identity SK.*

Proof. (i) implies (ii). $\rho(\{0\}) = \Delta_{\mathbf{L}}$ means that modally equivalent elements of L are identical. Now, let $\Box a \leq \Box b$ and $\Diamond a \leq \Diamond b$. By SDM, this implies that $a \wedge b$ and a are modally equivalent, whence $a \leq b$.

(ii) implies (iii). Using SDM, we have that for all $a, b \in L$,

$$\Box(a \wedge \Diamond b) = \Box a \wedge \Diamond b \leq \Box a \vee \Box b = \Box(\Box a \vee b);$$
$$\Diamond(a \wedge \Diamond b) = \Diamond a \wedge \Diamond b \leq \Box a \vee \Diamond b = \Diamond(\Box a \vee b).$$

By our assumption, then, $a \wedge \Diamond b \leq \Box a \vee b$.

(iii) implies (i). Suppose $\Box a = \Box b$ and $\Diamond a = \Diamond b$. Then

$$a = a \wedge \Diamond a = a \wedge \Diamond b \leq \Box a \vee b = \Box b \vee b = b.$$

Similarly, $b \leq a$, whence our conclusion. □

Let us call $\mathbb{SK}$ the subvariety of $\mathbb{SDM}$ that is axiomatised relative to $\mathbb{SDM}$ by the identity SK.

Theorem 4.16. *The* 0*-assertional logic of* $\mathbb{SDM}$ *is strongly algebraisable with equivalent variety semantics* $\mathbb{SDM}_\varepsilon = \mathbb{SK}$.

Proof. By Theorem 2.25, it suffices to establish that $\mathbb{SDM}_\varepsilon$ is a variety, which would follow if we were to show that $\mathbb{SDM}_\varepsilon = \mathbb{SK}$. By Theorem 4.12, whenever $\mathbf{L}$ belongs to $\mathbb{SDM}$, $\{0\}^\varepsilon = \rho(\{0\})$. Thus $\mathbf{L} \in \mathbb{SDM}_\varepsilon$ iff $\rho(\{0\}) = \Delta_{\mathbf{L}}$, and by Lemma 4.15, this happens exactly when $\mathbf{L} \in \mathbb{SK}$. □

By Theorems 2.23 and 4.11, in any member $\mathbf{L}$ of $\mathbb{SDM}$ the Ursini ideals of $\mathbf{L}$ coincide with its p-ideals and with the deductive filters on $\mathbf{L}$ of the 0-assertional logic of $\mathbb{SDM}$. By [13, Thm. 3.58], therefore, we obtain:

Corollary 4.17. *Let* $\mathbf{L} \in \mathbb{SDM}$. *Then the lattice of p-ideals of* $\mathbf{L}$ *is isomorphic to the lattice of all congruences* θ *on* $\mathbf{L}$ *such that* $\mathbf{L}/\theta \in \mathbb{SK}$.

Observe that, although SK implies SDM in the context of $\mathrm{V}(\mathbb{AOL})$ [15, Lm. 3.8] this is not the case in the more general context of $\mathbb{PBZL}^*$. In fact, consider the PBZ*-lattice $\mathbf{L}$ whose lattice reduct is the 5-element modular and non-distributive lattice $\mathbf{M}_3$ with atoms a, a', b, where $b = b'$ and $b^\sim = 0$. Then $\mathbf{L}$ satisfies SK but fails SDM - actually, it fails even WSDM because $a \in S_K(\mathbf{L})$ and $(a \wedge b)^\sim = 1$ but $a^\sim \vee b^\sim = a'$. For future reference, we make a note of the fact that $\mathbf{L}$ satisfies J2.

4.3 Ideals in $\mathrm{V}(\mathbb{AOL})$

Another subvariety of $\mathbb{PBZL}^*$ where our description of ideals can be considerably simplified is the variety $\mathrm{V}(\mathbb{AOL})$ generated by all antiortholattices. Bignall and Spinks first observed that the variety of distributive BZ-lattices is a binary discriminator variety [4]. We extend their observation by noticing that $\mathrm{V}(\mathbb{AOL})$ is itself a binary discriminator variety.

Proposition 4.18. $\mathrm{V}(\mathbb{AOL})$ *is a* 0*-binary discriminator variety.*

Proof. Referring to Definition 2.26 for notation and terminology, let $b_0^L(x,y) = x \wedge y^\sim$. Then for any antiortholattice $\mathbf{L}$ and, for any $a, c \in L$, $b_0^L(a,0) = a \wedge 0^\sim = a \wedge 1 = a$, while, if $c > 0$, then $b_0^L(a,c) = a \wedge c^\sim = a \wedge 0 = 0$. □

Among the consequences of this remark we have a very slender description of $\mathrm{V}(\mathbb{AOL})$-ideals. In fact, recall from [4] that if $\mathbf{A}$ is an algebra in a 0-binary discriminator variety $\mathbb{V}$ and $b_0(x, y)$ is the term witnessing this property for $\mathbb{V}$, $I \subseteq A$ is a $\mathbb{V}$-ideal of $\mathbf{A}$ exactly when, for any $a, c \in A$, if $c \in I$ and $b_0^{\mathbf{A}}(a, c) \in I$, then $a \in I$. Therefore:

Proposition 4.19. *Let* $\mathbf{L} \in \mathrm{V}(\mathbb{AOL})$. *For a lattice ideal* $I \subseteq L$ *the following are equivalent:*

(i) I *is a* $\mathrm{V}(\mathbb{AOL})$*-ideal.*

(ii) For any $a, b \in L$, *if* $b \in I$ *and* $a \wedge b^{\sim} \in I$, *then* $a \in I$.

(iii) I *is closed w.r.t. all interpretations in* $\mathbf{L}$ *of the* $\mathrm{V}(\mathbb{AOL})$*-ideal term (in* y, z*):*

$$u(x, y, z) = x \wedge \Diamond (y \vee z).$$

Proof. The equivalence of (i) and (ii) follows from the remarks immediately preceding this lemma. Suppose now that (ii) holds, and that $a, b \in I$. Then $a \vee b \in I$. On the other hand, for any c in L,

$$0 = c \wedge \Diamond (a \vee b) \wedge (a \vee b)^{\sim} \in I,$$

whence by our hypothesis $c \wedge \Diamond (a \vee b) \in I$. Conversely, under the assumption (iii), let $b \in I$ and $a \in L$ be such that $a \wedge b^{\sim} \in I$. Then, using WSDM and Lemma 2.14 several times,

$$\begin{aligned} u^{\mathbf{A}}(a, b, a \wedge b^{\sim}) &= a \wedge \Diamond (b \vee (a \wedge b^{\sim})) \\ &= a \wedge (\Diamond b \vee \Diamond (a \wedge b^{\sim})) \\ &= a \wedge (\Diamond b \vee (\Diamond a \wedge b^{\sim})) \\ &= a \wedge (\Diamond b \vee \Diamond a) \\ &= (a \wedge \Diamond b) \vee a = a, \end{aligned}$$

hence $a \in I$. □

5. Axiomatic Bases for Some Subvarieties

The goal of this final section is to simplify the axiomatisation of $\mathrm{V}(\mathbb{AOL})$ given in [14] and to solve a problem (here called the Join Problem) posed in [15], where it was observed that the varietal join $\mathbb{OML} \vee \mathrm{V}(\mathbb{AOL})$ in the lattice of subvarieties of $\mathbb{PBZL}^*$ was strictly included in $\mathbb{PBZL}^*$, but no axiomatic basis for such a join was given.

5.1 A Streamlined Axiomatisation for $\mathrm{V}(\mathbb{AOL})$

In Theorem 2.15.(i) we recalled that an equational basis for $\mathrm{V}(\mathbb{AOL})$ relative to $\mathbb{PBZL}^*$ is given by the identities AOL1-AOL3, here reproduced for the reader's convenience:

$$\begin{aligned}
&(\mathrm{AOL1})\ (x^{\sim} \vee y^{\sim}) \wedge (\Diamond x \vee z^{\sim}) \approx ((x^{\sim} \vee y) \wedge (\Diamond x \vee z))^{\sim}\,;\\
&(\mathrm{AOL2})\ x \approx (x \wedge y^{\sim}) \vee (x \wedge \Diamond y)\,;\\
&(\mathrm{AOL3})\ x \approx (x \vee y^{\sim}) \wedge (x \vee \Diamond y)\,.
\end{aligned}$$

The aim of this subsection is showing that AOL2 suffices to derive the remaining two axioms. For a start, we notice that Lemma 2.14 does not depend on AOL1, whence it holds for any subvariety of $\mathbb{PBZL}^*$ that satisfies AOL2 and AOL3.

Lemma 5.1. *Let* $\mathbf{L}$ *be a member of* $\mathbb{PBZL}^*$ *that satisfies AOL2 and AOL3. Then, for any* $a, b, c \in L$*: (i)* $a \wedge b \leq (a \wedge c^{\sim}) \vee (b \wedge \Diamond c)$*; (ii)* $(a \wedge \Diamond b)^{\sim} \vee \Diamond b = 1$*; (iii)* $\Diamond a \leq (b \wedge a^{\sim})^{\sim}$*.*

Proof. (i) In fact, using Lemma 2.14,

$$a \wedge b \leq (a \vee c^{\sim}) \wedge (b \vee \Diamond c) \wedge (a \vee b) = (a \wedge c^{\sim}) \vee (b \wedge \Diamond c)\,.$$

(ii) Since $a \wedge \Diamond b \leq \Diamond b$, it follows that $b^{\sim} \leq (a \wedge \Diamond b)^{\sim}$, whence $1 = b^{\sim} \vee \Diamond b \leq (a \wedge \Diamond b)^{\sim} \vee \Diamond b$.

(iii) Since $b \wedge a^{\sim} \leq a^{\sim}$, our conclusion follows. □

Lemma 5.2. *Let* $\mathbf{L}$ *be a member of* $\mathbb{PBZL}^*$ *that satisfies AOL2 and AOL3. Then* $\mathbf{L}$ *satisfies AOL1 iff it satisfies WSDM.*

Proof. From left to right, WSDM can be obtained by taking $y = 0$ and applying Lemma 2.14.(iii). Conversely, let $\mathbf{L}$ satisfy WSDM, and let $a, b, c \in L$. Then:

$$\begin{aligned}
((a^{\sim} \vee b) \wedge (\Diamond a \vee c))^{\sim} &= ((a^{\sim} \wedge c) \vee (\Diamond a \wedge b) \vee (b \wedge c))^{\sim} && \text{Lm. 2.14}\\
&= ((a^{\sim} \wedge c) \vee (\Diamond a \wedge b))^{\sim} && \text{Lm. 5.1.(i)}\\
&= (a^{\sim} \wedge c)^{\sim} \wedge (\Diamond a \wedge b)^{\sim} && \text{Lm. 2.6.(iii)}\\
&= (a^{\sim} \vee b^{\sim}) \wedge (\Diamond a \vee c^{\sim}) && \text{WSDM}
\end{aligned}$$

□

Theorem 5.3. *An equational basis for* $\mathrm{V}(\mathbb{AOL})$ *relative to* $\mathbb{PBZL}^*$ *is given by the single identity AOL2.*

Proof. By Theorem 2.15.(i), an equational basis for $\mathrm{V}(\mathbb{AOL})$ relative to $\mathbb{PBZL}^*$ is given by the identities AOL1-AOL3. To attain our conclusion, taking into account Lemma 5.2, it will suffice to show that: i) any subvariety of $\mathbb{PBZL}^*$ that satisfies AOL2 and AOL3 also satisfies WSDM; ii) any subvariety of $\mathbb{PBZL}^*$ that satisfies AOL2 also satisfies AOL3. We establish these claims in reverse order.

i) Let $\mathbf{L}$ belong to any subvariety of $\mathbb{PBZL}^*$ that satisfies AOL2, and let $a, b \in L$. Then $a' = (a' \wedge b^{\sim}) \vee (a' \wedge \Diamond b)$, whence

$$a = \left(\left(a' \wedge b^{\sim}\right) \vee \left(a' \wedge \Diamond b\right)\right)' = \left(a' \wedge b^{\sim}\right)' \wedge \left(a' \wedge \Diamond b\right)' = (a \vee \Diamond b) \wedge (a \vee b^{\sim}) .$$

ii) Let $\mathbf{L}$ belong to any subvariety of $\mathbb{PBZL}^*$ that satisfies AOL2 (thus also AOL3, by the previous item), and let $a, b \in L$. Then:

$$\begin{array}{lll} a^{\sim} \vee \Diamond b & = ((a \wedge b^{\sim}) \vee (a \wedge \Diamond b))^{\sim} \vee \Diamond b & \text{AOL2} \\ & = ((a \wedge b^{\sim})^{\sim} \wedge (a \wedge \Diamond b)^{\sim}) \vee \Diamond b & \text{Lm. 2.6.(iii)} \\ & = ((a \wedge b^{\sim})^{\sim} \vee \Diamond b) \wedge ((a \wedge \Diamond b)^{\sim} \vee \Diamond b) & \text{Lm. 2.14} \\ & = ((a \wedge b^{\sim})^{\sim} \vee \Diamond b) & \text{Lm. 5.1.(ii)} \\ & = (a \wedge b^{\sim})^{\sim} & \text{Lm. 5.1.(iii).} \end{array}$$

□

Taking into account Lemma 3.5 and Theorem 3.8, we have that:

Corollary 5.4. *(i)* $\mathrm{V}(\mathbb{AOL})$ *is the class of all PBZ*-lattices* $\mathbf{L}$ *such that* $C_p(\mathbf{L}) = S_K(\mathbf{L})$.

(ii) $\mathrm{V}(\mathbb{AOL})$ *is the class of all PBZ*-lattices* $\mathbf{L}$ *that satisfy WSDM and are such that* $C_{pbz}(\mathbf{L}) = S_K(\mathbf{L})$.

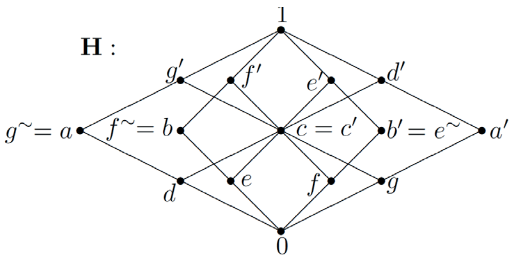
(iii) The class of the directly indecomposable members of $\mathrm{V}(\mathbb{AOL})$ *is* $\mathbb{AOL}$.

5.2 The Join Problem

We round off this paper by axiomatising the variety $\mathbb{OML} \vee \mathrm{V}(\mathbb{AOL})$, as well as some of its notable subvarieties. Let:

- $\mathbb{V}_1$ be the variety of PBZ*-lattices that is axiomatised relative to $\mathbb{PBZL}^*$ by the identities J2 and WSDM;
- $\mathbb{V}_2$ be the variety of PBZ*-lattices that is axiomatised relative to $\mathbb{PBZL}^*$ by the identities J2 and SDM;
- $\mathbb{V}_3$ be the variety of PBZ*-lattices that is axiomatised relative to $\mathbb{PBZL}^*$ by the identities J2, WSDM, and SK.

Taking into account the results in [15] and [14], as well as Theorem 5.3, in $V(\mathbb{AOL})$, SK implies SDM, that is AOL2 and SK imply SDM, while AOL2 and SDM do not imply SK. We observed in Section 4 that J2 and SK do not imply WSDM (all the more so, thus, SDM). The following PBZ*-lattice:



satisfies SK and SDM and fails J2. The 4-element antiortholattice chain $\mathbf{D}_4$ fails SK but satisfies SDM and AOL2, thus also J2. Therefore, in the sets of axioms {J2, SK, SDM} and {J2, SK, WSDM}, each axiom is independent from the other two.

Given any $\mathbf{L} \in \mathbb{PBZL}^*$, it will be expedient to denote by $T(\mathbf{L})$ the set $\{x \in L : x^{\sim} = 0\} \cup \{0\}$.

Lemma 5.5. *Let* $\mathbf{L}$ *be a PBZ*-lattice that satisfies WSDM and such that* $S_K(\mathbf{L}) \cup T(\mathbf{L}) = L$*. Then if* $b \in S_K(\mathbf{L})$ *and* $c \notin S_K(\mathbf{L})$*, it follows that either* $b = 1$ *or* $b \leq c$*.*

Proof. If $b \wedge c \in S_K(\mathbf{L})$, then

$$b \wedge c = \Diamond (b \wedge c) = \Diamond b \wedge \Diamond c = b \wedge 1 = b,$$

where WSDM can be applied to obtain the second equality because $b \in S_K(\mathbf{L})$, while the third equality follows from the fact that $c \notin S_K(\mathbf{L})$, whence $c^{\sim} = 0$. On the other hand, if $b \wedge c \notin S_K(\mathbf{L})$, then we apply again WSDM (as $b \in S_K(\mathbf{L})$) and the assumption that $S_K(\mathbf{L}) \cup T(\mathbf{L}) = L$, obtaining

$$0 = (b \wedge c)^{\sim} = b^{\sim} \vee c^{\sim} = b^{\sim} \vee 0 = b^{\sim},$$

whereby $b = 1$ since $b \in S_K(\mathbf{L})$. □

Proposition 5.6. *Any directly indecomposable* $\mathbf{L} \in \mathbb{V}_1$ *is either orthomodular or an antiortholattice.*

Proof. Let $\mathbf{L}$ be as in the statement of the proposition, and suppose that $\mathbf{L}$ is directly indecomposable, but is neither orthomodular nor an antiortholattice. By Theorem 3.8, the only PBZ*-central elements of $\mathbf{L}$ are 0 and 1. By WSDM and Lemma 3.5.(iii) we conclude that 0 and 1 are the only sharp elements a such that $b = (b \wedge a^{\sim}) \vee (b \wedge \Diamond a)$ for all $b \in L$.

Now, we want to show that $S_K(\mathbf{L}) \cup T(\mathbf{L}) = L$. Let $x \in L$. The element $\Diamond (x \wedge x')$ is sharp and, by J2, we have that

$$b = \left(b \wedge (x \wedge x')^{\sim}\right) \vee \left(b \wedge \Diamond (x \wedge x')\right) = \left(b \wedge \Diamond (x \wedge x')^{\sim}\right) \vee \left(b \wedge \Diamond\Diamond (x \wedge x')\right)$$

for all $b \in L$. So, $\Diamond (x \wedge x') \in \{0, 1\}$. If $\Diamond (x \wedge x') = 0$, then $x \wedge x' \leq \Diamond (x \wedge x') = 0$, whence $x \in S_K(\mathbf{L})$. If $\Diamond (x \wedge x') = 1$, then

$$x^{\sim} \leq x^{\sim} \vee \Box x = (x \wedge x')^{\sim} = 0,$$

and $x \in T(\mathbf{L})$. Our claim is therefore settled.

Recall that $\mathbf{L}$ is directly indecomposable but fails to be an antiortholattice — whence by Corollary 5.4.(iii) there exist $a, b \in L$ such that $a > (a \wedge b^{\sim}) \vee (a \wedge \Diamond b)$. Also, recall throughout the remainder of this proof that $S_K(\mathbf{L}) \cup T(\mathbf{L}) = L$. If $b \notin S_K(\mathbf{L})$, then

$$a > (a \wedge b^{\sim}) \vee (a \wedge \Diamond b) = a,$$

a contradiction. Therefore $b \in S_K(\mathbf{L})$ and we can apply Lemma 5.5: either $b = 1$, or $b \leq x$ for every $x \notin S_K(\mathbf{L})$. If $b = 1$, then $a > a$, a contradiction again. If there is some $c \notin S_K(\mathbf{L})$, then $b \leq c$, whence $b = \Box b \leq \Box c = 0$, which yields again the contradiction $a > a$. Therefore $L = S_K(\mathbf{L})$ and $\mathbf{L}$ is orthomodular, against our assumption. □

Theorem 5.7. *(i)* $\mathbb{V}_1 = \mathbb{OML} \vee \mathrm{V}(\mathbb{AOL})$.

(ii) $\mathbb{V}_2 = \mathbb{OML} \vee \mathbb{SAOL}$.

(iii) $\mathbb{V}_3 = \mathbb{OML} \vee \mathrm{V}(\mathbf{D}_3)$.

Proof. (i) It will suffice to show that any subdirectly irreducible $\mathbf{L} \in \mathbb{V}_1$ is either an orthomodular lattice or an antiortholattice. However, since $\mathbf{L}$ is directly indecomposable, Proposition 5.6 applies and we obtain our conclusion.

(ii) Any subdirectly irreducible, and thus directly indecomposable, member of $\mathbb{V}_2$ is either orthomodular, or an antiortholattice satisfying SDM; since $\mathbb{SAOL}$ is generated by such antiortholattices, our claim follows.

(iii) This follows, as above, from the fact that $\mathrm{V}(\mathbf{D}_3)$ is axiomatised by SK relative to $\mathrm{V}(\mathbb{AOL})$ [15, Cor. 3.3]. □

An upshot of this theorem is that $\mathbb{V}_3 \subset \mathbb{V}_2 \subset \mathbb{V}_1 \subset \mathbb{SDM} \vee \mathrm{V}(\mathbb{AOL})$, where the last strict inclusion is witnessed by the PBZ*-lattice $\mathbf{H}$ above which satisfies SDM, thus also WSDM, but fails J2, thus showing in passing that $\mathbb{V}_2 \subset \mathbb{SDM}$. Note, also, that $\mathbb{V}_3$ is the unique cover of $\mathbb{OML}$ in the lattice of subvarieties of $\mathbb{PBZL}^*$, because any member of such which is not included in $\mathbb{OML}$ contains $\mathbf{D}_3$ [14, Thm. 5.5].

Acknowledgements

All authors gratefully acknowledge the following funding sources: the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 689176 (project "Syntax Meets Semantics: Methods, Interactions, and Connections in Substructural logics"); the project "Proprietà d'ordine in matematica e in fisica", Regione Autonoma della Sardegna, L. R. 7/2007, n. 7, 2015, CUP: F72F16002920002; the project "Theory and applications of resource sensitive logics", PRIN 2017, Prot. 20173WKCM5. Discussions with Antonio Ledda and Matthew Spinks were extremely helpful in shaping up this paper. Finally, we thank an anonymous referee for her/his precious comments.

References

[1] P. Aglianò, A. Ursini, On subtractive varieties II: General properties, *Algebra Universalis* **36** (1996), 222–259.

[2] P. Aglianò, A. Ursini, On subtractive varieties III: From ideals to congruences, *Algebra Universalis* **37** (1997), 296–333.

[3] P. Aglianò, A. Ursini, On subtractive varieties IV: Definability of principal ideals, *Algebra Universalis* **38** (1997), 355–389.

[4] R. Bignall, M. Spinks, On binary discriminator varieties, I, II, and III, typescript.

[5] G. Bruns, J. Harding, Algebraic aspects of orthomodular lattices, In: Current Research in Operational Quantum Logic (Eds. B. Coecke et al.), Springer, Berlin, 2000, pp. 37–65.

[6] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Graduate Texts in Mathematics **78**, Springer–Verlag, New York–Berlin, 1981.

[7] G. Cattaneo, M.L. Dalla Chiara, R. Giuntini, Some algebraic structures for many-valued logicst, *Tatra Mountains Mathematical Publications* **15** (1998), 173–196.

[8] G. Cattaneo, R. Giuntini, R. Pilla, BZMV and Stonian MV algebras (applications to fuzzy sets and rough approximations), *Fuzzy Sets and Systems* **108** (1999), 201–222.

[9] G. Cattaneo, G. Nisticò, Brouwer-Zadeh posets and three-valued Łukasiewicz posets, *Fuzzy Sets and Systems* **33:2** (1989), 165–190.

[10] I. Chajda, A note on pseudo-Kleene algebras, *Acta Univ. Palacky Olomouc* **55:1** (2016), 39–45.

[11] I. Chajda, R. Halaš, I.G. Rosenberg, Ideals and the binary discriminator in universal algebra, *Algebra Universalis* **42** (1999), 239–251.

[12] M.L. Dalla Chiara, R. Giuntini, R.J. Greechie, Reasoning in Quantum Theory, Kluwer, Dordrecht, 2004.

[13] J.M. Font, Abstract Algebraic Logic: An Introductory Textbook, College Publications, London, 2016.

[14] R. Giuntini, A. Ledda, F. Paoli, A new view of effects in a Hilbert space, *Studia Logica* **104** (2016), 1145–1177.

[15] R. Giuntini, A. Ledda, F. Paoli, On some properties of PBZ*-lattices, *International Journal of Theoretical Physics* **56:12** (2017), 3895–3911.

[16] R. Giuntini, C. Muresan, F. Paoli, On PBZ*-lattices, In: Mathematics, Logic, and Their Philosophies: Essays in Honour of Mohammad Ardeshir (Eds. M. Saleh Zarepour, S. Rahman, M. Mojtahedi), Springer, Berlin, forthcoming.

[17] G. Grätzer, Universal Algebra, Second Edition, Springer Science+Business Media, LLC, New York, 2008.

[18] H.F. de Groote, On a canonical lattice structure on the effect algebra of a von Neumann algebra, arXiv:math-ph/0410018v2, 2005.

[19] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley, New York, 1978.

[20] C. Mureşan, A note on direct products, subreducts and subvarieties of PBZ*-lattices, Mathematica Slovaca, forthcoming, ArXiv:1904.10093v2 [math.RA], 2019.

[21] M.P. Olson, The self-adjoint operators of a von Neumann algebra form a conditionally complete lattice, *Proceedings of the American Mathematical Society* **28** (1971), 537–544.

[22] A. Salibra, A. Ledda, F. Paoli, T. Kowalski, Boolean-like algebras, *Algebra Universalis* **69:2** (2013), 113–138.

[23] D.W. Stroock, A Concise Introduction to the Theory of Integration (3rd ed.), Birkhäuser, Basel, 1998.

[24] A. Ursini, On subtractive varieties, I, *Algebra Universalis* **31** (1994), 204–222.

[25] A. Ursini, On subtractive varieties, V: congruence modularity and the commutators, *Algebra Universalis* **43** (2000), 51–78.

[26] H. Werner, Discriminator Algebras, Studien zur Algebra und ihre Anwendungen, Band 6, Akademie-Verlag, Berlin, 1978.

Roberto Giuntini, Francesco Paoli
Department of Pedagogy, Psychology, Philosophy
University of Cagliari
09123 Cagliari, Italy
giuntini@unica.it, paoli@unica.it

Claudia Muresan
Faculty of Mathematics and Computer Science
University of Bucharest
Academiei 14, RO 010014, Bucharest, Romania
cmuresan@fmi.unibuc.ro