REPORTS ON MATHEMATICAL LOGIC
54 (2019), 3–43
doi:10.4467/20842589RM.19.001.10649

Andrzej ROSŁANOWSKI and Saharon SHELAH

BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS

A b s t r a c t. We study the existence of Borel sets $B \subseteq {}^{\omega}2$ admitting a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of ${}^{\omega}2$ such that $\big|(\eta_\alpha + B) \cap (\eta_\beta + B)\big| \geq 6$ for all $\alpha, \beta < \lambda$ but with no perfect set of such η's. Our result implies that under the Martin Axiom, if $\aleph_\alpha < \mathfrak{c}$, $\alpha < \omega_1$ and $3 \leq \iota < \omega$, then there exists a Σ^0_2 set $B \subseteq {}^{\omega}2$ which has $\aleph_\alpha$ many pairwise 2ι–nondisjoint translations but not a perfect set of such translations. Our arguments closely follow Shelah [7, Section 1].

1. Introduction

Shelah [7] analyzed the question whether there are Borel sets in the plane which contain large squares but no perfect squares. A rank on models with

Received 16 June 2018
Publication 1138 of the second author.
Keywords and phrases: Forcing, Borel sets, Cantor space, perfect set of overlapping translations, non-disjointness rank.
AMS subject classification: Primary 03E35; Secondary 03E15, 03E50.

a countable vocabulary was introduced and was used to define a cardinal λ_{ω_1} (the first λ such that there is no model with universe λ, countable vocabulary and rank $< \omega_1$). It was shown in [7, Claim 1.12] that every Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ which contains a λ_{ω_1}–square must contain a perfect square. On the other hand, by [7, Theorem 1.13], if $\mu = \mu^{\aleph_0} < \lambda_{\omega_1}$ then some ccc forcing notion forces that (the continuum is arbitrarily large and) some Borel set contains a μ–square but no μ^+–square.

We would like to understand what the results mentioned above mean for general relations. Natural first step is to ask about Borel sets with $\mu \geq \aleph_1$ pairwise disjoint translations but without any perfect set of such translations, as motivated e.g. by Balcerzak, Rosłanowski and Shelah [1] (were we studied the σ–ideal of subsets of ${}^\omega 2$ generated by Borel sets with a perfect set of pairwise disjoint translations) or Elekes and Keleti [3] (see Question 4.5 there). A generalization of this direction could follow Zakrzewski [8] who introduced perfectly k–small sets.

However, preliminary analysis of the problem revealed that another, somewhat orthogonal to the one described above, direction is more natural in the setting of [7]. Thus we investigate Borel sets with many, but not too many, pairwise overlapping intersections.

Easily, every uncountable Borel subset B of ${}^\omega 2$ has a perfect set of pairwise non-disjoint translations (just consider a perfect set $P \subseteq B$ and note that for $x, y \in P$ we have $\mathbf{0}, x+y \in (B+x)\cap(B+y)$). The problem of many non-disjoint translations becomes more interesting if we demand that the intersections have more elements. Note that in ${}^\omega 2$, if $x + b_0 = y + b_1$ then also $x + b_1 = y + b_0$, so $x \neq y$ and $|(B + x) \cap (B + y)| < \omega$ imply that $|(B + x) \cap (B + y)|$ is even.

In the present paper we study the case when the intersections $(B+x)\cap(B + y)$ have at least 6 elements. We show that for $\lambda < \lambda_{\omega_1}$ there is a ccc forcing notion $\mathbb{P}$ adding a Σ^0_2 subset B of the Cantor space ${}^\omega 2$ such that

- for some $H \subseteq {}^\omega 2$ of size λ, $|(B + h) \cap (B + h')| \geq 6$ for all $h, h' \in H$, but
- for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with $|(B + x) \cap (B + x')| < 6$.

We fully utilize the algebraic properties of $({}^\omega 2, +)$, in particular the fact that all elements of ${}^\omega 2$ are self-inverse.

In Section 2 of the paper we recall the rank from [7]. We give the relevant definitions, state and prove all the properties needed for our results later. In the third section we analyze when a Σ^0_2 subset of ${}^\omega 2$ has a perfect set of pairwise overlapping translations. The main consistency result concerning adding a Borel set with no perfect set of overlapping translations is given in the fourth section.

Notation. Our notation is rather standard and compatible with that of classical textbooks (like Jech [4] or Bartoszyński and Judah [2]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

1. For a set u we let

$$u^{\langle 2\rangle} = \{(x,y) \in u \times u : x \neq y\}.$$

2. The Cantor space ${}^\omega 2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology and the group operation of coordinate-wise addition + modulo 2.

3. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ as well as ξ. Finite ordinals (non-negative integers) will be denoted by letters $a, b, c, d, i, j, k, \ell, m, n, M$ and ι.

4. The Greek letters κ, λ will stand for uncountable cardinals.

5. For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.

2. The rank

We will remind some basic facts from [7, Section 1] concerning a rank (on models with countable vocabulary) which will be used in the construction of a forcing notion in the fourth section. For the convenience of the reader we provide proofs for most of the claims, even though they were given in [7]. Our rank rk is the rk^0 of [7] and rk^* is the rk^2 there.

Let λ be a cardinal and $\mathbb{M}$ be a model with the universe λ and a countable vocabulary τ.

Definition 2.1. 1. By induction on ordinals δ, for finite non-empty sets $w \subseteq \lambda$ we define when $\mathrm{rk}(w, \mathbb{M}) \geq \delta$. Let $w = \{\alpha_0, \ldots, \alpha_n\} \subseteq \lambda$, $|w| = n+1$.

(a) $\mathrm{rk}(w) \geq 0$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n]$ then the set

$$\big\{\alpha \in \lambda : \mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_n]\big\}$$

is uncountable;

(b) if δ is limit, then $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ if and only if $\mathrm{rk}(w, \mathbb{M}) \geq \gamma$ for all $\gamma < \delta$;

(c) $\mathrm{rk}(w, \mathbb{M}) \geq \delta + 1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n]$ then there is $\alpha^* \in \lambda \setminus w$ such that

$$\mathrm{rk}(w \cup \{\alpha^*\}, \mathbb{M}) \geq \delta \text{ and } \mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*, \alpha_{k+1}, \ldots, \alpha_n].$$

2. Similarly, for finite non-empty sets $w \subseteq \lambda$ we define when $\mathrm{rk}^*(w, \mathbb{M}) \geq \delta$ (by induction on ordinals δ). Let $w = \{\alpha_0, \ldots, \alpha_n\} \subseteq \lambda$. We take clauses (a) and (b) above and

(c)* $\mathrm{rk}^*(w, \mathbb{M}) \geq \delta + 1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n]$ then there are pairwise distinct $\langle \alpha^*_\zeta : \zeta < \omega_1 \rangle \subseteq \lambda \setminus (w \setminus \{\alpha_k\})$ such that $\alpha^*_0 = \alpha_k$ and for all $\varepsilon < \zeta < \omega_1$ we have

$$\begin{aligned} &\mathrm{rk}^*(w \setminus \{\alpha_k\} \cup \{\alpha^*_\varepsilon, \alpha^*_\zeta\}, \mathbb{M}) \geq \delta \\ &\qquad \text{and} \quad \mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_\zeta, \alpha_{k+1}, \ldots, \alpha_n]. \end{aligned}$$

By a straightforward induction on α one easily shows the following observation.

Observation 2.2. *If $\emptyset \neq v \subseteq w$ then*

- $\mathrm{rk}(w, \mathbb{M}) \geq \delta \geq \gamma$ *implies* $\mathrm{rk}(v, \mathbb{M}) \geq \gamma$, *and*
- $\mathrm{rk}^*(w, \mathbb{M}) \geq \delta \geq \gamma$ *implies* $\mathrm{rk}^*(v, \mathbb{M}) \geq \gamma$.

Hence we may define the rank functions on finite non-empty subsets of λ.

Definition 2.3. The ranks $\mathrm{rk}(w, \mathbb{M})$ and $\mathrm{rk}^*(w, \mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ are defined as:

- $\mathrm{rk}(w, \mathbb{M}) = -1$ if $\neg(\mathrm{rk}(w, \mathbb{M}) \geq 0)$, and
  $\mathrm{rk}^*(w, \mathbb{M}) = -1$ if $\neg(\mathrm{rk}^*(w, \mathbb{M}) \geq 0)$,
- $\mathrm{rk}(w, \mathbb{M}) = \infty$ if $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ for all ordinals δ, and
  $\mathrm{rk}^*(w, \mathbb{M}) = \infty$ if $\mathrm{rk}^*(w, \mathbb{M}) \geq \delta$ for all ordinals δ,
- for an ordinal δ: $\mathrm{rk}(w, \mathbb{M}) = \delta$ if $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ but $\neg(\mathrm{rk}(w, \mathbb{M}) \geq \delta + 1)$,
  and $\mathrm{rk}^*(w, \mathbb{M}) = \delta$ if $\mathrm{rk}^*(w, \mathbb{M}) \geq \delta$ but $\neg(\mathrm{rk}^*(w, \mathbb{M}) \geq \delta + 1)$.

Definition 2.4. 1. For an ordinal ε and a cardinal λ let $\mathrm{NPr}_\varepsilon(\lambda)$ be the following statement: "there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that $\sup\{\mathrm{rk}(w, \mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$."

2. The statement $\mathrm{NPr}^*_\varepsilon(\lambda)$ is defined similarly but using the rank rk^*.
3. $\mathrm{Pr}_\varepsilon(\lambda)$ and $\mathrm{Pr}^*_\varepsilon(\lambda)$ are the negations of $\mathrm{NPr}_\varepsilon(\lambda)$ and $\mathrm{NPr}^*_\varepsilon(\lambda)$, respectively.

Observation 2.5. *1. If a model $\mathbb{M}^+$ (on λ) is an expansion*[1] *of the model $\mathbb{M}$, then $\mathrm{rk}^*(w, \mathbb{M}^+) \leq \mathrm{rk}(w, \mathbb{M}^+) \leq \mathrm{rk}(w, \mathbb{M})$.*

2. If λ is uncountable and $\mathrm{NPr}_\varepsilon(\lambda)$, then there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that

- *$\mathrm{rk}(\{\alpha\}, \mathbb{M}^*) \geq 0$ for all $\alpha \in \lambda$ and*
- *$\mathrm{rk}(w, \mathbb{M}^*) < \varepsilon$ for every finite non-empty set $w \subseteq \lambda$.*

Proposition 2.6 (See [7, Claim 1.7])**.** 1. $\mathrm{NPr}_1(\omega_1)$.

2. *If $\mathrm{NPr}_\varepsilon(\lambda)$, then $\mathrm{NPr}_{\varepsilon+1}(\lambda^+)$.*
3. *If $\mathrm{NPr}_\varepsilon(\mu)$ for $\mu < \lambda$ and $\mathrm{cf}(\lambda) = \omega$, then $\mathrm{NPr}_{\varepsilon+1}(\lambda)$.*

---

[1] So $\mathbb{M}^+$ is a model with a countable vocabulary $\tau^* \supseteq \tau$, with the universe λ, and the interpretation of symbols from τ in $\mathbb{M}^+$ is the same as in $\mathbb{M}$.

4. $\mathrm{NPr}_\varepsilon(\lambda)$ *implies* $\mathrm{NPr}^*_\varepsilon(\lambda)$.

Proof. (1) Let Q be a binary relational symbol and let $\mathbb{M}_1$ be a model with the universe ω_1, the vocabulary $\tau(\mathbb{M}_1)=\{Q\}$ and such that $Q^{\mathbb{M}_1}=\{(\alpha,\beta)\in\omega_1\times\omega_1:\alpha<\beta\}$. Then for each $\alpha_0<\alpha_1<\omega_1$ we have $\mathbb{M}_1\models Q[\alpha_0,\alpha_1]$ but the set $\{\alpha<\omega_1:\mathbb{M}_1\models Q[\alpha,\alpha_1]\}$ is countable. Hence $\mathrm{rk}(w,\mathbb{M}_1)=-1$ whenever $|w|\geq 2$ and $\mathrm{rk}(\{\alpha\},\mathbb{M}_1)=0$ for $\alpha\in\omega_1$. Consequently, $\mathbb{M}_1$ witnesses $\mathrm{NPr}_1(\omega_1)$.

(2) Assume $\mathrm{NPr}_\varepsilon(\lambda)$ holds true as witnessed by a model $\mathbb{M}$ with the universe λ and a countable vocabulary τ. We may assume that $\tau=\{R_i:i<\omega\}$, where each R_i is a relational symbol of arity $n(i)$. Let S be a new binary relational symbol, T be a new unary relational symbol, and Q_i be a new $(n(i)+1)$–ary relational symbol (for $i<\omega$). Let $\tau^+=\{R_i,Q_i:i<\omega\}\cup\{S,T\}$.

For each $\gamma\in[\lambda,\lambda^+)$ fix a bijection $f_\gamma:\gamma\xrightarrow{1-1}\lambda$. We define a model $\mathbb{M}^+$:

- the vocabulary of $\mathbb{M}^+$ is τ^+ and the universe of $\mathbb{M}^+$ is λ^+,
- $R_i^{\mathbb{M}^+}=R_i^{\mathbb{M}}\subseteq\lambda^{n(i)}$,
- $Q_i^{\mathbb{M}^+}=\{\big(\alpha_0,\ldots,\alpha_{n(i)-1},\alpha_{n(i)}\big):\lambda\leq\alpha_{n(i)}<\lambda^+\ \&\ (\forall\ell<n(i))(\alpha_\ell<\alpha_{n(i)})\ \&\ \big(f_{\alpha_{n(i)}}(\alpha_0),\ldots,f_{\alpha_{n(i)}}(\alpha_{n(i)-1})\big)\in R_i^{\mathbb{M}}\}$,
- $S^{\mathbb{M}^+}=\{(\alpha_0,\alpha_1)\in\lambda^+\times\lambda^+:\alpha_0<\alpha_1\}$ and $T^{\mathbb{M}^+}=[\lambda,\lambda^+)$.

Claim 2.6.1. (i) *If* $\lambda\leq\gamma<\lambda^+$, $\emptyset\neq w\subseteq\gamma$, *then* $\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+)\leq\mathrm{rk}(f_\gamma[w],\mathbb{M})$ *and thus* $\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+)<\varepsilon$.

(ii) *If* $\emptyset\neq w\subseteq\lambda$, *then* $\mathrm{rk}(w,\mathbb{M}^+)\leq\mathrm{rk}(w,\mathbb{M})$ *and thus* $\mathrm{rk}(w,\mathbb{M}^+)<\varepsilon$.

(iii) *If* $\lambda\leq\gamma<\lambda^+$, *then* $\mathrm{rk}(\{\gamma\},\mathbb{M}^+)\leq\varepsilon$.

Proof of the Claim. (i) By induction on α we show that $\alpha\leq\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+)$ implies $\alpha\leq\mathrm{rk}(f_\gamma[w],\mathbb{M})$ (for all sets $w\subseteq\gamma$ with fixed $\gamma\in[\lambda,\lambda^+)$).

$(*)_0$ Assume $\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+)\geq 0$, $w=\{\alpha_0,\ldots,\alpha_n\}$ and $k\leq n$. Let $\varphi(x_0,\ldots,x_n)$ be a quantifier free formula in the vocabulary τ such that

$$\mathbb{M}\models\varphi[f_\gamma(\alpha_0),\ldots,f_\gamma(\alpha_k),\ldots,f_\gamma(\alpha_n)].$$

Let $\varphi^*(x_0,\ldots,x_n,x_{n+1})$ be a quantifier free formula in the vocabulary τ^+ obtained from φ by replacing each $R_i(y_0,\ldots,y_{n(i)-1})$ (where $\{y_0,\ldots,y_{n(i)-1}\} \subseteq \{x_0,\ldots,x_n\}$) with $Q_i(y_0,\ldots,y_{n(i)-1},x_{n+1})$ and let φ^+ be

$$\varphi^*(x_0,\ldots,x_n,x_{n+1}) \ \wedge\ S(x_0,x_{n+1}) \ \wedge \ldots \wedge S(x_n,x_{n+1}).$$

Then $\mathbb{M}^+ \models \varphi^+[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n,\gamma]$. By our assumption on $w\cup\{\gamma\}$ we know that the set

$$A = \{\beta < \lambda^+ : \mathbb{M}^+ \models \varphi^+[\alpha_0,\ldots,\alpha_{k-1},\beta,\alpha_{k+1},\ldots,\alpha_n,\gamma]\}$$

is uncountable. Clearly $A \subseteq \gamma$ (note $S(x_k,x_{n+1})$ in φ^+) and thus the set $f_\gamma[A]$ is an uncountable subset of λ. For each $\beta\in A$ we have

$$\mathbb{M} \models \varphi[f_\gamma(\alpha_0),\ldots,f_\gamma(\beta),\ldots,f_\gamma(\alpha_n)],$$

so now we may conclude that $\mathrm{rk}(f_\gamma[w],\mathbb{M}) \geq 0$.

$(*)_1$ Assume $\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+) \geq \alpha+1$. Let $\varphi(x_0,\ldots,x_n)$ be a quantifier free formula in the vocabulary τ, $k \leq n$ and $w=\{\alpha_0,\ldots,\alpha_n\}$, and suppose that $\mathbb{M} \models \varphi[f_\gamma(\alpha_0),\ldots,f_\gamma(\alpha_k),\ldots,f_\gamma(\alpha_n)]$. Let φ^* and φ^+ be defined exactly as in $(*)_0$. Then $\mathbb{M}^+ \models \varphi^+[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n,\gamma]$. By our assumption there is $\beta^* \in \lambda^+ \setminus (w\cup\{\gamma\})$ such that $\mathbb{M}^+ \models \varphi^+[\alpha_0,\ldots,\beta^*,\ldots,\alpha_n,\gamma]$ and $\mathrm{rk}(w\cup\{\gamma,\beta^*\},\mathbb{M}^+) \geq \alpha$. Necessarily $\beta^* < \gamma$, and by the inductive hypothesis $\mathrm{rk}(f_\gamma[w\cup\{\beta^*\}],\mathbb{M}) \geq \alpha$. Clearly $\mathbb{M} \models \varphi[f_\gamma(\alpha_0),\ldots,f_\gamma(\beta^*),\ldots,f_\gamma(\alpha_n)]$ and we may conclude $\mathrm{rk}(f_\gamma[w],\mathbb{M}) \geq \alpha+1$.

$(*)_2$ If α is limit and $\mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+) \geq \alpha$ then, by the inductive hypothesis, for each $\beta<\alpha$ we have $\beta \leq \mathrm{rk}(w\cup\{\gamma\},\mathbb{M}^+) \leq \mathrm{rk}(f_\gamma[w],\mathbb{M})$. Hence $\alpha \leq \mathrm{rk}(f_\gamma[w],\mathbb{M})$.

(ii) Induction similar to part (i). For a quantifier free formula $\varphi(x_0,\ldots,x_n)$ in the vocabulary τ, let φ^* be the formula $\varphi(x_0,\ldots,x_n) \ \wedge\ \neg T(x_0) \ \wedge \ldots \wedge \neg T(x_n)$ (so φ^* is a quantifier free formula in the vocabulary τ^+). If φ witnesses that $\neg(\mathrm{rk}(w,\mathbb{M}) \geq 0)$, then φ^* witnesses $\neg(\mathrm{rk}(w,\mathbb{M}^+) \geq 0)$, and similarly with $\alpha+1$ in place of 0.

(iii) Suppose towards contradiction that $\varepsilon+1 \leq \mathrm{rk}(\{\gamma\},\mathbb{M}^+)$. Since $\mathbb{M}^+ \models T[\gamma]$, we may find $\gamma' \neq \gamma$ such that $\mathrm{rk}(\{\gamma,\gamma'\},\mathbb{M}^+) \geq \varepsilon$ and $\mathbb{M}^+ \models T[\gamma']$. Let $\{\gamma,\gamma'\} = \{\gamma_0,\gamma_1\}$ where $\gamma_0<\gamma_1$. It follows from part (i) that $\mathrm{rk}(\{\gamma_0,\gamma_1\},\mathbb{M}^+) < \varepsilon$, a contradiction. □

It follows from Claim 2.6.1 (and Observation 2.2) that $\mathrm{rk}(w, \mathbb{M}^+) \le \varepsilon$ for every non-empty set $w \subseteq \lambda^+$. Consequently, the model $\mathbb{M}^+$ witnesses $\mathrm{NPr}_{\varepsilon+1}(\lambda^+)$.

(3) Let $\langle \mu_n : n < \omega \rangle$ be an increasing sequence cofinal in λ. For each n fix a model $\mathbb{M}_n$ with a countable vocabulary $\tau(\mathbb{M}_n)$ consisting of relational symbols only and with the universe μ_n and such that $\mathrm{rk}(w, \mathbb{M}_n) < \varepsilon$ for nonempty finite $w \subseteq \mu_n$. We also assume that $\tau(\mathbb{M}_n) \cap \tau(\mathbb{M}_m) = \emptyset$ for $n < m < \omega$. Let P_n (for $n < \omega$) be new unary relational symbols and let $\tau = \bigcup\{\tau(\mathbb{M}_n) : n < \omega\} \cup \{P_n : n < \omega\}$. Consider a model $\mathbb{M}$ in vocabulary τ with the universe λ and such that

- $P_n^{\mathbb{M}} = \mu_n$ for $n < \omega$, and
- for each $n < \omega$ and $S \in \tau(\mathbb{M}_n)$ we have $S^{\mathbb{M}} = S^{\mathbb{M}_n}$.

Claim 2.6.2. *If w is a finite non-empty subset of μ_n, $n < \omega$, then $\mathrm{rk}(w, \mathbb{M}) \le \mathrm{rk}(w, \mathbb{M}_n) < \varepsilon$.*

Proof of the Claim. Similar to the proofs in Claim 2.6.1. □

(4) Follows from Observation 2.5(1). □

Proposition 2.7. (See [7, Conclusion 1.8].) *Assume $\beta < \alpha < \omega_1$, $\mathbb{M}$ is a model with a countable vocabulary τ and the universe μ, $m, n < \omega$, $n > 0$, $A \subseteq \mu$ and $|A| \ge \beth_{\omega\cdot\alpha}$. Then there is $w \subseteq A$ with $|w| = n$ and $\mathrm{rk}^*(w, \mathbb{M}) \ge \omega \cdot \beta + m$* [2].

Proof. Induction on $\alpha < \omega_1$.

Step $\alpha = 1$ (and $\beta = 0$): Let $\mathbb{M}, \mu, n, m$ be as in the assumptions, $A \subseteq \mu$ and $|A| \ge \beth_\omega$. Using the Erdős–Rado theorem we may choose a sequence $\langle \alpha_\varepsilon : \varepsilon < \omega_2 \rangle$ of distinct elements of A such that:

(a) the quantifier free type of $\langle \alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n}} \rangle$ in $\mathbb{M}$ is constant for $\varepsilon_0 < \ldots < \varepsilon_{m+n} < \omega_2$, and

(b) for each $k \le m + n$ the value of $\min\{\omega, \mathrm{rk}^*(\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n-k}}\}, \mathbb{M})\}$ is constant for $\varepsilon_0 < \ldots < \varepsilon_{m+n-k} < \omega_2$.

[2] " $\cdot$ " stands for the ordinal multiplication.

Let $\zeta_\ell = \omega_1 \cdot (\ell + 1)$ (for $\ell = -1, 0, \ldots, m+n$). Suppose $\phi(x_0, \ldots, x_{m+n}) \in \mathcal{L}(\tau)$ is a quantifier free formula, $k \le m+n$ and

$$\mathbb{M} \models \phi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_k}, \ldots, \alpha_{\zeta_{m+n}}].$$

It follows from the property stated in (a) above that for every ε in the (uncountable) interval (ζ_{k-1}, ζ_k) we have

$$\mathbb{M} \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{k-1}}, \alpha_\varepsilon, \alpha_{\zeta_{k+1}}, \ldots, \alpha_{\zeta_{m+n}}].$$

Consequently, $\mathrm{rk}^*\big(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{m+n}}\}, \mathbb{M}\big) \ge 0$, and the homogeneity stated in (b) implies that for every nonempty set $w \subseteq \omega_2$ with at most $m+n+1$ elements we have $\mathrm{rk}^*(\{\alpha_\varepsilon : \varepsilon \in w\}, \mathbb{M}) \ge 0$. Now, by induction on $k \le m+n$ we will argue that

$(*)_k$ for every nonempty set $w \subseteq \omega_2$ with at most $m+n+1-k$ elements we have $\mathrm{rk}^*(\{\alpha_\varepsilon : \varepsilon \in w\}, \mathbb{M}) \ge k$.

We have already justified $(*)_0$. For the inductive step assume $(*)_k$ and $k < m+n$. Let $\zeta_\ell = \omega_1 \cdot (\ell+1)$ and suppose that $\varphi(x_0, \ldots, x_{m+n-k-1})$ is a quantifier free formula, $\mathbb{M} \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_z}, \ldots, \alpha_{\zeta_{m+n-k-1}}]$ and $0 \le z \le m+n-k-1$. By the homogeneity stated in (a), for every ε in the uncountable interval (ζ_{z-1}, ζ_z) we have

$$\mathbb{M} \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_\varepsilon, \alpha_{\zeta_{z+1}}, \ldots, \alpha_{\zeta_{m+n-k-1}}].$$

The inductive hypothesis $(*)_k$ implies that

$$\mathrm{rk}^*(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_\varepsilon, \alpha_\xi, \alpha_{\zeta_{z+1}}, \ldots \alpha_{\zeta_{m+n-k-1}}\}, \mathbb{M}) \ge k$$

(for any $\zeta_{z-1} < \varepsilon < \xi \le \zeta_z$). Now we easily conclude that $k+1 \le \mathrm{rk}^*(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{m+n-k-1}}\}, \mathbb{M})$ and $(*)_{k+1}$ follows by the homogeneity given by (b).

Finally note that $(*)_{m+1}$ gives the desired conclusion: taking any $\varepsilon_0 < \ldots < \varepsilon_{n-1} < \omega_2$ we will have $m+1 \le \mathrm{rk}^*\big(\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{n-1}}\}, \mathbb{M}\big)$.

Step $\alpha = \gamma + 1$: Let $\mathbb{M}, \mu, n, m$ be as in the assumptions, $A \subseteq \mu$ and $|A| \ge \beth_{\omega\cdot\gamma+\omega}$. By the Erdős–Rado theorem we may choose a sequence $\langle \alpha_\varepsilon : \varepsilon < \beth_{\omega\cdot\gamma} \rangle$ of distinct elements of A such that the following two demands are satisfied.

(c) The quantifier free type of $\langle\alpha_{\varepsilon_0},\ldots,\alpha_{\varepsilon_{m+n}}\rangle$ in $\mathbb{M}$ is constant for $\varepsilon_0<\ldots<\varepsilon_{m+n}<\beth_{\omega\cdot\gamma}$.

(d) For each $k\leq m+n$ the value of $\min\{\omega\cdot(\gamma+1),{\rm rk}^*(\{\alpha_{\varepsilon_0},\ldots,\alpha_{\varepsilon_{m+n-k}}\},\mathbb{M})\}$ is constant for $\varepsilon_0<\ldots<\varepsilon_{m+n-k}<\beth_{\omega\cdot\gamma}$.

For any $\ell<\omega$ and $\gamma'<\gamma$, we may apply the inductive hypothesis to $\{\alpha_\varepsilon:\varepsilon<\beth_{\omega\cdot\gamma}\}$, ℓ, $m+n+1$ and γ' to find $\varepsilon_0<\ldots<\varepsilon_{m+n}<\beth_{\omega\cdot\gamma}$ such that ${\rm rk}^*(\{\alpha_{\varepsilon_0},\ldots,\alpha_{\varepsilon_{m+n}}\},\mathbb{M})\geq\omega\cdot\gamma'+\ell$. By the homogeneity in (d) this implies that

$(**)_0$ for all $\varepsilon_0<\ldots<\varepsilon_{m+n}<\beth_{\omega\cdot\gamma}$ we have

$$ {\rm rk}^*(\{\alpha_{\varepsilon_0},\ldots,\alpha_{\varepsilon_{m+n}}\},\mathbb{M})\geq\omega\cdot\gamma. $$

Now, by induction on $k\leq m+n$ we argue that

$(**)_k$ for each $\varepsilon_0<\ldots<\varepsilon_{m+n-k}<(\beth_{\omega\cdot\gamma})^+$ we have

$$ \omega\cdot\gamma+k\leq{\rm rk}^*(\{\alpha_{\varepsilon_0},\ldots,\alpha_{\varepsilon_{m+n-k}}\},\mathbb{M}). $$

So assume $(**)_k$, $k<m+n$ and let $\zeta_\ell=\omega_1\cdot(\ell+1)$ (for $\ell=-1,0,\ldots,m+n$) and $0\leq z\leq m+n-k-1$. Suppose that $\mathbb{M}\models\varphi[\alpha_{\zeta_0},\ldots,\alpha_{\zeta_z},\ldots,\alpha_{\zeta_{m+n-k-1}}]$. Then by the homogeneity in (c), for every ε in the uncountable interval (ζ_{z-1},ζ_z) we have $\mathbb{M}\models\varphi[\alpha_{\zeta_0},\ldots,\alpha_{\zeta_{z-1}},\alpha_\varepsilon,\alpha_{\zeta_{z+1}},\ldots,\alpha_{\zeta_{m+n-k-1}}]$. By the inductive hypothesis $(**)_k$ we know

$$ \omega\cdot\gamma+k\leq{\rm rk}^*(\{\alpha_{\zeta_0},\ldots,\alpha_{\zeta_{z-1}},\alpha_\varepsilon,\alpha_\xi,\alpha_{\zeta_{z+1}},\ldots\alpha_{\zeta_{m+n-k-1}}\},\mathbb{M}) $$

(for $\zeta_{z-1}<\varepsilon<\xi\leq\zeta_z$). Now we easily conclude that $\omega\cdot\gamma+k+1\leq{\rm rk}^*(\{\alpha_{\zeta_0},\ldots\alpha_{\zeta_{m+n-k-1}}\},\mathbb{M})$, and $(**)_{k+1}$ follows by the homogeneity in (d).

Finally note that $(**)_{m+1}$ gives the desired conclusion: taking any $\zeta_0<\ldots<\zeta_{n-1}<\beth_{\omega\cdot\gamma}$ we will have ${\rm rk}^*\big(\{\alpha_{\zeta_0},\ldots,\alpha_{\zeta_{n-1}}\},\mathbb{M}\big)\geq\omega\cdot\gamma+m+1$.

Step α is limit: Straightforward. □

Definition 2.8. Let λ_{ω_1} be the smallest cardinal λ such that ${\rm Pr}_{\omega_1}(\lambda)$ and $\lambda^*_{\omega_1}$ be the smallest cardinal λ such that ${\rm Pr}^*_{\omega_1}(\lambda)$.

Corollary 2.9. 1. *If* $\alpha<\omega_1$, *then* ${\rm NPr}_{\omega_1}(\aleph_\alpha)$.

2. $\mathrm{Pr}^*_{\omega_1}(\beth_{\omega_1})$ *holds and hence also* $\mathrm{Pr}_{\omega_1}(\beth_{\omega_1})$.

3. $\aleph_{\omega_1} \leq \lambda_{\omega_1} \leq \lambda^*_{\omega_1} \leq \beth_{\omega_1}$.

Proof. (1) Immediately from Proposition 2.6, by induction on $\alpha < \omega_1$.
(2) Follows from Proposition 2.7 (and 2.6(4)).
(3) By clauses (1), (2) above. □

Proposition 2.10. (See [7, Claim 1.10(1)].) *If* $\mathbb{P}$ *is a ccc forcing notion and* λ *is a cardinal such that* $\mathrm{Pr}^*_{\omega_1}(\lambda)$ *holds, then* $\Vdash_{\mathbb{P}}$ *"* $\mathrm{Pr}^*_{\omega_1}(\lambda)$ *and hence also* $\mathrm{Pr}_{\omega_1}(\lambda)$ *".*

Proof. Suppose towards contradiction that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}} \mathrm{NPr}^*_{\omega_1}(\lambda)$. Let $\tau = \{R_{n,\zeta} : n, \zeta < \omega\}$ where $R_{n,\zeta}$ is an n–ary relation symbol (for $n, \zeta < \omega$). Then we may pick a name $\underset{\sim}{\mathbb{M}}$ for a model on λ in vocabulary τ and an ordinal $\alpha_0 < \omega_1$ such that

$$
\begin{array}{ll}
p \Vdash & \text{“}\underset{\sim}{\mathbb{M}} = (\lambda, \{R^{\underset{\sim}{\mathbb{M}}}_{n,\zeta}\}_{n,\zeta<\omega}) \text{ is a model such that} \\
& \text{(a) for every } n \text{ and a quantifier free formula } \varphi(x_0,\ldots,x_{n-1}) \in \mathcal{L}(\tau) \\
& \qquad \text{there is } \zeta < \omega \text{ such that for all } \gamma_0,\ldots,\gamma_{n-1} \\
& \qquad \underset{\sim}{\mathbb{M}} \models \varphi[\gamma_0,\ldots,\gamma_{n-1}] \Leftrightarrow R_{n,\zeta}[\gamma_0,\ldots,\gamma_{n-1}] \\
& \text{(b) } \sup\{\mathrm{rk}(w,\underset{\sim}{\mathbb{M}}) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \alpha_0 \text{ ”.}
\end{array}
$$

Now, let $S_{n,\zeta,\beta,k}$ be an n–ary predicate (for $k < n, \zeta < \omega$ and $-1 \leq \beta < \alpha_0$) and let $\tau^* = \{S_{n,\zeta,\beta,k} : k < n < \omega,\ \zeta < \omega \text{ and } -1 \leq \beta < \alpha_0\}$. (So τ^* is a countable vocabulary.) We define a model $\mathbb{M}^*$ in the vocabulary τ^*. The universe of $\mathbb{M}^*$ is λ and for $k < n$, $\zeta < \omega$ and $-1 \leq \beta < \alpha_0$:

$$
\begin{array}{rl}
S^{\mathbb{M}^*}_{n,\zeta,\beta,k} = & \big\{(\gamma_0,\ldots,\gamma_{n-1}) \in {}^n\lambda : \gamma_0 < \ldots < \gamma_{n-1} \text{ and} \\
& \qquad \text{some condition } q \geq p \text{ forces that} \\
& \text{“}\underset{\sim}{\mathbb{M}} \models R_{n,\zeta}[\gamma_0,\ldots,\gamma_{n-1}] \text{ and } \mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{n-1}\},\underset{\sim}{\mathbb{M}}) = \beta \text{ and} \\
& R_{n,\zeta}, k \text{ witness that } \neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{n-1}\},\underset{\sim}{\mathbb{M}}) \geq \beta+1\big) \text{ ”}\big\}.
\end{array}
$$

Claim 2.10.1. *For every* n *and every increasing tuple* $(\gamma_0,\ldots,\gamma_{n-1}) \in {}^n\lambda$ *there are* $\zeta < \omega$ *and* $-1 \leq \beta < \alpha_0$ *and* $k < n$ *such that* $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[\gamma_0,\ldots,\gamma_{n-1}]$.

Proof of the Claim. Clear. □

Claim 2.10.2. *If* $(\gamma_0,\ldots,\gamma_{n-1})\in{}^n\lambda$ *and* $\mathbb{M}^*\models S_{n,\zeta,\beta,k}[\gamma_0,\ldots,\gamma_{n-1}]$, *then*

$$\mathrm{rk}^*\big(\{\gamma_0,\ldots,\gamma_{n-1}\},\mathbb{M}^*\big)\leq\beta.$$

Proof of the Claim. First let us deal with the case of $\beta=-1$. Assume towards contradiction that $\mathbb{M}^*\models S_{n,\zeta,-1,k}[\gamma_0,\ldots,\gamma_{n-1}]$, but $\mathrm{rk}^*\big(\{\gamma_0,\ldots,\gamma_{n-1}\},\mathbb{M}^*\big)\geq 0$. Then we may find distinct $\langle\delta_\varepsilon:\varepsilon<\omega_1\rangle\subseteq\lambda\setminus\{\gamma_0,\ldots,\gamma_{n-1}\}$ such that

$(\otimes)_1$ $\mathbb{M}^*\models S_{n,\zeta,-1,k}[\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}]$ for all $\varepsilon<\omega_1$.

For $\varepsilon<\omega_1$ let $p_\varepsilon\in\mathbb{P}$ be such that $p_\varepsilon\geq p$ and

$$\begin{aligned}p_\varepsilon\Vdash\text{``}\ &\underset{\sim}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\delta_\varepsilon,\ldots,\gamma_{n-1}]\text{ and}\\ &\mathrm{rk}^*(\{\gamma_0,\ldots,\delta_\varepsilon,\ldots,\gamma_{n-1}\},\underset{\sim}{\mathbb{M}})=-1\text{ and}\\ &R_{n,\zeta},k\text{ witness that}\\ &\qquad\neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}\},\underset{\sim}{\mathbb{M}})\geq 0\big)\text{''}\end{aligned}$$

Let $\underset{\sim}{Y}$ be a name $\mathbb{P}$–name such that $p\Vdash\underset{\sim}{Y}=\{\varepsilon<\omega_1:p_\varepsilon\in\underset{\sim}{G}_{\mathbb{P}}\}$. Since $\mathbb{P}$ satisfies ccc, we may pick $p^*\geq p$ such that $p^*\Vdash$ "$\underset{\sim}{Y}$ is uncountable". Since

$$p^*\Vdash\big(\forall\varepsilon\in\underset{\sim}{Y}\big)\big(\underset{\sim}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}]\big),$$

then also

$$p^*\Vdash\big\{\delta<\lambda:\underset{\sim}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\gamma_{k-1},\delta,\gamma_{k+1},\ldots,\gamma_{n-1}]\big\}\text{ is uncountable.}$$

But

$$\begin{aligned}&p^*\Vdash\big(\forall\varepsilon\in\underset{\sim}{Y}\big)\\ &\qquad\big(R_{n,\zeta},k\text{ witness }\neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}\},\underset{\sim}{\mathbb{M}})\geq 0\big)\big),\end{aligned}$$

and hence

$$p^*\Vdash\big\{\delta<\lambda:\underset{\sim}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\gamma_{k-1},\delta,\gamma_{k+1},\ldots,\gamma_{n-1}]\big\}\text{ is countable,}$$

a contradiction.

Next we continue the proof of the Claim by induction on $\beta<\alpha_0$, so we assume that $0\leq\beta$ and for $\beta'<\beta$ our claim holds true (for any n,ζ,k). Assume towards contradiction that $\mathbb{M}^*\models S_{n,\zeta,\beta,k}[\gamma_0,\ldots,\gamma_{n-1}]$, but $\mathrm{rk}^*\big(\{\gamma_0,\ldots,\gamma_{n-1}\},\mathbb{M}^*\big)\geq\beta+1$. Then we may find distinct $\langle\delta_\varepsilon:\varepsilon<\omega_1\rangle\subseteq\lambda\setminus(w\setminus\{\gamma_k\})$ such that

$(\oplus)_1$ $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}]$ for all $\varepsilon<\omega_1$, $\delta_0=\gamma_k$ and

$(\oplus)_2$ $\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\delta_\zeta,\gamma_{k+1},\ldots,\gamma_{n-1}\},\mathbb{M}^*)\geq\beta$ for all $\varepsilon<\zeta<\omega_1$.

For $\varepsilon<\omega_1$ let $p_\varepsilon\in\mathbb{P}$ be such that $p_\varepsilon\geq p$ and

$$\begin{aligned} p_\varepsilon \Vdash \text{“}&\underaccent{\tilde}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\delta_\varepsilon,\ldots,\gamma_{n-1}]\\ &\text{and } \mathrm{rk}^*(\{\gamma_0,\ldots,\delta_\varepsilon,\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})=\beta\\ &\text{and } R_{n,\zeta},k \text{ witness that}\\ &\quad\neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})\geq\beta+1\big)\text{”}\end{aligned}$$

Take $p^*\geq p$ such that

$$p^*\Vdash \text{“}\underaccent{\tilde}{Y}\overset{\text{def}}{=}\{\varepsilon<\omega_1: p_\varepsilon\in\underaccent{\tilde}{G}_{\mathbb{P}}\}\text{ is uncountable”}.$$

Since

$$\begin{aligned} p^*\Vdash\quad &(\forall\varepsilon\in\underaccent{\tilde}{Y})\Big(\ \underaccent{\tilde}{\mathbb{M}}\models R_{n,\zeta}[\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\gamma_{k+1},\ldots,\gamma_{n-1}]\ \wedge\\ &\qquad R_{n,\zeta},k\text{ witness that }\neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\delta_\varepsilon,\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})\geq\beta+1\big)\Big),\end{aligned}$$

we see that

$$p^*\nVdash\big(\forall\varepsilon,\zeta\in\underaccent{\tilde}{Y}\big)\big(\varepsilon\neq\zeta\Rightarrow\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_\varepsilon,\delta_\zeta,\gamma_{k+1},\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})\geq\beta\big).$$

Consequently we may pick $q\geq p^*$, $\varepsilon_0,\zeta_0<\omega_1$ and $\gamma<\beta$ and $\xi<\omega$ and $\ell\leq n$ such that $\delta_{\varepsilon_0}<\delta_{\zeta_0}$ and

$$\begin{aligned} q\Vdash\quad\text{“}&p_{\varepsilon_0},p_{\zeta_0}\in\underaccent{\tilde}{G}_{\mathbb{P}}\text{ and }\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_{\varepsilon_0},\delta_{\zeta_0},\gamma_{k+1},\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})=\gamma\\ &\text{and } R_{n+1,\xi}\text{ and }\ell\text{ witness that}\\ &\neg\big(\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_{\varepsilon_0},\delta_{\zeta_0},\gamma_{k+1},\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})\geq\gamma+1\big)\text{”}.\end{aligned}$$

Then $\mathbb{M}^*\models S_{n+1,\xi,\gamma,\ell}[\gamma_0,\ldots,\gamma_{k-1},\delta_{\varepsilon_0},\delta_{\zeta_0},\gamma_{k+1},\ldots,\gamma_{n-1}]$ and by the inductive hypothesis $\mathrm{rk}^*(\{\gamma_0,\ldots,\gamma_{k-1},\delta_{\varepsilon_0},\delta_{\zeta_0},\gamma_{k+1},\ldots,\gamma_{n-1}\},\underaccent{\tilde}{\mathbb{M}})\leq\gamma$, contradicting clause $(\oplus)_2$ above. □

□

Corollary 2.11. *Let $\mu=\beth_{\omega_1}\leq\kappa$ and $\mathbb{C}_\kappa$ be the forcing notion adding κ Cohen reals. Then $\Vdash_{\mathbb{C}_\kappa}\lambda_{\omega_1}\leq\mu\leq\mathfrak{c}$.*

3. Spectrum of translation non-disjointness

Definition 3.1. Let $B \subseteq {}^{\omega}2$ and $1 \leq \kappa \leq \mathfrak{c}$.

1. We say that B is *perfectly orthogonal to κ–small* (or a κ–**pots**–set) if there is a perfect set $P \subseteq {}^{\omega}2$ such that $|(B+x) \cap (B+y)| \geq \kappa$ for all $x, y \in P$.
   The set B is *a κ–**npots**–set* if it is not κ–**pots**.

2. We say that B has λ *many pairwise κ–nondisjoint translations* if for some set $X \subseteq {}^{\omega}2$ of cardinality λ, for all $x, y \in X$ we have $\big|(B+x) \cap (B+y)\big| \geq \kappa$.

3. We define the *spectrum of translation κ–non-disjointness of B* as

$$\mathrm{stnd}_\kappa(B) = \{(x, y) \in {}^{\omega}2 \times {}^{\omega}2 : |(B+x) \cap (B+y)| \geq \kappa\}.$$

Remark 3.2. 1. Note that if $B \subseteq {}^{\omega}2$ is an uncountable Borel set, then there is a perfect set $P \subseteq B$. For B, P as above for every $x, y \in P$ we have $0 = x + x = y + y \in (B+x) \cap (B+y)$ and $x + y \in (B+x) \cap (B+y)$. Consequently every uncountable Borel subset of ${}^{\omega}2$ is a 2–**pots**–set.

2. Assume $B \subseteq {}^{\omega}2$ and $x, y \in {}^{\omega}2$. If $b_x, b_y \in B$ and $b_x + x = b_y + y \in (B+x) \cap (B+y)$, then also $b_x + y = b_y + x \in (B+x) \cap (B+y)$. Consequently, if $(B+x) \cap (B+y) \neq \emptyset$ is finite, then it has an even number of elements.

Proposition 3.3. *1. Let $1 \leq \kappa \leq \mathfrak{c}$. A set $B \subseteq {}^{\omega}2$ is a κ–**pots**–set if and only if there is a perfect set $P \subseteq {}^{\omega}2$ such that $P \times P \subseteq \mathrm{stnd}_\kappa(B)$.*

2. Assume $k < \omega$. If B is Σ^0_2, then $\mathrm{stnd}_k(B)$ is Σ^0_2 as well. If B is Borel, then $\mathrm{stnd}_k(B)$ and $\mathrm{stnd}_\omega(B)$ are Σ^1_1 and $\mathrm{stnd}_{\mathfrak{c}}(B)$ is Δ^1_2.

3. Let $\mathfrak{c} < \lambda \leq \mu$ and let $\mathbb{C}_\mu$ be the forcing notion adding μ Cohen reals. Then, remembering Definition 3.1(2),

$$\Vdash_{\mathbb{C}_\mu} \text{“if a Borel set } B \subseteq {}^{\omega}2 \text{ has } \lambda \text{ many pairwise } \kappa\text{–non-disjoint translates, then } B \text{ is a } \kappa\text{–}\mathbf{pots}\text{–set”}.$$

4. *If $k<\omega$, B is a (code for) Σ^0_2 k–**npots**–set and $\mathbb{P}$ is a forcing notion, then $\Vdash_{\mathbb{P}}$ " B is a (code for) k–**npots**–set ".*

5. *Assume $\Pr_{\omega_1}(\lambda)$. If $\kappa\leq\omega$ and a Borel set $B\subseteq{}^\omega 2$ has λ many pairwise κ–nondisjoint translates, then it is a κ–**pots**–set.*

Proof. (2) Let $B=\bigcup\limits_{n<\omega}F_n$, where each F_n is a closed subset of ${}^\omega 2$. Then

$$(x,y)\in\mathrm{stnd}_k(B)\Leftrightarrow$$
$$\big(\exists n_0,\ldots,n_{k-1},m_0,\ldots,m_{k-1},N<\omega\big)\big(\exists z_0,\ldots,z_{k-1}\in{}^\omega 2\big)\big(\forall i,j<k\big)\Big(\\ (i\neq j\Rightarrow z_i{\restriction}N\neq z_j{\restriction}N)\ \wedge\ z_i+x\in F_{n_i}\ \wedge\ z_i+y\in F_{m_i}\Big)$$

The formula

$$\big(\forall i,j<k\big)\big((i\neq j\Rightarrow z_i{\restriction}N\neq z_j{\restriction}N)\ \wedge\ z_i+x\in F_{n_i}\ \wedge\ z_i+y\in F_{m_i}\big)$$

represents a compact subset of $\big({}^\omega 2\big)^{k+2}$ and hence easily the assertion follows.

(3) This is a consequence of (1,2) above and Shelah [7, Fact 1.16].

(4) If B is a Σ^0_2 set then the formula "there is a perfect set $P\subseteq{}^\omega 2$ such that for all $x,y\in P$ we have $(x,y)\in\mathrm{stnd}_k(B)$ " is Σ^1_2 (remember (2) above).

(5) By [7, Claim 1.12(1)]. □

We want to analyze k–**pots**–sets in more detail, restricting ourselves to Σ^0_2 subsets of ${}^\omega 2$ and even $k<\omega$. For the rest of this section we assume the following Hypothesis.

Hypothesis 3.4. 1. $T_n\subseteq{}^{\omega>}2$ is a tree with no maximal nodes (for $n<\omega$);

2. $B=\bigcup\limits_{n<\omega}\lim(T_n)$, $\bar{T}=\langle T_n:n<\omega\rangle$;

3. $2\leq\iota<\omega$, $k=2\iota$.

Definition 3.5. Let $\mathbf{M}_{\bar{T},k}$ consist of all tuples

$$\mathbf{m}=(\ell_{\mathbf{m}},u_{\mathbf{m}},\bar{h}_{\mathbf{m}},\bar{g}_{\mathbf{m}})=(\ell,u,\bar{h},\bar{g})$$

such that:

(a) $0<\ell<\omega$, $u\subseteq {}^{\ell}2$ and $2\leq |u|$;

(b) $\bar{h}=\langle h_i:i<\iota\rangle$, $\bar{g}=\langle g_i:i<\iota\rangle$ and for each $i<\iota$ we have

$$h_i:u^{\langle 2\rangle}\longrightarrow\omega \quad \text{and} \quad g_i:u^{\langle 2\rangle}\longrightarrow\bigcup_{n<\omega}(T_n\cap {}^{\ell}2)$$

(remember $u^{\langle 2\rangle}=\{(\eta,\nu)\in u\times u:\eta\neq\nu\}$);

(c) $g_i(\eta,\nu)\in T_{h_i(\eta,\nu)}\cap {}^{\ell}2$ for all $(\eta,\nu)\in u^{\langle 2\rangle}$, $i<\iota$;

(d) if $(\eta,\nu)\in u^{\langle 2\rangle}$ and $i<\iota$, then $\eta+g_i(\eta,\nu)=\nu+g_i(\nu,\eta)$;

(e) for any $(\eta,\nu)\in u^{\langle 2\rangle}$, there are no repetitions in the sequence $\langle g_i(\eta,\nu),g_i(\nu,\eta):i<\iota\rangle$.

Definition 3.6. Assume $\mathbf{m}=(\ell,u,\bar{h},\bar{g})\in\mathbf{M}_{\bar{T},k}$ and $\rho\in {}^{\ell}2$. We define $\mathbf{m}+\rho=(\ell',u',\bar{h}',\bar{g}')$ by

- $\ell'=\ell$, $u'=\{\eta+\rho:\eta\in u\}$,
- $\bar{h}'=\langle h'_i:i<\iota\rangle$ where $h'_i:(u')^{\langle 2\rangle}\longrightarrow\omega$ are such that $h'_i(\eta+\rho,\nu+\rho)=h_i(\eta,\nu)$ for $(\eta,\nu)\in u^{\langle 2\rangle}$,
- $\bar{g}'=\langle g'_i:i<\iota\rangle$ where $g'_i:(u')^{\langle 2\rangle}\longrightarrow\bigcup\limits_{n<\omega}(T_n\cap {}^{\ell}2)$ are such that $g'_i(\eta+\rho,\nu+\rho)=g_i(\eta,\nu)$ for $(\eta,\nu)\in u^{\langle 2\rangle}$.

Also if $\rho\in {}^{\omega}2$, then we set $\mathbf{m}+\rho=\mathbf{m}+(\rho{\restriction}\ell)$.

Observation 3.7. *1. If $\mathbf{m}\in\mathbf{M}_{\bar{T},k}$ and $\rho\in {}^{\ell_{\mathbf{m}}}2$, then $\mathbf{m}+\rho\in\mathbf{M}_{\bar{T},k}$.*

2. For each $\rho\in {}^{\omega}2$ the mapping

$$\mathbf{M}_{\bar{T},k}\longrightarrow\mathbf{M}_{\bar{T},k}:\mathbf{m}\mapsto\mathbf{m}+\rho$$

is a bijection.

Definition 3.8. Assume $\mathbf{m},\mathbf{n}\in\mathbf{M}_{\bar{T},k}$. We say that $\mathbf{n}$ *extends* $\mathbf{m}$ ($\mathbf{m}\sqsubseteq\mathbf{n}$ in short) if and only if:

- $\ell_{\mathbf{m}}\leq\ell_{\mathbf{n}}$, $u_{\mathbf{m}}=\{\eta{\restriction}\ell_{\mathbf{m}}:\eta\in u_{\mathbf{n}}\}$, and

- for every $(\eta,\nu)\in(u_{\mathbf{n}})^{\langle 2\rangle}$ such that $\eta\restriction\ell_{\mathbf{m}}\neq\nu\restriction\ell_{\mathbf{m}}$ and each $i<\iota$ we have

$$h_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}},\nu\restriction\ell_{\mathbf{m}})=h_i^{\mathbf{n}}(\eta,\nu)\quad\text{and}\quad g_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}},\nu\restriction\ell_{\mathbf{m}})=g_i^{\mathbf{n}}(\eta,\nu)\restriction\ell_{\mathbf{m}}.$$

Definition 3.9. We define a function[3] ndrk : $\mathbf{M}_{\bar{T},k}\longrightarrow \mathrm{ON}\cup\{\infty\}$ declaring inductively when $\mathrm{ndrk}(\mathbf{m})\geq\alpha$ (for an ordinal α).

- $\mathrm{ndrk}(\mathbf{m})\geq 0$ always;
- if α is a limit ordinal, then

$$\mathrm{ndrk}(\mathbf{m})\geq\alpha\Leftrightarrow(\forall\beta<\alpha)(\mathrm{ndrk}(\mathbf{m})\geq\beta);$$

- if $\alpha=\beta+1$, then $\mathrm{ndrk}(\mathbf{m})\geq\alpha$ if and only if for every $\nu\in u_{\mathbf{m}}$ there is $\mathbf{n}\in\mathbf{M}_{\bar{T},k}$ such that $\ell_{\mathbf{n}}>\ell_{\mathbf{m}}$, $\mathbf{m}\sqsubseteq\mathbf{n}$ and $\mathrm{ndrk}(\mathbf{n})\geq\beta$ and

$$|\{\eta\in u_{\mathbf{n}}:\nu\vartriangleleft\eta\}|\geq 2;$$

- $\mathrm{ndrk}(\mathbf{m})=\infty$ if and only if $\mathrm{ndrk}(\mathbf{m})\geq\alpha$ for all ordinals α.

We also define

$$\mathrm{NDRK}(\bar{T})=\sup\{\mathrm{ndrk}(\mathbf{m})+1:\mathbf{m}\in\mathbf{M}_{\bar{T},k}\}.$$

Lemma 3.10. 1. *The relation* $\sqsubseteq$ *is a partial order on* $\mathbf{M}_{\bar{T},k}$.

2. *If* $\mathbf{m},\mathbf{n}\in\mathbf{M}_{\bar{T},k}$ *and* $\mathbf{m}\sqsubseteq\mathbf{n}$ *and* $\alpha\leq\mathrm{ndrk}(\mathbf{n})$, *then* $\alpha\leq\mathrm{ndrk}(\mathbf{m})$.
3. *The function* ndrk *is well defined.*
4. *If* $\mathbf{m}\in\mathbf{M}_{\bar{T},k}$ *and* $\rho\in{}^{\omega}2$ *then* $\mathrm{ndrk}(\mathbf{m})=\mathrm{ndrk}(\mathbf{m}+\rho)$.
5. *If* $\mathbf{m}\in\mathbf{M}_{\bar{T},k}$, $\nu\in u_{\mathbf{m}}$ *and* $\mathrm{ndrk}(\mathbf{m})\geq\omega_1$, *then there is an* $\mathbf{n}\in\mathbf{M}_{\bar{T},k}$ *such that* $\mathbf{m}\sqsubseteq\mathbf{n}$, $\mathrm{ndrk}(\mathbf{n})\geq\omega_1$, *and*

$$|\{\eta\in u_{\mathbf{n}}:\nu\vartriangleleft\eta\}|\geq 2.$$

6. *If* $\mathbf{m}\in\mathbf{M}_{\bar{T},k}$ *and* $\infty>\mathrm{ndrk}(\mathbf{m})=\beta>\alpha$, *then there is* $\mathbf{n}\in\mathbf{M}_{\bar{T},k}$ *such that* $\mathbf{m}\sqsubseteq\mathbf{n}$ *and* $\mathrm{ndrk}(\mathbf{n})=\alpha$.

[3] ndrk stands for **n**on**d**isjointness **r**an**k**.

7. *If* $\mathrm{NDRK}(\bar{T}) \geq \omega_1$*, then* $\mathrm{NDRK}(\bar{T}) = \infty$.

8. *Assume* $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ *and* $u' \subseteq u_{\mathbf{m}}$, $|u'| \geq 2$. *Put* $\ell' = \ell_m$, $h'_i = h^{\mathbf{m}}_i \restriction u^{\langle 2 \rangle}$ *and* $g'_i = g^{\mathbf{m}}_i \restriction u^{\langle 2 \rangle}$ *(for* $i < \iota$*), and let* $\mathbf{m} \restriction u' = (\ell', u', \bar{h}', \bar{g}')$. *Then* $\mathbf{m} \restriction u' \in \mathbf{M}_{\bar{T},k}$ *and* $\mathrm{ndrk}(\mathbf{m}) \leq \mathrm{ndrk}(\mathbf{m} \restriction u')$.

Proof. (1) Straightforward.

(2) Induction on α. If $\alpha = \alpha_0 + 1$ and $\mathbf{n}' \sqsupseteq \mathbf{n}$ is one of the witnesses used to claim that $\mathrm{ndrk}(\mathbf{n}) \geq \alpha_0 + 1$, then this $\mathbf{n}'$ can also be used for $\mathbf{m}$. Hence we can argue the successor step of the induction. The limit steps are even easier.

(3) One has to show that if $\beta < \alpha$ and $\mathrm{ndrk}(\mathbf{m}) \geq \alpha$, then $\mathrm{ndrk}(\mathbf{m}) \geq \beta$. This can be shown by induction on α: at the successor stage if $\mathbf{n}$ is one of the witnesses used to claim that $\mathrm{ndrk}(\mathbf{m}) \geq \alpha + 1$, then $\mathrm{ndrk}(\mathbf{n}) \geq \alpha$. By (2) we get $\mathrm{ndrk}(\mathbf{m}) \geq \alpha$ and by the inductive hypothesis $\mathrm{ndrk}(\mathbf{m}) \geq \gamma$ for $\gamma \leq \alpha$. Limit stages are easy too.

(4) Clear.

(5) Let $\mathcal{N}$ be the collection of all $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and $|\{\eta \in u_{\mathbf{n}} : \nu \vartriangleleft \eta\}| \geq 2$. If $\mathrm{ndrk}(\mathbf{n}_0) \geq \omega_1$ for some $\mathbf{n}_0 \in \mathcal{N}$, then we are done. So suppose towards contradiction that there is no such $\mathbf{n}_0$. Then, as $\mathcal{N}$ is countable,

$$\alpha_0 \stackrel{\mathrm{def}}{=} \sup\{\mathrm{ndrk}(\mathbf{n}) + 1 : \mathbf{n} \in \mathcal{N}\} < \omega_1.$$

But $\mathrm{ndrk}(\mathbf{m}) \geq \alpha_0 + 1$ implies that $\mathrm{ndrk}(\mathbf{n}_1) \geq \alpha_0$ for some $\mathbf{n}_1 \in \mathcal{N}$, a contradiction.

(6) Induction on ordinals β (for all $\alpha < \beta$). The main point is that if $\mathrm{ndrk}(\mathbf{m}) = \beta$, then for some $\nu \in u_{\mathbf{m}}$ we cannot find $\mathbf{n}$ as needed for witnessing $\mathrm{ndrk}(\mathbf{m}) \geq \beta + 1$, but for each $\gamma < \beta$ we can find $\mathbf{n}$ needed for $\mathrm{ndrk}(\mathbf{m}) \geq \gamma + 1$. Therefore for each $\gamma < \beta$ we may find $\mathbf{n} \sqsupseteq \mathbf{m}$ such that $\gamma \leq \mathrm{ndrk}(\mathbf{n}) < \beta$.

(7) Follows from (6) above.

(8) Clearly $(\ell', u', \bar{h}', \bar{g}') \in \mathbf{M}_{\bar{T},k}$. By a straightforward induction on α for all $\mathbf{m}$ and restrictions $\mathbf{m} \restriction u'$, one shows that

$$\alpha \leq \mathrm{ndrk}(\mathbf{m}) \ \Rightarrow\ \alpha \leq \mathrm{ndrk}(\mathbf{m} \restriction u').$$

$\square$

Proposition 3.11. *The following conditions are equivalent.*

(a) $\mathrm{NDRK}(\bar{T}) \geq \omega_1$.

(b) $\mathrm{NDRK}(\bar{T}) = \infty$.

(c) There is a perfect set $P \subseteq {}^{\omega}2$ *such that*

$$\big(\forall \eta, \nu \in P\big)\big(|(B+\eta)\cap(B+\nu)| \geq k\big).$$

(d) In some ccc forcing extension, there is $A \subseteq {}^{\omega}2$ *of cardinality* λ_{ω_1} *such that*

$$\big(\forall \eta, \nu \in A\big)\big(|(B+\eta)\cap(B+\nu)| \geq k\big).$$

Proof. (a) $\Rightarrow$ (b) This is Lemma 3.10(7).

(b) $\Rightarrow$ (c) If $\mathrm{NDRK}(\bar{T}) = \infty$ then there is $\mathbf{m}_0 \in \mathbf{M}_{\bar{T},k}$ with $\mathrm{ndrk}(\mathbf{m}_0) \geq \omega_1$. Using Lemma 3.10(5) we may now choose a sequence $\langle \mathbf{m}_j : j < \omega\rangle \subseteq \mathbf{M}_{\bar{T},k}$ such that for each $j<\omega$:

(i) $\mathbf{m}_j \sqsubseteq \mathbf{m}_{j+1}$,

(ii) $\mathrm{ndrk}(\mathbf{m}_j) \geq \omega_1$,

(iii) $|\{\eta \in u_{\mathbf{m}_{j+1}} : \nu \lhd \eta| \geq 2$ for each $\nu \in u_{\mathbf{m}_j}$.

Let $P = \{\rho \in {}^{\omega}2 : (\forall j<\omega)(\rho{\restriction}\ell_{\mathbf{m}_j} \in u_{\mathbf{m}_j})\}$. Clearly, P is a perfect set. For $\eta,\nu \in P$, $\eta \neq \nu$, let j_0 be the smallest such that $\eta{\restriction}\ell_{\mathbf{m}_{j_0}} \neq \nu{\restriction}\ell_{\mathbf{m}_{j_0}}$ and let

$$G_i(\eta,\nu) = \bigcup\big\{g_i^{\mathbf{m}_j}(\eta{\restriction}\ell_{\mathbf{m}_j}, \nu{\restriction}\ell_{\mathbf{m}_j}) : j \geq j_0\big\} \in \lim\Big(T_{h_i^{\mathbf{m}_{j_0}}(\eta{\restriction}\ell_{\mathbf{m}_{j_0}}, \nu{\restriction}\ell_{\mathbf{m}_{j_0}})}\Big)$$

for $i<\iota$. Then $G_i : P^{\langle 2\rangle} \longrightarrow B$ and for $(\eta,\nu) \in P^{\langle 2\rangle}$ and $i<\iota$:

$$\eta + G_i(\eta,\nu) = \nu + G_i(\nu,\eta) \quad \text{and} \quad \eta + G_i(\nu,\eta) = \nu + G_i(\eta,\nu).$$

Moreover, there are no repetitions in the sequence $\langle G_i(\eta,\nu), G_i(\nu,\eta){:}i<\iota\rangle$. Hence, for distinct $\eta,\nu\in P$ we have $|(B+\eta)\cap(B+\nu)| \geq 2\iota = k$.

(c) $\Rightarrow$ (d) Assume (c). Let $\kappa = \beth_{\omega_1}$. By Corollary 2.11 we know that $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mathfrak{c}$. Remembering Proposition 3.3(1,2), we note that the formula "$P \times P \subseteq \mathrm{stnd}_k(B)$" is Π^1_1, so it holds in the forcing extension by $\mathbb{C}_\kappa$. Now we easily conclude (d).

(d) ⇒ (a) Assume (d) and let $\mathbb{P}$ be the ccc forcing notion witnessing this assumption, $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$. Let us work in $\mathbf{V}[G]$.

Let $\langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle$ be a sequence of distinct elements of ${}^\omega 2$ such that

$$\big(\forall \alpha < \beta < \lambda_{\omega_1}\big)\big(|(B+\eta_\alpha) \cap (B+\eta_\beta)| \geq k\big).$$

Let $\tau = \{R_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_{\bar{T},k}\}$ be a (countable) vocabulary where each $R_{\mathbf{m}}$ is a $|u_{\mathbf{m}}|$–ary relational symbol. Let $\mathbb{M} = \big(\lambda_{\omega_1}, \{R^{\mathbb{M}}_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{M}_{\bar{T},k}}\big)$ be the model in the vocabulary τ, where for $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},k}$ the relation $R^{\mathbb{M}}_{\mathbf{m}}$ is defined by

$$\begin{aligned} R^{\mathbb{M}}_{\mathbf{m}} = \Big\{ &(\alpha_0, \ldots, \alpha_{|u|-1}) \in (\lambda_{\omega_1})^{|u|} : \{\eta_{\alpha_0}{\restriction}\ell, \ldots, \eta_{|u|-1}{\restriction}\ell\} = u \text{ and} \\ &\text{for distinct } j_1, j_2 < |u| \text{ there are } G_i(\alpha_{j_1}, \alpha_{j_2}) \text{ (for } i < \iota) \text{ such that} \\ &g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) \lhd G_i(\alpha_{j_1}, \alpha_{j_2}) \in \lim\big(T_{h_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell)}\big) \text{ and} \\ &\eta_{\alpha_{j_1}} + G_i(\alpha_{j_1}, \alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2}, \alpha_{j_1}) \Big\}. \end{aligned}$$

Claim 3.11.1. 1. *If $\alpha_0, \alpha_1, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$ are distinct, $j \geq 2$, then for sufficiently large $\ell < \omega$ there is $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ such that*

$$\ell_{\mathbf{m}} = \ell, \quad u_{\mathbf{m}} = \{\eta_{\alpha_0}{\restriction}\ell, \ldots, \eta_{\alpha_{j-1}}{\restriction}\ell\} \quad \textit{and} \quad \mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}].$$

2. *Assume that $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$, $j < |u_{\mathbf{m}_0}|$, $\alpha_0, \alpha_1, \ldots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ and $\alpha^* < \lambda_{\omega_1}$ are all pairwise distinct and such that*

$$\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_j, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$$

and

$$\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots \alpha_{|u_{\mathbf{m}}|-1}].$$

Then for every sufficiently large $\ell > \ell_{\mathbf{m}}$ there is $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and

$$\begin{aligned} \ell_{\mathbf{n}} = \ell, \quad & u_{\mathbf{n}} = \{\eta_{\alpha_0}{\restriction}\ell, \ldots, \eta_{\alpha_{|u_{\mathbf{m}}|-1}}{\restriction}\ell, \eta_{\alpha^*}{\restriction}\ell\} \\ \textit{and} \quad & \mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*]. \end{aligned}$$

3. *If $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$, then*

$$\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \leq \mathrm{ndrk}(\mathbf{m}).$$

Proof of the Claim. (1) For distinct $j_1, j_2 < j$ let $G_i(\alpha_{j_1}, \alpha_{j_2}) \in B$ (for $i < \iota$) be such that

$$\eta_{\alpha_{j_1}} + G_i(\alpha_{j_1}, \alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2}, \alpha_{j_1})$$

and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1}, \alpha_{j_2}), G_i(\alpha_{j_2}, \alpha_{j_1}) : i < \iota\rangle$. (Remember, $x \in (B+\eta_{\alpha_{j_1}}) \cap (B+\eta_{\alpha_{j_2}})$ if and only if $x+(\eta_{\alpha_{j_1}}+\eta_{\alpha_{j_2}}) \in (B + \eta_{\alpha_{j_1}}) \cap (B + \eta_{\alpha_{j_2}})$, so the choice of $G_i(\alpha_{j_1}, \alpha_{j_2})$ is possible by the assumptions on η_α's.) Suppose that $\ell < \omega$ is such that for any distinct $j_1, j_2 < j$ we have $\eta_{\alpha_{j_1}}{\restriction}\ell \neq \eta_{\alpha_{j_2}}{\restriction}\ell$ and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1}, \alpha_{j_2}){\restriction}\ell, G_i(\alpha_{j_2}, \alpha_{j_1}){\restriction}\ell : i < \iota\rangle$. Now let $u = \{\eta_{\alpha_{j'}}{\restriction}\ell : j' < j\}$, and for $i < \iota$ let $g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) = G_i(\alpha_{j_1}, \alpha_{j_2}){\restriction}\ell$, and let $h_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) < \omega$ be such that $G_i(\alpha_{j_1}, \alpha_{j_2}) \in \lim\big(T_{h_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell)}\big)$. This defines $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},k}$ and easily $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}]$.

(2) An obvious modification of the argument above.

(3) By induction on β we show that *for every* $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and *all* $\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$:

$$\beta \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \text{ implies } \beta \le \mathrm{ndrk}(\mathbf{m}).$$

Steps $\beta = 0$ and β is limit: Straightforward.

Step $\beta = \gamma + 1$: Suppose $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ are such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\gamma + 1 \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$. Let $\nu \in u_{\mathbf{m}}$, so $\nu = \eta_{\alpha_j}{\restriction}\ell_{\mathbf{m}}$ for some $j < |u_{\mathbf{m}}|$. Since

$$\gamma + 1 \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$$

we may find $\alpha^* \in \lambda_{\omega_1} \setminus \{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}$ such that

$$\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u|-1}]$$

and $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u|-1}, \alpha^*\}, \mathbb{M}) \ge \gamma$. Taking sufficiently large ℓ we may use clause (2) to find $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$, $\ell_{\mathbf{n}} = \ell$ and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*]$ and $|\{\eta \in u_{\mathbf{n}} : \nu \vartriangleleft \eta\}| \ge 2$. By the inductive hypothesis we have also $\gamma \le \mathrm{ndrk}(\mathbf{n})$. Now we may easily conclude that $\gamma + 1 \le \mathrm{ndrk}(\mathbf{m})$. □

By the definition of λ_{ω_1},

($\odot$) $\sup\{\mathrm{rk}(w,\mathbb{M}) : \emptyset \neq w \in [\lambda_{\omega_1}]^{<\omega}\} \geq \omega_1$

Now, suppose that $\beta < \omega_1$. By ($\odot$), there are distinct $\alpha_0,\ldots,\alpha_{j-1} < \lambda_{\omega_1}$, $j \geq 2$, such that $\mathrm{rk}(\{\alpha_0,\ldots,\alpha_{j-1}\},\mathbb{M}) \geq \beta$. By Claim 3.11.1(1) we may find $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1}]$. Then by Claim 3.11.1(3) we also have $\mathrm{ndrk}(\mathbf{m}) \geq \beta$. Consequently, $\mathrm{NDRK}(\bar{T}) \geq \omega_1$.

All the considerations above where carried out in $\mathbf{V}[G]$. However, the rank function ndrk is absolute, so we may also claim that in $\mathbf{V}$ we have $\mathrm{NDRK}(\bar{T}) \geq \omega_1$. $\square$

Corollary 3.12. *Assume that $\varepsilon \leq \omega_1$ and $\mathrm{Pr}_\varepsilon(\lambda)$. If there is $A \subseteq {}^{\omega}2$ of cardinality λ such that*

$$\big(\forall \eta,\nu \in A\big)\big(|(B+\eta)\cap(B+\nu)| \geq k\big),$$

then $\mathrm{NDRK}(\bar{T}) \geq \varepsilon$.

Proof. This is essentialy shown by the proof of the implication (d) $\Rightarrow$ (a) of Proposition 3.11. $\square$

4. The forcing

In this section we construct a forcing notion adding a sequence $\bar{T}$ of subtrees of ${}^{\omega>}2$ such that $\mathrm{NDRK}(\bar{T}) < \omega_1$. The sequence $\bar{T}$ will be added by finite approximations, so it will be convenient to have finite version of Definition 3.5.

Definition 4.1. Assume that

- $2 \leq \iota < \omega$, $k = 2\iota$, and $0 < n, M < \omega$,
- $\bar{t} = \langle t_m : m < M\rangle$, and each t_m is a subtree of ${}^{n\geq}2$ in which all terminal branches are of length n,
- $T_j \subseteq {}^{\omega>}2$ (for $j < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_j : j < \omega\rangle$ and $t_m = T_m \cap {}^{n\geq}2$ for $m < M$,
- $\mathbf{M}_{\bar{T},k}$ is defined as in Definition 3.5.

1. Let $\mathbf{M}^n_{\bar{t},k}$ consist of all tuples $\mathbf{m} = (\ell_\mathbf{m}, u_\mathbf{m}, \bar{h}_\mathbf{m}, \bar{g}_\mathbf{m}) \in \mathbf{M}_{\bar{T},k}$ such that $\ell_\mathbf{m} \leq n$ and $\mathrm{rng}(h^\mathbf{m}_i) \subseteq M$ for each $i < \iota$.

2. Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}^n_{\bar{t},k}$. We say that $\mathbf{m}$, $\mathbf{n}$ *are essentially the same* ($\mathbf{m} \doteqdot \mathbf{n}$ in short) if and only if:

   - $\ell_\mathbf{m} = \ell_\mathbf{n}$, $u_\mathbf{m} = u_\mathbf{n}$ and
   - for each $(\eta, \nu) \in (u_\mathbf{m})^{\langle 2 \rangle}$ we have
   $$\big\{\{g^\mathbf{m}_i(\eta,\nu), g^\mathbf{m}_i(\nu,\eta)\} : i < \iota\big\} = \big\{\{g^\mathbf{n}_i(\eta,\nu), g^\mathbf{n}_i(\nu,\eta)\} : i < \iota\big\},$$
   and for $i, j < \iota$:
   if $g^\mathbf{m}_i(\eta,\nu) = g^\mathbf{n}_j(\eta,\nu)$, then $h^\mathbf{m}_i(\eta,\nu) = h^\mathbf{n}_j(\eta,\nu)$,
   if $g^\mathbf{m}_i(\eta,\nu) = g^\mathbf{n}_j(\nu,\eta)$, then $h^\mathbf{m}_i(\eta,\nu) = h^\mathbf{n}_j(\nu,\eta)$.

3. Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}^n_{\bar{t},k}$. We say that $\mathbf{n}$ *essentially extends* $\mathbf{m}$ ($\mathbf{m} \sqsubseteq^* \mathbf{n}$ in short) if and only if:

   - $\ell_\mathbf{m} \leq \ell_\mathbf{n}$, $u_\mathbf{m} = \{\eta \restriction \ell_\mathbf{m} : \eta \in u_\mathbf{n}\}$, and
   - for every $(\eta, \nu) \in (u_\mathbf{n})^{\langle 2 \rangle}$ such that $\eta \restriction \ell_\mathbf{m} \neq \nu \restriction \ell_\mathbf{m}$ we have
   $$\begin{aligned} &\big\{\{g^\mathbf{m}_i(\eta\restriction\ell_\mathbf{m}, \nu\restriction\ell_\mathbf{m}), g^\mathbf{m}_i(\nu\restriction\ell_\mathbf{m}, \eta\restriction\ell_\mathbf{m})\} : i < \iota\big\} \\ = \; &\big\{\{g^\mathbf{n}_i(\eta,\nu)\restriction\ell_\mathbf{m}, g^\mathbf{n}_i(\nu,\eta)\restriction\ell_\mathbf{m}\} : i < \iota\big\}, \end{aligned}$$
   and for $i, j < \iota$:
   if $g^\mathbf{m}_i(\eta\restriction\ell_\mathbf{m}, \nu\restriction\ell_\mathbf{m}) = g^\mathbf{n}_j(\eta,\nu)\restriction\ell_\mathbf{m}$, then $h^\mathbf{m}_i(\eta\restriction\ell_\mathbf{m}, \nu\restriction\ell_\mathbf{m}) = h^\mathbf{n}_j(\eta,\nu)$,
   if $g^\mathbf{m}_i(\eta\restriction\ell_\mathbf{m}, \nu\restriction\ell_\mathbf{m}) = g^\mathbf{n}_j(\nu,\eta)\restriction\ell_\mathbf{m}$, then $h^\mathbf{m}_i(\eta\restriction\ell_\mathbf{m}, \nu\restriction\ell_\mathbf{m}) = h^\mathbf{n}_j(\nu,\eta)$.

Observation 4.2. *If* $\mathbf{m} \in \mathbf{M}^n_{\bar{t},k}$ *and* $\rho \in {}^{\ell_\mathbf{m}}2$, *then* $\mathbf{m} + \rho \in \mathbf{M}^n_{\bar{t},k}$ *(remember Definition 3.6).*

Lemma 4.3. *Let* $0 < \ell < \omega$ *and let* $\mathcal{B} \subseteq {}^\ell 2$ *be a linearly independent set of vectors (in* $({}^\ell 2, +)$ *over* $(2, +_2, \cdot_2)$*).*

1. *If* $\mathcal{A} \subseteq {}^\ell 2$, $|\mathcal{A}| \geq 5$ *and* $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, *then for a unique* $x \in {}^\ell 2$ *we have* $\mathcal{A} + x \subseteq \mathcal{B}$.

2. *Let* $b^* \in \mathcal{B}$. *Suppose that* $\rho^0_i, \rho^1_i \in \big(\mathcal{B} \cup (b^* + \mathcal{B})\big) \setminus \{\mathbf{0}, b^*\}$ *(for* $i < 3$*) are such that*

   (a) there are no repetitions in $\langle \rho^0_i, \rho^1_i : i < 3 \rangle$, *and*

(b) $\rho^0_i+\rho^1_i=\rho^0_j+\rho^1_j$ *for* $i<j<3$.

Then $\big\{\{\rho^0_i,\rho^1_i\}:i<3\}\subseteq\{\{b,b+b^*\}:b\in\mathcal{B},\ b\neq b^*\big\}$.

Proof. Easy, for (1) see e.g. [5, Lemma 2.3]. □

Theorem 4.4. *Assume* $\mathrm{NPr}_{\omega_1}(\lambda)$ *and let* $3\leq\iota<\omega$. *Then there is a ccc forcing notion* $\mathbb{P}$ *of size* λ *such that*

$\Vdash_{\mathbb{P}}$ *"for some* Σ^0_2 2ι–**npots**–*set* $B\subseteq{}^\omega 2$ *there is a sequence* $\langle\eta_\alpha:\alpha<\lambda\rangle$ *of distinct elements of* ${}^\omega 2$ *such that* $\big|(\eta_\alpha+B)\cap(\eta_\beta+B)\big|\geq 2\iota$ *for all* $\alpha,\beta<\lambda$*".*

Proof. If $Q\subseteq{}^\omega 2$ is a countable infinite subgroup of ${}^\omega 2$ then Q is npots but Q has ω–many pairwise ω–nondisjoint translations. So we may assume that λ is uncountable.

Fix a countable vocabulary $\tau=\{R_{n,\zeta}:n,\zeta<\omega\}$, where $R_{n,\zeta}$ is an n–ary relational symbol (for $n,\zeta<\omega$). By the assumption on λ, we may fix a model $\mathbb{M}=(\lambda,\{R^{\mathbb{M}}_{n,\zeta}\}_{n,\zeta<\omega})$ in the vocabulary τ with the universe λ and an ordinal $\alpha^*<\omega_1$ such that:

$(\circledast)_{\rm a}$ for every n and a quantifier free formula $\varphi(x_0,\ldots,x_{n-1})\in\mathcal{L}(\tau)$ there is $\zeta<\omega$ such that for all $a_0,\ldots,a_{n-1}\in\lambda$,

$$\mathbb{M}\models\varphi[a_0,\ldots,a_{n-1}]\Leftrightarrow R_{n,\zeta}[a_0,\ldots,a_{n-1}],$$

$(\circledast)_{\rm b}$ $\sup\{\mathrm{rk}(v,\mathbb{M}):\emptyset\neq v\in[\lambda]^{<\omega}\}<\alpha^*$,

$(\circledast)_{\rm c}$ the rank of every singleton is at least 0.

For a nonempty finite set $v\subseteq\lambda$ let $\mathrm{rk}(v)=\mathrm{rk}(v,\mathbb{M})$, and let $\zeta(v)<\omega$ and $k(v)<|v|$ be such that $R_{|v|,\zeta(v)},k(v)$ witness the rank of v. Thus letting $\{a_0,\ldots,a_k,\ldots a_{n-1}\}$ be the increasing enumeration of v and $k=k(v)$ and $\zeta=\zeta(v)$, we have

$(\circledast)_{\rm d}$ if $\mathrm{rk}(v)\geq 0$, then $\mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_k,\ldots,a_{n-1}]$ but there is no $a\in\lambda\setminus v$ such that

$$\mathrm{rk}(v\cup\{a\})\geq\mathrm{rk}(v)\ \text{ and }\ \mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_{k-1},a,a_{k+1},\ldots,a_{n-1}],$$

$(\circledast)_{\rm e}$ if $\mathrm{rk}(v) = -1$, then $\mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_k, \ldots, a_{n-1}]$ but the set

$$\big\{a \in \lambda : \mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1}]\big\}$$

is countable.

Without loss of generality we may also require that (for $\zeta = \zeta(v)$, $n = |v|$)

$(\circledast)_{\rm f}$ for every $b_0, \ldots, b_{n-1} < \lambda$

$$\text{if } \ \mathbb{M} \models R_{n,\zeta}[b_0, \ldots, b_{n-1}] \text{ then } \ b_0 < \ldots < b_{n-1}.$$

Now we will define a forcing notion $\mathbb{P}$. *A condition p* in $\mathbb{P}$ is a tuple

$$\big(w^p, n^p, M^p, \bar{\eta}^p, \bar{t}^p, \bar{r}^p, \bar{h}^p, \bar{g}^p, \mathcal{M}^p\big) = \big(w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}\big)$$

such that the following demands $(*)_1$–$(*)_{11}$ are satisfied.

$(*)_1$ $w \in [\lambda]^{<\omega}$, $|w| \geq 5$, $0 < n, M < \omega$.

$(*)_2$ $\bar{\eta} = \langle \eta_\alpha : \alpha \in w\rangle$ is a sequence of linearly independent vectors in ${}^n 2$ (over the field $\mathbb{Z}_2$); so in particular $\eta_\alpha \in {}^n 2$ are pairwise distinct non-zero sequences (for $\alpha \in w$).

$(*)_3$ $\bar{t} = \langle t_m : m < M\rangle$, where $\emptyset \neq t_m \subseteq {}^{n\geq} 2$ for $m < M$ is a tree in which all terminal branches are of length n and $t_m \cap t_{m'} \cap {}^n 2 = \emptyset$ for $m < m' < M$.

$(*)_4$ $\bar{r} = \langle r_m : m < M\rangle$, where $0 < r_m \leq n$ for $m < M$.

$(*)_5$ $\bar{h} = \langle h_i : i < \iota\rangle$, where $h_i : w^{\langle 2\rangle} \longrightarrow M$.

$(*)_6$ $\bar{g} = \langle g_i : i < \iota\rangle$, where $g_i : w^{\langle 2\rangle} \longrightarrow \bigcup\limits_{m<M} (t_m \cap {}^n 2)$, and $g_i(\alpha, \beta) \in t_{h_i(\alpha,\beta)}$ and $\eta_\alpha + g_i(\alpha, \beta) = \eta_\beta + g_i(\beta, \alpha)$ for $(\alpha, \beta) \in w^{\langle 2\rangle}$ and $i < \iota$.

$(*)_7$ There are no repetitions in the list

$$\langle g_i(\alpha, \beta) : i < \iota,\ (\alpha, \beta) \in w^{\langle 2\rangle}\rangle.$$

$(*)_8$ $\mathcal{M}$ consists of all those $\mathbf{m} \in \mathbf{M}^n_{\bar{t},k}$ (see Definition 4.1) that for some ℓ_*, w_* we have

$(*)^{\rm a}_8$ $w_*\subseteq w$, $5\leq |w_*|$, $0<\ell_{\mathbf m}=\ell_*\leq n$, and for each $(\alpha,\beta)\in (w_*)^{\langle 2\rangle}$ and $i<\iota$ we have $r_{h_i(\alpha,\beta)}\leq\ell_*$,

$(*)^{\rm b}_8$ $u_{\mathbf m}=\{\eta_\alpha{\restriction}\ell_*:\alpha\in w_*\}$ and $\eta_\alpha{\restriction}\ell_*\neq\eta_\beta{\restriction}\ell_*$ for distinct $\alpha,\beta\in w_*$,

$(*)^{\rm c}_8$ $\bar h_{\mathbf m}=\langle h^{\mathbf m}_i:i<\iota\rangle$, where

$$h^{\mathbf m}_i:(u_{\mathbf m})^{\langle 2\rangle}\longrightarrow M:(\eta_\alpha{\restriction}\ell_*,\eta_\beta{\restriction}\ell_*)\mapsto h_i(\alpha,\beta),$$

$(*)^{\rm d}_8$ $\bar g_{\mathbf m}=\langle g^{\mathbf m}_i:i<\iota\rangle$, where

$$g^{\mathbf m}_i:(u_{\mathbf m})^{\langle 2\rangle}\longrightarrow\bigcup_{m<M}(t_m\cap{}^{\ell_*}2):(\eta_\alpha{\restriction}\ell_*,\eta_\beta{\restriction}\ell_*)\mapsto g_i(\alpha,\beta){\restriction}\ell_*$$

In the above situation we will write $\mathbf m=\mathbf m(\ell_*,w_*)=\mathbf m^p(\ell_*,w_*)$. (Note that w_* is not determined uniquely by $\mathbf m$ and we may have $\mathbf m(\ell,w_0)=\mathbf m(\ell,w_1)$ for distinct $w_0,w_1\subseteq w$. Also, the conditions $(*)^{\rm a}_8$–$(*)^{\rm d}_8$ alone do not necessarily determine an element of $\mathbf M^n_{\bar t,k}$, but clearly for each $w_*\subseteq w$ of size ≥ 5 we have $\mathbf m^p(n^p,w_*)\in\mathcal M^p$.)

$(*)_9$ If $\mathbf m(\ell,w_0),\mathbf m(\ell,w_1)\in\mathcal M$, $\rho\in{}^\ell 2$ and $\mathbf m(\ell,w_0)\doteqdot\mathbf m(\ell,w_1)+\rho$, then ${\rm rk}(w_0)={\rm rk}(w_1)$, $\zeta(w_0)=\zeta(w_1)$, $k(w_0)=k(w_1)$ and if $\alpha\in w_0$, $\beta\in w_1$ are such that $|\alpha\cap w_0|=k(w_0)=k(w_1)=|\beta\cap w_1|$, then $(\eta_\alpha{\restriction}\ell)+\rho=\eta_\beta{\restriction}\ell$.

$(*)_{10}$ If $\mathbf m(\ell_*,w_*)\in\mathcal M$, $\alpha\in w_*$, $|\alpha\cap w_*|=k(w_*)$, ${\rm rk}(w_*)=-1$, and $\mathbf m(\ell_*,w_*)\sqsubseteq^*\mathbf n\in\mathcal M$, then $|\{\nu\in u_{\mathbf n}:(\eta_\alpha{\restriction}\ell_*)\trianglelefteq\nu\}|=1$.

$(*)_{11}$ If $\rho^0_i,\rho^1_i\in\bigcup\limits_{m<M}(t_m\cap{}^n2)$ (for $i<\iota$) are such that

(a) there are no repetitions in $\langle\rho^0_i,\rho^1_i:i<\iota\rangle$, and

(b) $\rho^0_i+\rho^1_i=\rho^0_j+\rho^1_j$ for $i<j<\iota$,

then for some $\alpha,\beta\in w$ we have

$$\big\{\{\rho^0_i,\rho^1_i\}:i<\iota\big\}=\big\{\{g_i(\alpha,\beta),g_i(\beta,\alpha)\}:i<\iota\big\}.$$

To define *the order* $\leq$ *of* $\mathbb P$ we declare for $p,q\in\mathbb P$ that $p\leq q$ if and only if

- $w^p\subseteq w^q$, $n^p\leq n^q$, $M^p\leq M^q$, and
- $t^p_m=t^q_m\cap{}^{n^p\geq}2$ and $r^p_m=r^q_m$ for all $m<M^p$, and

- $\eta^p_\alpha \trianglelefteq \eta^q_\alpha$ for all $\alpha \in w^p$, and
- $h^q_i {\restriction} (w^p)^{\langle 2\rangle} = h^p_i$ and $g^p_i(\alpha,\beta) \trianglelefteq g^q_i(\alpha,\beta)$ for $i<\iota$ and $(\alpha,\beta)\in (w^p)^{\langle 2\rangle}$.

Claim 4.4.1. *Assume $p = \big(w,n,M,\bar\eta,\bar t,\bar h,\bar g,\mathcal{M}\big)\in\mathbb{P}$. If $\mathbf{m}\in \mathbf{M}^n_{\bar t,k}$ is such that $\ell_{\mathbf{m}} = n$ and $|u_{\mathbf{m}}|\geq 5$, then for some $\rho\in {}^n2$ and $\mathbf{n}\in\mathcal{M}$ we have $(\mathbf{m}+\rho)\doteqdot \mathbf{n}$.*

Proof of the Claim. Let $\mathbf{m}\in \mathbf{M}^n_{\bar t,k}$ be such that $\ell_{\mathbf{m}} = n$. It follows from Definition 3.5(d,e) and clauses $(*)_6 + (*)_{11}$ that

($\boxdot$) for every $(\nu,\eta)\in (u_{\mathbf{m}})^{\langle 2\rangle}$ there is $(\alpha,\beta)\in w^{\langle 2\rangle}$ such that $\nu+\eta = \eta_\alpha+\eta_\beta$.

By Lemma 4.3 for some ρ we have $u_{\mathbf{m}}+\rho\subseteq\{\eta_\alpha:\alpha\in w\}$. Let $w_0=\{\alpha\in w:\eta_\alpha+\rho\in u_{\mathbf{m}}\}$ and $\mathbf{n}=\mathbf{m}^p(n,w_0)\in\mathcal{M}$. Using clauses $(*)_{11}$ and $(*)_6$ we easily conclude $(\mathbf{m}+\rho)\doteqdot\mathbf{n}$. (Note that since $t_m\cap t_{m'}\cap {}^n2=\emptyset$ for $m<m'<M$, $h^{\mathbf{m}}_i(\eta,\nu)$ is determined by $g^{\mathbf{m}}_i(\eta,\nu)$.) $\square$

Claim 4.4.2. 1. *$\mathbb{P}\neq\emptyset$ and $(\mathbb{P},\leq)$ is a partial order.*

2. *For each $\beta<\lambda$ and $n_0,M_0<\omega$ the set*

$$D^{n_0,M_0}_\beta=\big\{p\in\mathbb{P}: n^p>n_0\ \wedge\ M^p>M_0\ \wedge\ \beta\in w^p\big\}$$

is open dense in $\mathbb{P}$.

Proof of the Claim. (1) Straightforward.

(2) Let $p\in\mathbb{P}$, $\beta\in\lambda\setminus w^p$. Put $N=|w^p|\cdot\iota+2$.

We will define a condition $q\in\mathbb{P}$ such that $q\geq p$ and

$$w^q=w^p\cup\{\beta\},\quad n^q=n^p+N>n^p+1,\quad M^q=M^p+N-2>M^p+1.$$

For $\alpha\in w^p$ we set $\eta^q_\alpha=\eta^p_\alpha{}^\frown\langle\underbrace{0,\dots,0}_{N}\rangle$ and we also let

$$\eta^q_\beta=\langle\underbrace{0,\dots,0}_{n^p+1}\rangle^\frown\langle\underbrace{1,\dots,1}_{N-1}\rangle.$$

Next, if $(\alpha_0,\alpha_1)\in (w^p)^{\langle 2\rangle}$, then for all $i<\iota$

$$h^q_i(\alpha_0,\alpha_1)=h^p_i(\alpha_0,\alpha_1)\quad\text{and}\quad g^q_i(\alpha_0,\alpha_1)=g^p_i(\alpha_0,\alpha_1)^\frown\langle\underbrace{0,\dots,0}_{N}\rangle.$$

If $\alpha\in w^p$ and $j=|w^p\cap\alpha|$, then for $i<\iota$:

- $g^q_i(\alpha,\beta)=\langle\underbrace{0,\ldots,0}_{n^p}\rangle^\frown\langle 1\rangle^\frown\langle\underbrace{0,\ldots,0}_{j\iota+i+1}\rangle^\frown\langle\underbrace{1,\ldots,1}_{N-j\iota-i-2}\rangle$,
- $g^q_i(\beta,\alpha)=\eta^p_\alpha{}^\frown\langle\underbrace{1,\ldots,1}_{j\iota+i+2}\rangle^\frown\langle\underbrace{0,\ldots,0}_{N-j\iota-i-2}\rangle$,
- $h^q_i(\beta,\alpha)=h^q_i(\alpha,\beta)=M^p+j\iota+i$.

We also set:

- if $m<M^p$, then $r^q_m=r^p_m$ and

  $$t^q_m=\{\eta\in {}^{n^q\geq}2:\eta\restriction n^p\in t^p_m\ \wedge\ (\forall j<n^q)(n^p\leq j<|\eta|\Rightarrow\eta(j)=0)\}$$

  and
- if $M^p\leq m<M^q$, $m=M^p+j\iota+i$, $i<\iota$ and $j<|w^p|$, then $r^q_m=n^q$ and

  $$t^q_m=\{g^q_i(\alpha,\beta)\restriction\ell, g^q_i(\beta,\alpha)\restriction\ell:\ell\leq n^q\},$$

  where $\alpha\in w^p$ is such that $|\alpha\cap w^p|=j$.

Now letting $\mathcal{M}^q$ be defined as in $(*)_8$ we check that

$$q=\big(w^q,n^q,M^q,\bar{\eta}^q,\bar{t}^q,\bar{r}^q,\bar{h}^q,\bar{g}^q,\mathcal{M}^p\big)\in\mathbb{P}.$$

Demands $(*)_1$–$(*)_8$ are pretty straightforward.

RE $(*)_9$ **:** To justify clause $(*)_9$, suppose that $\mathbf{m}^q(\ell,w_0),\mathbf{m}^q(\ell,w_1)\in\mathcal{M}^q$, $\rho\in{}^{\ell}2$ and $\mathbf{m}^q(\ell,w_0)\doteqdot\mathbf{m}^q(\ell,w_1)+\rho$, and consider the following two cases.

CASE 1: $\beta\notin w_0\cup w_1$
Then letting $\ell^*=\min(\ell,n^p)$ and $\rho^*=\rho\restriction\ell^*$ we see that $\mathbf{m}^p(\ell^*,w_0)\doteqdot\mathbf{m}^p(\ell^*,w_1)+\rho^*$ (and both belong to $\mathcal{M}^p$). Hence clause $(*)_9$ for p applies.

CASE 2: $\beta\in w_0\cup w_1$
Say, $\beta\in w_0$. If $\alpha\in w_0\setminus\{\beta\}$, then $h^q_i(\alpha,\beta)=h^q_i(\beta,\alpha)\geq M^p$ and $r^q_{h^q_i(\alpha,\beta)}=n^q$. Consequently, $\ell=n^q$. Moreover,

$$(\gamma,\delta)\in(w^q)^{\langle 2\rangle}\ \wedge\ h^q_j(\gamma,\delta)=h^q_i(\alpha,\beta)\quad\Rightarrow\quad\{\gamma,\delta\}=\{\alpha,\beta\}.$$

Therefore, $\beta\in w_1$ and $w_1=w_0$ and since $|w_1|\geq 5$, the linear independence of $\bar{\eta}$ implies $\rho=\mathbf{0}$.

RE $(*)_{10}$: Concerning clause $(*)_{10}$, suppose that $\mathbf{m}^q(\ell_0, w_0), \mathbf{m}^q(\ell_1, w_1) \in \mathcal{M}^q$, $\alpha \in w_0$, $|\alpha \cap w_0| = k(w_0)$, $\mathrm{rk}(w_0) = -1$, and $\mathbf{m}^q(\ell_0, w_0) \sqsubseteq^* \mathbf{m}^q(\ell_1, w_1)$. Assume towards contradiction that there are $\alpha_0, \alpha_1 \in w_1$ such that

$$\eta^q_{\alpha_0}{\restriction}\ell_1 \neq \eta^q_{\alpha_1}{\restriction}\ell_1 \ \wedge\ \eta^q_\alpha{\restriction}\ell_0 \lhd \eta^q_{\alpha_0} \ \wedge\ \eta^q_\alpha{\restriction}\ell_0 \lhd \eta^q_{\alpha_1}.$$

Suppose $\beta \in w_0 \cup w_1$. Then looking at the function h^q_i in a manner similar to considerations for clause $(*)_9$ we get $\beta \in w_0 \cap w_1$. Let $\beta' \in w_0 \setminus \{\beta\}$. Then $h^q_0(\beta, \beta') \geq M^p$ and hence $r^q_{h_0(\beta,\beta')} = n^q = \ell_0 = \ell_1$, contradicting our assumptions. Therefore $\beta \notin w_0 \cup w_1$. But then we immediately get contradiction with clause $(*)_{10}$ for p.

RE $(*)_{11}$: Let us argue that $(*)_{11}$ is satisfied as well and for this suppose that $\rho^0_i, \rho^1_i \in \bigcup\limits_{m<M^q} (t_m \cap {}^{n^q}2)$ (for $i < \iota$) are such that

(a) there are no repetitions in $\langle \rho^0_i, \rho^1_i : i < \iota \rangle$, and

(b) $\rho^0_i + \rho^1_i = \rho^0_j + \rho^1_j$ for $i < j < \iota$.

Clearly, if

$(\odot)_1$ all ρ^0_i, ρ^1_i are from $\bigcup\limits_{m<M^p} t_m$,

then we may use the condition $(*)_{11}$ for p and conclude that for some $\alpha_0, \alpha_1 \in w^p$ we have

$$\big\{\{\rho^0_i, \rho^1_i\} : i < \iota\big\} = \big\{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\big\}.$$

Now note that if $\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup\limits_{m<M^q} (t_m \cap {}^{n^q}2)$, $\rho_0 + \rho_1 = \rho_2 + \rho_3$ and $\rho_0 \in \bigcup\limits_{m<M^p} (t_m \cap {}^{n^q}2)$ but $\rho_1 \notin \bigcup\limits_{m<M^p} (t_m \cap {}^{n^q}2)$, then $\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}$. Hence easily, if $(\odot)_1$ fails we must have

$(\odot)_2$ $\rho^0_i, \rho^1_i \in \bigcup\limits_{m=M^p}^{M^q-1} (t_m \cap {}^{n^q}2)$ for $i < \iota$.

But then necessarily

$$\begin{aligned} &\big\{\{\rho^0_i{\restriction}[n^p, n^q), \rho^1_i{\restriction}[n^p, n^q)\} : i < \iota\big\} \\ &\quad\subseteq\ \big\{\{g_i(\alpha, \beta){\restriction}[n^p, n^q), g_i(\beta, \alpha){\restriction}[n^p, n^q)\} : i < \iota,\ \alpha \in w^p\big\}. \end{aligned}$$

(Use Lemma 4.3(2), remember $\iota \geq 3$.) Since $\big(g_i(\alpha,\beta)+g_i(\beta,\alpha)\big){\restriction} n^p = \eta^p_\alpha$ we easily conclude that for some $\alpha \in w^p$ we have

$$\big\{\{\rho^0_i,\rho^1_i\} : i<\iota\big\} = \big\{\{g_i(\alpha,\beta),g_i(\beta,\alpha)\} : i<\iota\big\}.$$

One easily verifies that the condition q is stronger than p. □

Claim 4.4.3. *The forcing notion $\mathbb{P}$ has the Knaster property.*

Proof of the Claim. Suppose that $\langle p_\xi : \xi < \omega_1\rangle$ is a sequence of pairwise distinct conditions from $\mathbb{P}$ and let

$$p_\xi = \big(w_\xi, n_\xi, M_\xi, \bar\eta_\xi, \bar t_\xi, \bar r_\xi, \bar h_\xi, \bar g_\xi, \mathcal{M}_\xi\big)$$

where $\bar\eta_\xi = \langle \eta^\xi_\alpha : \alpha \in w_\xi\rangle$, $\bar t_\xi = \langle t^\xi_m : m < M_\xi\rangle$, $\bar r_\xi = \langle r^\xi_m : m < M_\xi\rangle$, and $\bar h_\xi = \langle h^\xi_i : i<\iota\rangle$, $\bar g_\xi = \langle g^\xi_i : i < \iota\rangle$. By a standard Δ–system cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that the following demands $(*)_{12}$–$(*)_{15}$ are satisfied.

$(*)_{12}$ $\{w_\xi : \xi \in A\}$ forms a Δ–system.

$(*)_{13}$ If $\xi,\varsigma \in A$, then $|w_\xi| = |w_\varsigma|$, $n_\xi = n_\varsigma$, $M_\xi = M_\varsigma$, and $t^\xi_m = t^\varsigma_m$ and $r^\xi_m = r^\varsigma_m$ (for $m < M_\xi$).

$(*)_{14}$ If $\xi < \varsigma$ are from A and $\pi : w_\xi \longrightarrow w_\varsigma$ is the order isomorphism, then

(a) $\pi(\alpha) = \alpha$ for $\alpha \in w_\xi \cap w_\varsigma$,

(b) if $\emptyset \neq v \subseteq w_\xi$, then $\mathrm{rk}(v) = \mathrm{rk}(\pi[v])$, $\zeta(v) = \zeta(\pi[v])$ and $k(v) = k(\pi[v])$,

(c) $\eta^\xi_\alpha = \eta^\varsigma_{\pi(\alpha)}$ (for $\alpha \in w_\xi$),

(d) $g_i(\alpha,\beta) = g_i(\pi(\alpha),\pi(\beta))$ and $h_i(\alpha,\beta) = h_i(\pi(\alpha),\pi(\beta))$ for $(\alpha,\beta) \in (w_\xi)^{\langle 2\rangle}$ and $i < \iota$,

and

$(*)_{15}$ $\mathcal{M}_\xi = \mathcal{M}_\varsigma$ (this actually follows from the previous demands).

Following the pattern of Claim 4.4.2(2) we will argue that for distinct ξ,ς from A the conditions p_ξ, p_ς are compatible. So let $\xi,\varsigma \in A$, $\xi < \varsigma$ and let $\pi : w_\xi \longrightarrow w_\varsigma$ be the order isomorphism. We will define $q =$

$\big(w, n, M, \bar\eta, \bar t, \bar r, \bar h, \bar g, \mathcal{M}\big)$ where $\bar\eta = \langle \eta_\alpha : \alpha \in w\rangle$, $\bar t = \langle t_m : m < M\rangle$, $\bar r = \langle r_m : m < M\rangle$, and $\bar h = \langle h_i : i < \iota\rangle$, $\bar g = \langle g_i : i < \iota\rangle$.

Let $w_\xi \cap w_\varsigma = \{\alpha_0, \ldots, \alpha_{k-1}\}$, $w_\xi \setminus w_\varsigma = \{\beta_0, \ldots, \beta_{\ell-1}\}$ and $w_\varsigma \setminus w_\xi = \{\gamma_0, \ldots, \gamma_{\ell-1}\}$ be the increasing enumerations.

We set $N_0 = \iota \cdot \ell(\ell+k) + \iota \cdot \frac{\ell(\ell-1)}{2} + 1$, $N = N_0 + \ell + 1$, and we define

$(*)_{16}$ $w = w_\xi \cup w_\varsigma$, $n = n_\xi + N$, and $M = M_\xi + 1$;

$(*)_{17}$ $\eta_\alpha = \eta^\xi_\alpha{}^\frown\langle\underbrace{0,\ldots,0}_{N}\rangle$ for $\alpha \in w_\xi$ and we also let for $c < \ell$

$$\eta_{\gamma_c} = \eta^\varsigma_{\gamma_c}{}^\frown\langle 0\rangle^\frown\langle\underbrace{1,\ldots,1}_{N_0}\rangle^\frown\langle\underbrace{0,\ldots,0}_{c}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-c}\rangle.$$

Next we are going to define $h_i(\alpha,\beta)$ and $g_i(\alpha,\beta)$ for $(\alpha,\beta) \in w^{\langle 2\rangle}$. For $d < N_0$ let

$$\nu_d = \langle\underbrace{0,\ldots,0}_{d}\rangle^\frown\langle 1\rangle^\frown\langle\underbrace{0,\ldots,0}_{N_0-d-1}\rangle \in {}^{N_0}2, \quad \text{and} \quad \nu^*_d = \mathbf{1} + \nu_d \in {}^{N_0}2$$

and note that $\{\nu_d : d < N_0 - 1\} \cup \{\mathbf{1}\}$ are linearly independent in ${}^{N_0}2$. Fix a bijection

$$\Theta : (k\times\ell\times\iota\times\{0\})\cup(\{(a,b)\in\ell^2 : a<b\}\times\iota\times\{1\})\cup(\ell\times\ell\times\iota\times\{2\}) \longrightarrow N_0-1$$

and define h_i, g_i as follows.

$(*)^{\rm a}_{18}$ If $(\alpha,\beta) \in (w_\xi)^{\langle 2\rangle}$ and $i < \iota$, then
$h_i(\alpha,\beta) = h^\xi_i(\alpha,\beta)$ and $g_i(\alpha,\beta) = g^\xi_i(\alpha,\beta)^\frown\langle\underbrace{0,\ldots,0}_{N}\rangle$.

$(*)^{\rm b}_{18}$ If $a < k$, $c < \ell$ and $i < \iota$, then $h_i(\alpha_a,\gamma_c) = h^\varsigma_i(\alpha_a,\gamma_c)$ and $h_i(\gamma_c,\alpha_a) = h^\varsigma_i(\gamma_c,\alpha_a)$, and

$$g_i(\alpha_a,\gamma_c) = g^\varsigma_i(\alpha_a,\gamma_c)^\frown\langle 1\rangle^\frown\nu_{\Theta(a,c,i,0)}{}^\frown\langle\underbrace{0,\ldots,0}_{\ell}\rangle \quad \text{and}$$

$$g_i(\gamma_c,\alpha_a) = g^\varsigma_i(\gamma_c,\alpha_a)^\frown\langle 1\rangle^\frown\nu^*_{\Theta(a,c,i,0)}{}^\frown\langle\underbrace{0,\ldots,0}_{c}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-c}\rangle.$$

$(*)^{\rm c}_{18}$ If $b<c<\ell$ and $i<\iota$, then $h_i(\gamma_b,\gamma_c)=h^\varsigma_i(\gamma_b,\gamma_c)$, $h_i(\gamma_c,\gamma_b)=h^\varsigma_i(\gamma_c,\gamma_b)$, and

$$g_i(\gamma_b,\gamma_c)=g^\varsigma_i(\gamma_b,\gamma_c)^\frown\langle 1\rangle^\frown\nu_{\Theta(b,c,i,1)}{}^\frown\langle\underbrace{0,\ldots,0}_{b}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-b}\rangle \quad\text{and}$$
$$g_i(\gamma_c,\gamma_b)=g^\varsigma_i(\gamma_c,\gamma_b)^\frown\langle 1\rangle^\frown\nu_{\Theta(b,c,i,1)}{}^\frown\langle\underbrace{0,\ldots,0}_{c}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-c}\rangle$$

(note: ν_Θ not ν^*_Θ).

$(*)^{\rm d}_{18}$ If $b<\ell$, $c<\ell$ and $b\neq c$ and $i<\iota$, then $h_i(\beta_b,\gamma_c)=h_i(\gamma_c,\beta_b)=M_\xi=M_\varsigma$, and

$$g_i(\beta_b,\gamma_c)=g^\xi_i(\beta_b,\beta_c)^\frown\langle 1\rangle^\frown\nu_{\Theta(b,c,i,2)}{}^\frown\langle\underbrace{0,\ldots,0}_{c}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-c}\rangle \quad\text{and}$$
$$g_i(\gamma_c,\beta_b)=g^\varsigma_i(\gamma_c,\gamma_b)^\frown\langle 1\rangle^\frown\nu^*_{\Theta(b,c,i,2)}{}^\frown\langle\underbrace{0,\ldots,0}_{\ell}\rangle.$$

$(*)^{\rm e}_{18}$ If $b<\ell$ and $i<\iota$, then $h_i(\beta_b,\gamma_b)=h_i(\gamma_b,\beta_b)=M_\xi=M_\varsigma$, and

$$g_i(\beta_b,\gamma_b)=\eta^\xi_{\beta_b}{}^\frown\langle 1\rangle^\frown\nu_{\Theta(b,b,i,2)}{}^\frown\langle\underbrace{0,\ldots,0}_{b}\rangle^\frown\langle\underbrace{1,\ldots,1}_{\ell-b}\rangle \quad\text{and}$$
$$g_i(\gamma_b,\beta_b)=\eta^\varsigma_{\gamma_b}{}^\frown\langle 1\rangle^\frown\nu^*_{\Theta(b,b,i,2)}{}^\frown\langle\underbrace{0,\ldots,0}_{\ell}\rangle.$$

We also set:

$(*)_{19}$ $r_m=r^\xi_m$ for $m<M_\xi$, $r_{M_\xi}=n$ and if $m<M_\xi$, then

$$\begin{aligned} t_m = \ & \big\{\eta\in{}^{n\geq}2:\eta{\restriction}n_\xi\in t^\xi_m\ \wedge\ (\forall j<n)(n\leq j<|\eta|\Rightarrow\eta(j)=0)\big\}\cup\\ & \big\{g_i(\delta,\varepsilon){\restriction}n':(\delta,\varepsilon)\in w^{\langle 2\rangle}, i<\iota,\ \text{and } n'\leq n \text{ and } h_i(\delta,\varepsilon)=m\big\}\end{aligned}$$

and

$$t_{M_\xi}=\big\{g_i(\delta,\varepsilon){\restriction}n':(\delta,\varepsilon)\in w^{\langle 2\rangle}, i<\iota,\ \text{and } n'\leq n \text{ and } h_i(\delta,\varepsilon)=M_\xi\big\}.$$

Now letting $\mathcal{M}$ be defined by $(*)_8$ we claim that

$$q=\big(w,n,M,\bar\eta,\bar t,\bar r,\bar h,\bar g,\mathcal{M}\big)\in\mathbb{P}.$$

Demands $(*)_1$–$(*)_8$ are pretty straightforward.

RE $(*)_9$: To justify clause $(*)_9$, suppose that $\mathbf{m}(\ell, w'), \mathbf{m}(\ell, w'') \in \mathcal{M}$, $\rho \in {}^{\ell}2$ and $\mathbf{m}(\ell, w') \doteqdot \mathbf{m}(\ell, w'') + \rho$, and consider the following three cases.

Case 1: $w' \subseteq w_\xi$
Then for each $(\delta, \varepsilon) \in (w')^{\langle 2 \rangle}$ we have $h_i(\delta, \varepsilon) < M_\xi$, so this also holds for $(\delta, \varepsilon) \in (w'')^{\langle 2 \rangle}$. Consequently, either $w'' \subseteq w_\xi$ or $w'' \subseteq w_\varsigma$.
If $w'' \subseteq w_\xi$, then let $\ell' = \min(\ell, n_\xi)$ and consider $\mathbf{m}^{p_\xi}(w', \ell'), \mathbf{m}^{p_\xi}(w'', \ell') \in \mathcal{M}_\xi$. Using clause $(*)_9$ for p_ξ we immediately obtain the desired conclusion.

If $w'' \subseteq w_\varsigma$, then we let $\ell' = \min(\ell, n_\xi)$ and we consider $\mathbf{m}^{p_\xi}(w', \ell')$ and $\mathbf{m}^{p_\xi}(\pi^{-1}[w''], \ell')$ (both from $\mathcal{M}_\xi$). By $(*)_{14}$, clause $(*)_9$ for p_ξ applies to them and we get

- $\mathrm{rk}(w') = \mathrm{rk}(\pi^{-1}[w''])$, $\zeta(w') = \zeta(\pi^{-1}[w''])$, $k(w') = k(\pi^{-1}[w''])$ and
- if $\delta \in w'$, $\varepsilon \in \pi^{-1}[w'']$ are such that $|\delta \cap w'| = k(w') = k(\pi^{-1}[w'']) = |\varepsilon \cap \pi^{-1}[w'']|$, then $(\eta^{p_\xi}_\delta \restriction \ell') + \rho = \eta^{p_\xi}_\varepsilon \restriction \ell'$.

By $(*)_{14}$ this immediately implies the desired conclusion.

Case 2: $w' \subseteq w_\varsigma$
Same as the previous case, just interchanging ξ and ς.

Case 3: $w' \setminus w_\xi \neq \emptyset \neq w' \setminus w_\varsigma$
Then for some $(\delta, \varepsilon) \in (w')^{\langle 2 \rangle}$ we have $h_i(\delta, \varepsilon) = M_\xi$, so necessarily $\ell = r_{M_\xi} = n$. Hence $\{\eta_\alpha : \alpha \in w'\} = \{\eta_\alpha + \rho : \alpha \in w''\}$ and since $|w'| \geq 5$, the linear independence of $\bar{\eta}$ implies $\rho = \mathbf{0}$ and $w' = w''$ and the desired conclusion follows.

RE $(*)_{10}$: Let us prove clause $(*)_{10}$ now.

Suppose that $\mathbf{m}(\ell_0, w'), \mathbf{m}(\ell_1, w'') \in \mathcal{M}$, $\delta \in w'$, $|\delta \cap w'| = k(w')$, $\mathrm{rk}(w') = -1$, and $\mathbf{m}(\ell_0, w') \sqsubseteq^* \mathbf{m}(\ell_1, w'')$. Assume towards contradiction that there are $\varepsilon_0, \varepsilon_1 \in w''$ such that

$(\otimes)_0$ $\eta_{\varepsilon_0} \restriction \ell_1 \neq \eta_{\varepsilon_1} \restriction \ell_1$ and $\eta_\delta \restriction \ell_0 \vartriangleleft \eta_{\varepsilon_0}$ and $\eta_\delta \restriction \ell_0 \vartriangleleft \eta_{\varepsilon_1}$.

Without loss of generality $|w''| = |w'| + 1 \geq 6$.

Since we must have $\ell_0 < n$, for no $\alpha, \beta \in w'$ we can have $h_i(\alpha, \beta) = M_\xi$. Therefore either $w' \subseteq w_\xi$ or $w' \subseteq w_\varsigma$. Also,

$(\otimes)_1$ if $(\alpha, \beta) \in (w'')^{\langle 2 \rangle} \setminus \{(\varepsilon_0, \varepsilon_1), (\varepsilon_1, \varepsilon_0)\}$ then $h_i(\alpha, \beta) < M_\xi$ for $i < \iota$.

Note that

$(\otimes)_2$ if $(\alpha,\beta)\in (w_\xi)^{\langle 2\rangle}\cup (w_\varsigma)^{\langle 2\rangle}$ then $\min(\{\ell:\eta_\alpha(\ell)\neq\eta_\beta(\ell)\})<n_\xi$ and there are no repetitions in the sequence $\langle g_i(\alpha,\beta){\restriction}n_\xi, g_i(\beta,\alpha){\restriction}n_\xi : i<\iota\rangle$.

Let $\ell^*=\min(\ell_1,n_\xi)$.

Now, if $w'\cup w''\subseteq w_\xi$, then considering $\mathbf{m}(\ell_0,w')$ and $\mathbf{m}(\ell^*,w'')$ (and remembering $(\otimes)_2$) we see that $\ell_0<n_\xi$, $\mathbf{m}^{p_\xi}(\ell_0,w')\sqsubseteq^*\mathbf{m}^{p_\xi}(\ell^*,w'')$ and they have the property contradicting $(*)_{10}$ for p_ξ.

If $w'\cup w''\subseteq w_\varsigma$, then in a similar manner we get contradiction with $(*)_{10}$ for p_ς.

If $w'\subseteq w_\xi$ and $w''\subseteq w_\varsigma$ then one easily verifies that $\ell_0<n_\xi$ and $\mathbf{m}^{p_\xi}(\ell_0,w')\sqsubseteq^*\mathbf{m}^{p_\xi}(\ell^*,\pi^{-1}[w''])$ provide a counterexample for $(*)_{10}$ for p_ξ. Similarly if $w'\subseteq w_\varsigma$ and $w''\subseteq w_\xi$.

Consequently, the only possibility left is that $w''\setminus w_\xi\neq\emptyset\neq w''\setminus w_\varsigma$ and it follows from $(\otimes)_1$ that $|w''\setminus w_\xi|=|w''\setminus w_\varsigma|=1$. Let $\{\beta_b\}=w''\setminus w_\varsigma$ and $\{\gamma_c\}=w''\setminus w_\xi$; then $\{\varepsilon_0,\varepsilon_1\}=\{\beta_b,\gamma_c\}$.

Assume $w'\subseteq w_\xi$ (the case when $w'\subseteq w_\varsigma$ can be handled similarly). If we had $b\neq c$, then $\eta_{\beta_b}{\restriction}n_\xi=\eta^{p_\xi}_{\beta_b}{\restriction}n_\xi\neq\eta^{p_\varsigma}_{\gamma_c}{\restriction}n_\xi=\eta_{\gamma_c}{\restriction}n_\xi$. Since $w''\subseteq(w_\xi\cap w_\varsigma)\cup\{\beta_b,\gamma_c\}$ we could see that $\ell_0<n_\xi$ and $\mathbf{m}^{p_\xi}(\ell_0,w')\sqsubseteq^*\mathbf{m}^{p_\xi}(\ell^*,\pi^{-1}[w''])$ would provide a counterexample for $(*)_{10}$ for p_ξ. Consequently, $b=c$ and $\ell_1>n_\xi$. Now, remembering $(\otimes)_0$, $\eta^{p_\xi}_\delta{\restriction}\ell_0=\eta^{p_\xi}_{\beta_b}{\restriction}\ell_0$ and $\mathbf{m}^{p_\xi}(\ell_0,w')\doteqdot\mathbf{m}^{p_\xi}(\ell_0,w''\setminus\{\gamma_b\})$, so by $(*)_9$ for p_ξ we conclude

$$\mathrm{rk}(w''\setminus\{\gamma_b\})=-1\quad\text{ and }\quad|\beta_b\cap(w''\setminus\{\gamma_b\})|=k(w''\setminus\{\gamma_b\}).$$

Let $\zeta^*=\zeta(w''\setminus\{\gamma_b\})$ and $k^*=k(w''\setminus\{\gamma_b\})$. For $\varepsilon\in A\setminus\{\xi\}$ let $\pi^\varepsilon:w_\xi\longrightarrow w_\varepsilon$ be the order isomorphism and let $\gamma(\varepsilon)\in\pi^\varepsilon\big[w''\setminus\{\gamma_b\}\big]$ be such that $|\pi^\varepsilon\big[w''\setminus\{\gamma_b\}\big]\cap\gamma(\varepsilon)|=k^*$ (necessarily $\gamma(\varepsilon)=\pi^\varepsilon(\beta_b)\in w_\varepsilon\setminus w_\xi$). Then

- $\pi^\varepsilon[w''\setminus\{\gamma_b\}]=\big(w''\cap(w_\xi\cap w_\varepsilon)\big)\cup\{\gamma(\varepsilon)\}=w''\setminus\{\beta_b,\gamma_b\}\cup\{\gamma(\varepsilon)\}$,
- $\mathrm{rk}\Big(\pi^\varepsilon[w''\setminus\{\gamma_b\}]\Big)=-1$, and $\zeta\Big(\pi^\varepsilon[w''\setminus\{\gamma_b\}]\Big)=\zeta^*$, and
- $k\Big(\pi^\varepsilon[w''\setminus\{\gamma_b\}]\Big)=k^*=|\pi^\varepsilon[w''\setminus\{\gamma_b\}]\cap\gamma(\varepsilon)|$.

Hence $\mathbb{M} \models R_{|w'|,\zeta^*}\big[w'' \setminus \{\beta_b, \gamma_b\} \cup \{\gamma(\varepsilon)\}\big]$ for each $\varepsilon \in A \setminus \{\xi\}$. Consequently, the set

$$\Big\{\alpha < \lambda : \mathbb{M} \models R_{|w'|,\zeta^*}\big[w'' \setminus \{\beta_b, \gamma_b\} \cup \{\alpha\}\big]\Big\}$$

is uncountable, contradicting $(\circledast)_{\rm e}$.

RE $(*)_{11}$: Let us argue that $(*)_{11}$ is satisfied as well and for this suppose that $\rho^0_i, \rho^1_i \in \bigcup\limits_{m<M} (t_m \cap {}^n2)$ (for $i < \iota$) are such that

(a) there are no repetitions in $\langle \rho^0_i, \rho^1_i : i < \iota\rangle$, and

(b) $\rho^0_i + \rho^1_i = \rho^0_j + \rho^1_j$ for $i < j < \iota$.

Clearly, if all ρ^0_i, ρ^1_i are form $\rho^\frown\langle\underbrace{0, \ldots, 0}_{N}\rangle$, then we may use condition $(*)_{11}$ for p_ξ and conclude that for some $\alpha_0, \alpha_1 \in w_\xi$ we have

$$\big\{\{\rho^0_i, \rho^1_i\} : i < \iota\big\} = \big\{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\big\}.$$

So assume that we are not in the situation when all ρ^0_i, ρ^1_i are form $\rho^\frown\langle\underbrace{0, \ldots, 0}_{N}\rangle$.

Note that if $\rho \in \bigcup\limits_{m<M} (t_m \cap {}^n2)$ and $\rho(n_\xi) = 0$, then $\rho{\restriction}[n_\xi, n) = \mathbf{0}$. Hence, remembering definitions in $(*)_{18}$, if $\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup\limits_{m<M} (t_m \cap {}^n2)$, $\rho_0 + \rho_1 = \rho_2 + \rho_3$ and $\rho_0(n_\xi) = 0$ but $\rho_1(n_\xi) = 1$, then $\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}$. Therefore, under current assumption, we must have $\rho^0_i(n_\xi) = \rho^1_i(n_\xi) = 1$ for all $i < \iota$. Define

$B = \{(\alpha_a, \gamma_c) : a < k \ \&\ c < \ell\}$,
$C = \{(\gamma_b, \gamma_c) : b < c < \ell\}$,
$D = \{(\beta_b, \gamma_c) : b < \ell \ \&\ c < \ell \ \&\ b \neq c\}$,
$E = \{(\beta_b, \gamma_b) : b < \ell\}$.

(These four sets correspond to clauses $(*)^{\rm b}_{18}$–$(*)^{\rm e}_{18}$ in the definition of g_i.) Clearly, $\rho^0_i(n_\xi) = \rho^1_i(n_\xi) = 1$ implies that

$$\rho^0_i, \rho^1_i \in \{g_j(\varepsilon_0, \varepsilon_1), g_j(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in B \cup C \cup D \cup E,\ j < \iota\}.$$

Note also that for each $d < N_0 - 1$,

$(\boxtimes)_a$ the set $\{\rho \in \bigcup\limits_{m<M} (t_m \cap {}^n2) : \rho{\restriction}\big(n_\xi, n_\xi + N_0\big] = \nu_d\}$ is not empty but it has at most two elements, and

$(\boxtimes)_b$ $|\{\rho \in \bigcup\limits_{m<M} (t_m \cap {}^n2) : \rho{\restriction}\big(n_\xi, n_\xi + N_0\big] = \nu_d\}| = 2$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$, and

$(\boxtimes)_c$ the set $\{\rho \in \bigcup\limits_{m<M} (t_m \cap {}^n2) : \rho{\restriction}\big(n_\xi, n_\xi + N_0\big] = \nu^*_d\}$ has at most one element, and

$(\boxtimes)_d$ $\{\rho \in \bigcup\limits_{m<M} (t_m \cap {}^n2) : \rho{\restriction}\big(n_\xi, n_\xi + N_0\big] = \nu^*_d\} = \emptyset$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$.

Now consider $\rho^0_i{\restriction}\big(n_\xi, n_\xi + N_0\big]$, $\rho^1_i{\restriction}\big(n_\xi, n_\xi + N_0\big]$ for $i < \iota$.

If for some $(i, x) \neq (j, y)$ we have $\rho^x_i{\restriction}\big(n_\xi, n_\xi + N_0\big] = \rho^y_j{\restriction}\big(n_\xi, n_\xi + N_0\big]$, then (using $(\boxtimes)_a$–$(\boxtimes)_d$ and the linear independence of ν_d's) we must have that

$$\rho^0_i{\restriction}\big(n_\xi, n_\xi + N_0\big] = \rho^1_i{\restriction}\big(n_\xi, n_\xi + N_0\big] \quad \text{for all } i < \iota.$$

Thus, for every $i < \iota$ there are $b < c < \ell$ and $j < \iota$ such that

$$\{\rho^0_i, \rho^1_i\} = \{g_j(\gamma_b, \gamma_c), g_j(\gamma_c, \gamma_b)\}.$$

Since for $b < c < \ell$ we have

$$\big(g_j(\gamma_b, \gamma_c) + g_j(\gamma_c, \gamma_b)\big){\restriction}(N_0, N_0 + \ell] = \langle\underbrace{0, \ldots, 0}_{b}\rangle^\frown\langle\underbrace{1, \ldots, 1}_{c-b}\rangle^\frown\langle\underbrace{0, \ldots, 0}_{\ell-c}\rangle$$

we immediately get that (in the current situation) for some $b < c < \ell$ we have

$$\big\{\{\rho^0_i, \rho^1_i\} : i < \iota\big\} = \big\{\{g_i(\gamma_b, \gamma_c), g_i(\gamma_c, \gamma_b)\} : i < \iota\big\}.$$

So let us assume that $\rho^x_i{\restriction}\big(n_\xi, n_\xi + N_0\big] \neq \rho^y_j{\restriction}\big(n_\xi, n_\xi + N_0\big]$ for all distinct $(i, x), (j, y) \in \iota \times 2$. Since $\{\mathbf{1}, \nu_0, \ldots, \nu_{N_0-2}\}$ are linearly independent we may use Lemma 4.3(2) to conclude that

$$\Big\{\{\rho^0_i{\restriction}\big(n_\xi, n_\xi + N_0\big], \rho^1_i{\restriction}\big(n_\xi, n_\xi + N_0\big]\} : i < \iota\Big\} \subseteq \Big\{\{\nu_d, \nu^*_d\} : d < N_0 - 1\Big\}.$$

Consequently, we easily deduce that

$$\big\{\{\rho^0_i,\rho^1_i\}: i<\iota\big\} \subseteq \big\{\{g_i(\varepsilon_0,\varepsilon_1), g_i(\varepsilon_1,\varepsilon_0)\}: i<\iota \ \&\ (\varepsilon_0,\varepsilon_1)\in B\cup D\cup E\big\}.$$

Using the linear independence of η^ξ_ε's and the definitions of g_i's in $(*)_{18}$ one checks that the three sets

$\{g_i(\varepsilon_0,\varepsilon_1)+g_i(\varepsilon_1,\varepsilon_0) : (\varepsilon_0,\varepsilon_1)\in B,\ i<\iota\}$,
$\{g_i(\varepsilon_0,\varepsilon_1)+g_i(\varepsilon_1,\varepsilon_0) : (\varepsilon_0,\varepsilon_1)\in D,\ i<\iota\}$,
$\{g_i(\varepsilon_0,\varepsilon_1)+g_i(\varepsilon_1,\varepsilon_0) : (\varepsilon_0,\varepsilon_1)\in E,\ i<\iota\}$

are pairwise disjoint. Therefore, $\big\{\{\rho^0_i,\rho^1_i\}: i<\iota\big\}$ must be included in (exactly) one of the sets

$\big\{\{g_i(\varepsilon_0,\varepsilon_1), g_i(\varepsilon_1,\varepsilon_0)\}: i<\iota \ \&\ (\varepsilon_0,\varepsilon_1)\in B\big\}$,
$\big\{\{g_i(\varepsilon_0,\varepsilon_1), g_i(\varepsilon_1,\varepsilon_0)\}: i<\iota \ \&\ (\varepsilon_0,\varepsilon_1)\in D\big\}$, or
$\big\{\{g_i(\varepsilon_0,\varepsilon_1), g_i(\varepsilon_1,\varepsilon_0)\}: i<\iota \ \&\ (\varepsilon_0,\varepsilon_1)\in E\big\}$.

But now we easily check that for some $(\varepsilon_0,\varepsilon_1)\in B\cup D\cup E$ we must have

$$\big\{\{\rho^0_i,\rho^1_i\}: i<\iota\big\} = \big\{\{g_i(\varepsilon_0,\varepsilon_1), g_i(\varepsilon_1,\varepsilon_0)\}: i<\iota\big\}.$$

This completes the verification that $q=\big(w,n,M,\bar\eta,\bar t,\bar r,\bar h,\bar g,\mathcal{M}\big)\in\mathbb{P}$, and clearly q is stronger than both p_ξ and p_ς. □

Define $\mathbb{P}$–names $\underset{\sim}{T}_m$ and $\underset{\sim}{\eta}_\alpha$ (for $m<\omega$ and $\alpha<\lambda$) by
$\Vdash_{\mathbb{P}}$ "$\underset{\sim}{T}_m = \bigcup\{t^p_m : p\in \underset{\sim}{G}_{\mathbb{P}}\ \wedge\ m<M^p\}$", and
$\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\alpha = \bigcup\{\eta^p_\alpha : p\in \underset{\sim}{G}_{\mathbb{P}}\ \wedge\ \alpha\in w^p\}$".

Claim 4.4.4. *1. For each $m<\omega$ and $\alpha<\lambda$,*

$\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\eta}_\alpha \in {}^\omega 2$ *and* $\underset{\sim}{T}_m \subseteq {}^{\omega>}2$ *is a tree without terminal nodes".*

2. $\Vdash_{\mathbb{P}}$ " $\bigcup\limits_{m<\omega}\lim(\underset{\sim}{T}_m)$ *is a* 2ι–**npots** *set".*

Proof of the Claim. (1) By Claim 4.4.2 (and the definition of the order in $\mathbb{P}$).

(2) Let $G\subseteq\mathbb{P}$ be a generic filter over $\mathbf{V}$ and let us work in $\mathbf{V}[G]$.
Let $k=2\iota$ and $\bar T=\langle(\underset{\sim}{T}_m)^G : m<\omega\rangle$.
Suppose towards contradiction that $B=\bigcup\limits_{m<\omega}\lim\big((\underset{\sim}{T}_m)^G\big)$ is a k–**pots** set. Then, by Proposition 3.11, $\mathrm{NDRK}(\bar T)=\infty$. Using Lemma 3.10(5), by induction on $j<\omega$ we choose $\mathbf{m}_j,\mathbf{m}^*_j\in\mathbf{M}_{\bar T,k}$ and $p_j\in G$ such that

(i) $\mathrm{ndrk}(\mathbf{m}_j)\geq\omega_1$, $|u_{\mathbf{m}_j}|>5$ and $\mathbf{m}_j\sqsubseteq\mathbf{m}^*_j\sqsubseteq\mathbf{m}_{j+1}$,

(ii) for each $\nu \in u_{\mathbf{m}^*_j}$ the set $\{\eta \in u_{\mathbf{m}_{j+1}} : \nu \vartriangleleft \eta\}$ has at least two elements,

(iii) $p_j \leq p_{j+1}$, $\ell_{\mathbf{m}_j} \leq \ell_{\mathbf{m}^*_j} = n^{p_j} < \ell_{\mathbf{m}_{j+1}}$ and $\mathrm{rng}(h^{\mathbf{m}_j}_i) \subseteq M^{p_j}$ for all $i < \iota$, and

(iv) $|\{\eta{\restriction} n^{p_j} : \eta \in u_{\mathbf{m}_{j+1}}\}| = |u_{\mathbf{m}_j}| = |u_{\mathbf{m}^*_j}|$.

Then, by (iii)+(iv), $\mathbf{m}_j, \mathbf{m}^*_j \in \mathbf{M}^{n^{p_j}}_{\bar{t}^{p_j},k}$. It follows from Claim 4.4.1 that for some $w_j \subseteq w^{p_j}$ and $\rho_j \in {}^{n^{p_j}}2$ we have $(\mathbf{m}^*_j + \rho_j) \doteqdot \mathbf{m}^{p_j}(n^{p_j}, w_j) \in \mathcal{M}^{p_j}$.

Fix j for a moment and consider $\mathbf{m}^{p_j}(n^{p_j}, w_j) \in \mathcal{M}^{p_j}$ and $\mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1}) \in \mathcal{M}^{p_{j+1}}$. Since

$$(\mathbf{m}^*_j + (\rho_{j+1}{\restriction} n^{p_j})) \sqsubseteq (\mathbf{m}^*_{j+1} + \rho_{j+1}) \doteqdot \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1}),$$

we may choose $w^*_j \subseteq w_{j+1}$ such that

$$(\mathbf{m}^*_j + (\rho_{j+1}{\restriction} n^{p_j})) \doteqdot \mathbf{m}^{p_{j+1}}(n^{p_j}, w^*_j) \sqsubseteq^* \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1})$$

(and the latter two belong to $\mathcal{M}^{p_{j+1}}$). Then also

$$\begin{aligned} \mathbf{m}^{p_{j+1}}(n^{p_j}, w^*_j) &\doteqdot \mathbf{m}^{p_j}(n^{p_j}, w_j) + (\rho_j + \rho_{j+1}{\restriction} n^{p_j}) \\ &= \mathbf{m}^{p_{j+1}}(n^{p_j}, w_j) + (\rho_j + \rho_{j+1}{\restriction} n^{p_j}), \end{aligned}$$

so by clause $(*)_9$ for p_{j+1} we have

$$\mathrm{rk}(w^*_j) = \mathrm{rk}(w_j).$$

Clause (ii) of the choice of $\mathbf{m}_{j+1}$ implies that

$$(\forall \gamma \in w^*_j)(\exists \delta \in w_{j+1} \setminus w^*_j)(\eta^{p_{j+1}}_\gamma{\restriction} n^{p_j} = \eta^{p_{j+1}}_\delta{\restriction} n^{p_j}).$$

Let $\delta(\gamma)$ be the smallest $\delta \in w_{j+1} \setminus w^*_j$ with the above property and let $w^*_j(\gamma) = (w^*_j \setminus \{\gamma\}) \cup \{\delta(\gamma)\}$. Then, for $\gamma \in w^*_j$, $\mathbf{m}^{p_{j+1}}(n^{p_j}, w^*_j(\gamma)) \in \mathcal{M}^{p_{j+1}}$ and

$$\mathbf{m}^{p_{j+1}}(n^{p_j}, w^*_j) \doteqdot \mathbf{m}^{p_{j+1}}(n^{p_j}, w^*_j(\gamma)) \sqsubseteq^* \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1}).$$

So by clause $(*)_9$ we know that for each $\gamma \in w_j$:

$$\mathrm{rk}(w^*_j(\gamma)) = \mathrm{rk}(w^*_j), \quad \zeta(w^*_j(\gamma)) = \zeta(w^*_j), \quad \text{and} \quad k(w^*_j(\gamma)) = k(w^*_j).$$

Let $n = |w^*_j|$, $\zeta = \zeta(w^*_j)$, $k = k(w^*_j)$, and let $w^*_j = \{\alpha_0, \ldots, \alpha_k, \ldots, \alpha_{n-1}\}$ be the increasing enumeration. Let $\alpha^*_k = \delta(\alpha_k)$. Then clause $(*)_9$ also gives that $w^*_j(\alpha_k) = \{\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_k, \alpha_{k+1}, \ldots, \alpha_{n-1}\}$ is the increasing enumeration. Now,

$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n-1}] \qquad \text{and}$$
$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_k, \alpha_{k+1}, \ldots, \alpha_{n-1}],$$

and consequently if $\mathrm{rk}(w^*_j) \geq 0$, then

$$\mathrm{rk}(w_{j+1}) \leq \mathrm{rk}(w^*_j \cup \{\alpha^*_k\}) < \mathrm{rk}(w^*_j) = \mathrm{rk}(w_j)$$

(remember $(\circledast)_{\mathrm{d}}$).

Now, unfixing j, suppose that we constructed w_{j+1}, w^*_j for all $j < \omega$. It follows from our considerations above that for some $j_0 < \omega$ we must have:

(a) $\mathrm{rk}(w^*_{j_0}) = -1$, and

(b) $\mathbf{m}^{p_{j_0+1}}(n^{p_{j_0}}, w^*_{j_0}) \sqsubseteq^* \mathbf{m}^{p_{j_0+1}}(n^{p_{j_0+1}}, w_{j_0+1})$
(and both belong to $\mathcal{M}^{p_{j_0+1}}$),

(c) for every $\alpha \in w^*_{j_0}$ we have

$$\big|\{\beta \in w_{j_0+1} : \eta^{p_{j_0+1}}_\alpha \restriction n^{p_{j_0}} \lhd \eta^{p_{j_0+1}}_\beta\}\big| > 1.$$

However, this contradicts clause $(*)_{10}$ (for p_{j_0+1}). □

□

Corollary 4.5. *Assume* **MA** *and* $\aleph_\alpha < \mathfrak{c}$, $\alpha < \omega_1$. *Let* $3 \leq \iota < \omega$. *Then there exists a* Σ^0_2 2ι–**npots**–*set* $B \subseteq {}^\omega 2$ *which has* $\aleph_\alpha$ *many pairwise* 2ι–*nondisjoint translations.*

Proof. Standard modification of the proof of Theorem 4.4. □

Corollary 4.6. *Assume* $\mathrm{NPr}_{\omega_1}(\lambda)$ *and* $\lambda = \lambda^{\aleph_0} < \mu = \mu^{\aleph_0}$, $3 \leq \iota < \omega$. *Then there is a ccc forcing notion* $\mathbb{Q}$ *of size* μ *forcing that*

(a) $2^{\aleph_0} = \mu$ *and*

(b) there is a Σ^0_2 2ι–**npots**–*set* $B \subseteq {}^\omega 2$ *which has* λ *many pairwise* 2ι–*nondisjoint translates but not* λ^+ *such translates.*

Proof. Let $\mathbb{P}$ be the forcing notion given by Theorem 4.4 and let $\mathbb{Q} = \mathbb{P} * \mathbb{C}_\mu$. Use Proposition 3.3(4) to argue that the set B added by $\mathbb{P}$ is a **npots**–set in $\mathbf{V}^{\mathbb{Q}}$. By 3.3(3) this set cannot have λ^+ pairwise 2ι–nondisjoint translates, but it does have λ many pairwise 2ι–nondisjoint translates (by absoluteness). $\square$

Remark 4.7. It follows from Proposition 3.3(1,2), that if there exists a Σ^0_2 **pots**–set $B \subseteq {}^\omega 2$ such that for some set $A \subseteq {}^\omega 2$ we have $(B+a) \cap (B+b) \neq \emptyset$ for all $a, b \in A$, then $\mathrm{stnd}(B) \subseteq {}^\omega 2 \times {}^\omega 2$ is a Σ^0_2 set which contains a $|A|$–square but no perfect square. Thus Corollary 4.6 is a slight generalization of Shelah [7, Theorem 1.13].

5. Further research

The case of $k = 4$ in Theorem 4.4 will be dealt with in a subsequent paper [6] alongside with further investigations of Σ^0_2 subsets of ${}^\omega 2$ with pregiven rank NDRK. In subsequent works we will also investigate the general case of Polish groups (not just ${}^\omega 2$). The following two problems are still open however.

Problem 5.1. Is is consistent to have a Borel set $B \subseteq {}^\omega 2$ such that

- for some uncountable set H, $(B + x) \cap (B + y)$ is uncountable for every $x, y \in H$, but
- for every perfect set P there are $x, y \in P$ with $(B + x) \cap (B + y)$ countable?

Problem 5.2. Is it consistent to have a Borel set $B \subseteq {}^\omega 2$ such that

- B has uncountably many pairwise disjoint translations, but
- there is no perfect of pairwise disjoint translations of B ?

References

[1] M. Balcerzak, A. Rosłanowski, and S. Shelah, Ideals without ccc, *Journal of Symbolic Logic* **63** (1998), 128–147, arxiv:math/9610219.

[2] T. Bartoszyński and H. Judah, Set Theory: On the Structure of the Real Line, A.K. Peters, Wellesley, Massachusetts, 1995.

[3] M. Elekes and T. Keleti, Decomposing the real line into Borel sets closed under addition, *MLQ Math. Log. Q.* **61** (2015), 466–473.

[4] T. Jech, Set theory, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, the third millennium edition, revised and expanded.

[5] A. Rosłanowski and V.V. Rykov, Not so many non-disjoint translations, *Proceedings of the American Mathematical Society, Series B* **5** (2018), 73–84, arxiv:1711.04058.

[6] A. Rosłanowski, and S. Shelah, Borel sets without perfectly many overlapping translations II. In preparation.

[7] S. Shelah, Borel sets with large squares, *Fundamenta Mathematicae* **159** (1999), 1–50, arxiv:math/9802134.

[8] P. Zakrzewski, On Borel sets belonging to every invariant ccc σ–ideal on $2^{\mathbb{N}}$, *Proc. Amer. Math. Soc.* **141** (2013), 1055–1065.

Department of Mathematics
University of Nebraska at Omaha
Omaha, NE 68182-0243, USA
aroslanowski@unomaha.edu

Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, USA
shelah@math.huji.ac.il
http://shelah.logic.at