REPORTS ON MATHEMATICAL LOGIC
54 (2019), 45–63
doi:10.4467/20842589RM.19.002.10650

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CONTINUOUS REDUCIBILITY: FUNCTIONS VERSUS RELATIONS

A b s t r a c t. It is proved that the Tang-Pequignot reducibility (or reducibility by relatively continuous relations) on a second countable, T_0 space X either coincides with the Wadge reducibility for the given topology, or there is no topology on X that can turn it into Wadge reducibility.

1. Introduction and basic definitions

For X a topological space, denote by $\leq_W^X$ the *Wadge reducibility*, or *reducibility by continuous functions*, on $\mathcal{P}(X)$, the powerset of X. It is the preorder defined by letting $A \leq_W^X B$ if there exists a continuous function $f : X \to X$ such that $A = f^{-1}(B)$. Let $\equiv_W^X$ be the associated equivalence relation, whose equivalence classes are the *Wadge degrees*. The relation $\leq_W^X$ induces a preorder, still denoted $\leq_W^X$, on the Wadge degrees.

Received 11 July 2018
Keywords and phrases: Wadge reducibility; relatively continuous relation.
AMS subject classificastion: Primary 03E15; Secondary 03D55.

Wadge reducibility and several variations of it have been extensively considered in the literature. This paper focuses on a notion of reducibility introduced by Tang in [8] for the Scott domain and recently generalised by Pequignot in [5] to any second countable, T_0 space X: for this reason in this paper I denote it by $\preceq^X_{TP}$ and call it *Tang-Pequignot reducibility*; the associated equivalence relation is denoted $\approxeq^X_{TP}$.

Notice that $\leq^X_W, \preceq^X_{TP}$ really mean $\leq^{\mathcal{T}}_W, \preceq^{\mathcal{T}}_{TP}$, respectively, where $\mathcal{T}$ is the topology of X. When several topologies are being considered on X, the latter notation will be preferred. Similarly for the associated equivalence relations.

In [5], the preorder $\preceq^X_{TP}$ is called *reducibility by relatively continuous relations*, since it can be defined and motivated by introducing the concept of a relatively continuous, everywhere defined relation on the space X. As every continuous function is a relatively continuous relation ([5, Theorem 2]), one has that $A \leq^X_W B$ implies $A \preceq^X_{TP} B$, that is $\leq^X_W \subseteq \preceq^X_{TP}$.

However, for practical purpose, $\preceq^X_{TP}$ can be defined as follows, without explicit mention to relations on X. Let X be a second countable, T_0 space. Recall from [10] that an *admissible representation* of X is a continuous function $\rho : Y \subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that for every continuous $\sigma : Z \subseteq \mathbb{N}^{\mathbb{N}} \to X$ there is a continuous $h : Z \to Y$ such that $\sigma = \rho h$. Then (see [5, Lemma 3]) given $A, B \in \mathcal{P}(X)$, let $A \preceq^X_{TP} B$ if for some (or any) admissible representation $\rho : Y \subseteq \mathbb{N}^{\mathbb{N}} \to X$ the relation $\rho^{-1}(A) \leq^Y_W \rho^{-1}(B)$ holds.

In [3], Duparc and Fournier prove that the conciliatory preorder $\leq_c$ (a hierarchy on subsets of $\mathbb{N}^{\leq\omega}$ introduced in [2] by a purely game theoretic definition) coincides with $\preceq^{\mathcal{T}}_{TP}$, where $\mathcal{T}$ is the prefix topology on $\mathbb{N}^{\leq\omega}$, and raise the question ([3, Question 1]) of whether there exists a topology τ on $\mathbb{N}^{\leq\omega}$ such that $\leq_c = \leq^\tau_W$.

This suggests the following more general problem: given a second countable, T_0 space $(X, \mathcal{T})$, when does there exist a topology τ on X such that $\leq^\tau_W = \preceq^{\mathcal{T}}_{TP}$? Section 2 addresses this question, proving that there is a positive answer if and only if one already has $\leq^X_W = \preceq^X_{TP}$ (for the original topology $\mathcal{T}$). In fact, the main result (Theorem 2.4) shows that there are three possibilities:

(0) There is no topology τ whatsoever on X such that $\leq^\tau_W = \preceq^{\mathcal{T}}_{TP}$.

(1) There is a unique topology τ on X such that $\leq^\tau_W = \preceq^{\mathcal{T}}_{TP}$, namely $\tau = \mathcal{T}$.

(2) There are exactly two topologies τ on X such that $\leq_W^\tau = \preceq_{TP}^{\mathcal{T}}$. In this case, X must have more than one point and $\mathcal{T}$ must be an Alexandrov topology, that is a topology where arbitrary intersections of open sets are open. Moreover:

- if $\mathcal{T}$ is the discrete topology, then case (2) occurs and the two topologies τ satisfying the equality are $\mathcal{T}$ and the trivial topology;
- if $\mathcal{T}$ is not discrete and case (2) occurs, then the two topologies τ satisfying the equality are $\mathcal{T}$ and $\mathbf{\Pi}_1^0(\mathcal{T})$, where $\mathbf{\Pi}_1^0(\mathcal{T})$ is the closed family of $\mathcal{T}$ — see below.

Theorem 2.4 suggests the question of what are the second countable, T_0 spaces X that satisfy the equality $\leq_W^X = \preceq_{TP}^X$. Section 3 settles the problem for Hausdorff spaces, showing in Theorem 3.4 that if X is second countable and Hausdorff, then $\leq_W^X = \preceq_{TP}^X$ if and only if X is zero-dimensional (such spaces are necessarily metrisable).

For spaces that are not Hausdorff, a satisfactory characterisation is still lacking, but partial information is provided in Section 4. In particular, a negative answer to the motivating question [3, Question 1] is given in Corollary 4.4.

In this paper, I stick to the classical topological and descriptive set theoretic notation. Notice that this does not coincide with the notation used in some papers like [7] or [5], as in such articles $\mathbf{\Sigma}_2^0$ sets and $\mathbf{\Pi}_2^0$ sets are defined differently.

A subset A of a set X is *proper* if $A \neq X$; this is denoted $A \subset X$. If A is a subset of a topological space X, the frontier of A in X is denoted $Fr_X(A)$: this is the set of points that belong to the closure of A but not to its interior. The set A is *self-dual* if $A \leq_W^X X \setminus A$, or equivalently $A \equiv_W^X X \setminus A$; as this concept is invariant under $\equiv_W^X$, the terminology extends to Wadge degrees. A set I is *countable* if $\mathrm{card}(I) = \aleph_0$; it is *at most countable* if $\mathrm{card}(I) \leq \aleph_0$.

For X a topological space, let $\mathbf{\Sigma}_1^0(X)$ be the family of open subsets of X. Denote $\mathbf{\Pi}_1^0(X)$ the family of closed subsets of X; by $\mathbf{\Sigma}_2^0(X)$ the family of F_σ subsets of X, that is unions of at most countably many closed sets; by $\mathbf{\Pi}_2^0(X)$ the family of G_δ subsets of X, that is intersections of at most countably many open sets. If Γ is a class of subsets of topological spaces, let $\check{\Gamma}$ be the dual class of Γ, that is the collection of all complements of elements of Γ; let also $\Delta(\Gamma)$ be the collection of sets that are both in Γ and in $\check{\Gamma}$: particular cases are $\mathbf{\Delta}^0_1(X) = \mathbf{\Sigma}^0_1(X) \cap \mathbf{\Pi}^0_1(X) = \Delta(\mathbf{\Sigma}^0_1)(X)$, the family of clopen subsets of X, and $\mathbf{\Delta}^0_2(X) = \mathbf{\Sigma}^0_2(X) \cap \mathbf{\Pi}^0_2(X) = \Delta(\mathbf{\Sigma}^0_2)(X)$. Finally, for α an ordinal, let $D_\alpha(\Gamma)$ be the family of all α-differences of members of Γ (see for instance [4, §22.E]).

I wish to thank Yann Pequignot for several helpful discussions on the subject of this paper and for his detailed criticism on an earlier draft of it.

2. The main theorem

A standard tool for the study of the Wadge hierarchy on the Baire space is the Wadge game, which was introduced in Wadge's thesis [9].

If X is a subspace of $\mathbb{N}^\mathbb{N}$, for $A, B \in \mathcal{P}(X)$ consider the following game $G^X_W(A, B)$, which is an adaptation of the Wadge game on the Baire space. The rules are the same: players *I* and *II* alternate moves, playing a natural number at each of their turns; player *II* is allowed to skip. Let $x \in \mathbb{N}^\mathbb{N}, y \in \mathbb{N}^{\leq\omega}$ be the sequences played by player *I* and *II*, respectively, in a run of the game; then player *II* wins if and only if one the following conditions is satisfied:

- $x \notin X$; or
- $x \in A, y \in B$; or
- $x \in X \setminus A, y \in X \setminus B$.

Player *II* has a winning strategy in $G^X_W(A, B)$ if and only if there exists a continuous function $f : X \to X$ such that $A = f^{-1}(B)$. Notice that if X, A, B are Borel subsets of $\mathbb{N}^\mathbb{N}$, then the game $G^X_W(A, B)$ is determined; this implies that either $A \leq^X_W B$ or $X \setminus B \leq^X_W A$, in other words, the *semi-linear ordering principle* for Borel subsets of X holds.

The following lemma describes a small initial segment of the Wadge hierarchy, for general topological spaces and for subspaces of $\mathbb{N}^\mathbb{N}$. Parts (1) through (3) are used in the proof of the main theorem, parts (4) and (5) are needed later.

Lemma 2.1. *Let X be any topological space. Then:*

(1) $\emptyset, X$ are the two bottom elements with respect to $\leq_W^X$, and they precede all other elements of $\mathcal{P}(X)$:

$$\forall A \in \mathcal{P}(X) \setminus \{\emptyset, X\} \ (\emptyset \leq_W^X A \wedge X \leq_W^X A).$$

(2) If $\mathbf{\Delta}_1^0(X) \setminus \{\emptyset, X\} \neq \emptyset$, then $\mathbf{\Delta}_1^0(X) \setminus \{\emptyset, X\}$ is a Wadge degree and precedes under $\leq_W^X$ every Wadge degree different from $\{\emptyset\}, \{X\}$:

$$\forall A \in \mathbf{\Delta}_1^0(X) \ \forall B \in \mathcal{P}(X) \setminus \{\emptyset, X\} \ A \leq_W^X B.$$

Assume now $X \subseteq \mathbb{N}^{\mathbb{N}}$. Then:

(3) If X is not discrete then $\mathbf{\Sigma}_1^0(X) \setminus \mathbf{\Delta}_1^0(X), \mathbf{\Pi}_1^0(X) \setminus \mathbf{\Delta}_1^0(X)$ are non-self-dual Wadge degrees, and precede every Wadge degree not included in $\mathbf{\Pi}_1^0(X), \mathbf{\Sigma}_1^0(X)$, respectively:

$$\forall A \in \mathbf{\Sigma}_1^0(X) \ \forall B \in \mathcal{P}(X) \setminus \mathbf{\Pi}_1^0(X) \ A \leq_W^X B,$$
$$\forall A \in \mathbf{\Pi}_1^0(X) \ \forall B \in \mathcal{P}(X) \setminus \mathbf{\Sigma}_1^0(X) \ A \leq_W^X B.$$

(4) $\mathcal{P}(X) \neq \mathbf{\Sigma}_1^0(X) \cup \mathbf{\Pi}_1^0(X)$ if and only if X has at least two limit points. In this case $\Delta(D_2(\mathbf{\Sigma}_1^0))(X) \setminus (\mathbf{\Sigma}_1^0(X) \cup \mathbf{\Pi}_1^0(X))$ is a self-dual Wadge degree and precedes under $\leq_W^X$ every Wadge degree not included in $\mathbf{\Sigma}_1^0(X) \cup \mathbf{\Pi}_1^0(X)$:

$$\forall A \in \Delta(D_2(\mathbf{\Sigma}_1^0))(X) \ \forall B \in \mathcal{P}(X) \setminus (\mathbf{\Sigma}_1^0(X) \cup \mathbf{\Pi}_1^0(X)) \ A \leq_W^X B.$$

(5) $\mathcal{P}(X) \neq \Delta(D_2(\mathbf{\Sigma}_1^0))(X)$ if and only if the set of limit points of X is not discrete. In this case, $D_2(\mathbf{\Sigma}_1^0)(X) \setminus \Delta(D_2(\mathbf{\Sigma}_1^0))(X), \check{D}_2(\mathbf{\Sigma}_1^0)(X) \setminus \Delta(D_2(\mathbf{\Sigma}_1^0))(X)$ are non-self-dual Wadge degrees, and precede every Wadge degree not included in $\check{D}_2(\mathbf{\Sigma}_1^0)(X), D_2(\mathbf{\Sigma}_1^0)(X)$, respectively:

$$\forall A \in D_2(\mathbf{\Sigma}_1^0)(X) \ \forall B \in \mathcal{P}(X) \setminus \check{D}_2(\mathbf{\Sigma}_1^0)(X) \ A \leq_W^X B,$$
$$\forall A \in \check{D}_2(\mathbf{\Sigma}_1^0)(X) \ \forall B \in \mathcal{P}(X) \setminus D_2(\mathbf{\Sigma}_1^0)(X) \ A \leq_W^X B.$$

Proof. (1) If $A \in \mathcal{P}(X) \setminus \{\emptyset\}$, any constant function with value in A reduces X to A and $\emptyset$ to $X \setminus A$. Similarly, if $A \in \mathcal{P}(X) \setminus \{X\}$, any constant function with value in $X \setminus A$ reduces $\emptyset$ to A and X to $X \setminus A$.

(2) If $A \in \mathbf{\Delta}^0_1(X), B \in \mathcal{P}(X) \setminus \{\emptyset, X\}$, then any two-value function sending all elements of A to a fixed element of B, and all elements of $X \setminus A$ to a fixed element of $X \setminus B$ is continuous and reduces A to B. This in particular shows that all non-empty, proper, clopen subsets of X are $\equiv^X_W$-equivalent, so that $\mathbf{\Delta}^0_1(X) \setminus \{\emptyset, X\}$ is indeed a $\equiv^X_W$-degree.

(3) Since X is not discrete, there are non-clopen points; as every point is closed, it follows that $\mathbf{\Pi}^0_1(X) \setminus \mathbf{\Delta}^0_1(X) \neq \emptyset$, and then $\mathbf{\Sigma}^0_1(X) \setminus \mathbf{\Delta}^0_1(X) \neq \emptyset$ as well. Consequently, it is enough to show that $\forall A \in \mathbf{\Sigma}^0_1(X)\ \forall B \in \mathcal{P}(X) \setminus \mathbf{\Pi}^0_1(X)\ A \leq^X_W B$. Let S be a subset of $\mathbb{N}^{<\omega}$ such that $A = \bigcup_{s \in S} N^X_s$, where $N^X_s = \{x \in X \mid s \subseteq x\}$, and pick $x \in Fr_X(B) \setminus B$. The following is a winning strategy for II in $G^X_W(A, B)$: as long as I's does not extend any element of S, player II enumerates x; if at some moment I's position extends an element of S, then player II continues until the end of the game by enumerating a fixed element of B extending his current position.

(4) If X is discrete, then $\mathcal{P}(X) = \mathbf{\Delta}^0_1(X)$. If X has one limit point, say x, then every set containing x is closed and every set not containing x is open. So assume that x, y are distinct limit points of X. Let $U \in \mathbf{\Delta}^0_1(\mathbb{N}^\mathbb{N})$ be such that $U \cap \{x, y\} = \{x\}$. Then

$$
\begin{aligned}
((U \cap X) \setminus \{x\}) \cup \{y\} = (X \setminus \{x\}) \cap (U \cup \{y\}) \in \\
\in \Delta(D_2(\mathbf{\Sigma}^0_1))(X) \setminus (\mathbf{\Sigma}^0_1(X) \cup \mathbf{\Pi}^0_1(X)) \neq \emptyset.
\end{aligned}
$$

Let $A \in \Delta(D_2(\mathbf{\Sigma}^0_1))(X), B \in \mathcal{P}(X) \setminus (\mathbf{\Sigma}^0_1(X) \cup \mathbf{\Pi}^0_1(X))$, so that there are pruned trees T_0, T_1, S_0, S_1 on $\mathbb{N}$ such that

$$
\begin{aligned}
&[T_0] \subseteq [T_1], \\
&[S_0] \subseteq [S_1], \\
&A = ([T_1] \setminus [T_0]) \cap X, \\
&X \setminus A = ([S_1] \setminus [S_0]) \cap X.
\end{aligned}
$$

Moreover, there are points $b \in Fr_X(B) \cap B, b' \in Fr_X(B) \setminus B$.

The following describes a winning strategy for II in $G^X_W(A, B)$. Player II first enumerates $b \cap b'$. Then he skips his turn as long as I's position stays in $T_0 \cap S_0$. If player I never leaves $T_0 \cap S_0$, then at the end of the game he will have produced a sequence $x \notin X$, so that II wins this run.

If player I reaches a position $s \notin T_0 \cap S_0$, there are various cases to consider.

(i) $s \in T_1 \setminus T_0$. Then player *II* continues the enumeration of b, as long as *I*'s position remains in T_1.

- If player *I* never leaves T_1, at the end of the game either his run is outside X, or belongs to A. In both cases player *II* wins, as $b \in B$.
- If player *I* reaches a position in $\mathbb{N}^{<\omega} \setminus T_1$, then *II* plays until the end of the game by enumerating an element in $X \setminus B$; this can be done, as $b \in Fr_X(B)$. Since the final sequence produced by *I* is not in A, player *II* wins.

(ii) Case (i) does not apply, and $s \in S_1 \setminus S_0$. Then player *II* continues the enumeration of b', as long as *I*'s position remains in S_1.

- If player *I* never leaves S_1, his final run is outside A, so *II* wins as $b' \notin B$.
- If player *I* reaches a position in $\mathbb{N}^{<\omega} \setminus S_1$, then *II* continues by producing at the end of the game an element in B. This can be done, as $b' \in Fr_X(B)$. Since the final sequence produced by *I* is either outside X or in A, player *II* wins.

(iii) Case (ii) does not apply, and $s \notin T_1$ (so case (i) does not apply either). Then player *II* continues with the enumeration of b' and wins the game, since the final sequence produced by *I* is not in A.

(iv) Cases (i) and (iii) do not apply, and $s \notin S_1$ (so case (ii) does not apply either). Then *II* plays by continuing the enumeration of b. Since the final sequence produced by *I* is either outside X or in A, player *II* wins.

(5) Let D be the set of limit points of X. Assume first that D is discrete. It can be assumed that $D \neq \emptyset$, otherwise X is discrete. Since every subset of D is closed in X and every subset of $X \setminus D$ is open in X, it follows that $\mathcal{P}(X) = \Delta(D_2(\mathbf{\Sigma}^0_1))(X)$.

Assume now that D has a limit point a, and fix a sequence a_n of distinct points of $D \setminus \{a\}$ converging to a. For every $n \in \mathbb{N}$, let D_n be an open neighbourhood of a_n such that $n \neq n' \Rightarrow D_n \cap D_{n'} = \emptyset$ and $\lim_{n\to\infty} diam(D_n) = 0$. Let $F = \{a\} \cup \bigcup_{n\in\mathbb{N}}(D_n \setminus \{a_n\})$. Then F is the union of the open set $\bigcup_{n\in\mathbb{N}}(D_n \setminus \{a_n\})$ and the closed set $\{a\}$, so $F \in \check{D}_2(\mathbf{\Sigma}^0_1)(X)$. However, F

is not the intersection of an open set with a closed set. Indeed every open set containing a contains also some D_n, and every closed set containing $D_n \setminus \{a_n\}$ contains a_n too, but $a_n \notin F$. So $\check{D}_2(\mathbf{\Sigma}^0_1)(X) \setminus \Delta(D_2(\mathbf{\Sigma}^0_1))(X) \neq \emptyset$, consequently also $D_2(\mathbf{\Sigma}^0_1)(X) \setminus \Delta(D_2(\mathbf{\Sigma}^0_1))(X) \neq \emptyset$. Notice that it can be assumed that:

- $n < m \Rightarrow a_n \cap a \subset a_m \cap a$
- $D_n = N^X_{s_n}$ for some $s_n \subseteq a_n$

It is enough now to show that

$$\forall A \in D_2(\mathbf{\Sigma}^0_1)(X)\ A \leq^X_W X \setminus F \tag{1}$$

$$,\forall B \in \mathcal{P}(X) \setminus D_2(\mathbf{\Sigma}^0_1)(X)\ F \leq^X_W B. \tag{2}$$

To prove (1), fix $A \in D_2(\mathbf{\Sigma}^0_1)(X)$. Let T_0, T_1 be pruned trees on $\mathbb{N}$ such that

$$[T_0] \subseteq [T_1],$$
$$A = ([T_1] \setminus [T_0]) \cap X.$$

The following describes a winning strategy for II in $G_W(A, X \setminus F)$. As long as I's position belongs to T_0, player II enumerates a, so that if I never leaves T_0 then II wins since I's run is not in A. If at any of his moves player I reaches a position in $T_1 \setminus T_0$, and as long as he stays there, player II continues by enumerating some a_n extending his current position: this is possible since $\lim_{n\to\infty} a_n = a$. Thus if player I goes on by playing in $T_1 \setminus T_0$ until the end of the game, then II wins as I's run will be either outside X or in A. Finally, if at some point player I plays outside T_1, player II continues until the end of the game by enumerating some element of $D_n \setminus \{a_n\}$ extending his current position; again II wins, as player I's run will not belong to A.

To establish (2), fix $B \in \mathcal{P}(X) \setminus D_2(\mathbf{\Sigma}^0_1)(X)$. Note first the following.

Claim. There exists $x \in B$ such that for every $n \in \mathbb{N}$ there is $y \in X$ with the following properties:

$$y|_n = x|_n,$$
$$y \notin B,$$
$$y \in Fr_X(B).$$

Proof of the claim. Deny. Then for every $x \in B$ there exists $n_x \in \mathbb{N}$ such that any $y \in X \setminus B$ with $y|_{n_x} = x|_{n_x}$ is in the exterior of B: this means that $B \cap N^X_{n_x}$ is closed in X. Thus

$$B = \overline{B} \cap \bigcup_{x \in B} N^X_{n_x} \in D_2(\mathbf{\Sigma}^0_1)(X),$$

a contradiction. □

It remains thus to show that player *II* has a winning strategy in $G^X_W(F, B)$. As long as *I* enumerates a, player *II* enumerates an element x as granted by the claim, so that if *I*'s final run is a, then *II* wins. If at some of his moves player *I* stops enumerating a, then *II* skips until one of the following situations occurs:

(i) *I* reaches a position incompatible with all s_n. Then *II* continues by extending his current position producing at the end of the game an element of $X \setminus B$. This can be done by the claim. Player *II* wins as *I*'s run does not belong to F.

(ii) *I* reaches a position s_n. Then:

- As long as *I*'s positions are initial segments of a_n, player *II* continues by enumerating an element $y \in Fr_X(B) \setminus B$, which can be done by the claim; thus if *I*'s final run is a_n, then *II* wins.
- If at some of his moves player *I* reaches a position incompatible with a_n, then *II* continues until the end of the game with the enumeration of some element of B; again, *II* wins, as *I*'s run is either outside X or in F.

□

The structure described in Lemma 2.1 can be visualised in Figure 1.

Although the small initial segment of Wadge hierarchy described in Lemma 2.1 suffices for the purpose of this paper, the natural question arises about the structure of the hierarchy on general zero-dimensional, separable, metrisable spaces (for positive dimensional ones see the strong result of [6, Theorem 2.14]).

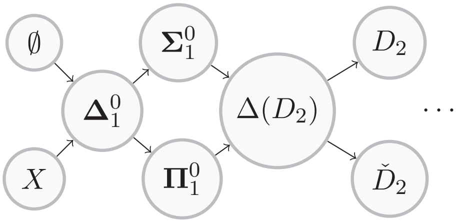

Figure 1: An initial segment of the Wadge hierarchy on a subspace of $\mathbb{N}^{\mathbb{N}}$ (Lemma 2.1).

Conjecture. Let X be a zero-dimensional, separable, metrisable space. Then the Wadge hierarchy on X is well-behaved with respect to $\boldsymbol{\Delta}^0_2$ sets, in the sense that

$$\forall A, B \in \boldsymbol{\Delta}^0_2(X)\ (A \leq^X_W B \vee (X \setminus B) \leq^X_W A),$$
$$\forall A \in \boldsymbol{\Delta}^0_2(X)\ \forall B \in \mathcal{P}(X) \setminus \boldsymbol{\Delta}^0_2(X)\ A \leq^X_W B.$$

Moreover, this good behaviour does not extend to F_σ or G_δ sets, in the sense that there exists a zero-dimensional, separable, metrisable space Y such that

$$\exists A \in \boldsymbol{\Sigma}^0_2(Y)\ \exists B \in \mathcal{P}(Y) \setminus \boldsymbol{\Pi}^0_2(Y)\ A \not\leq^Y_W B.$$

Of course, such a conjecture makes sense in ZFC, while its second part is false under AD.

Lemma 2.2. *Let X be a topological space such that $\boldsymbol{\Sigma}^0_1(X) = \boldsymbol{\Delta}^0_1(X)$. Then there are two possibilities:*

- *either X is discrete*
- *or X is not T_0, in which case X has exactly four Wadge degrees: $\{\emptyset\}, \{X\}, \boldsymbol{\Delta}^0_1(X) \setminus \{\emptyset, X\}, \mathcal{P}(X) \setminus \boldsymbol{\Delta}^0_1(X)$ (see Figure 2)*

In particular, all non-empty, proper subsets of X are self-dual.

Proof. Under the hypothesis, the Boolean algebra $\boldsymbol{\Delta}^0_1(X)$ is closed under arbitrary unions and intersections. So X is the disjoint union of clopen subsets that are atoms in this Boolean algebra: $X = \bigcup_{i \in I} D_i$.

Suppose that X is T_0: then for any $x \in X$, if D_i is such that $x \in D_i$, it must follow that $D_i = \{x\}$, which means that X is discrete.

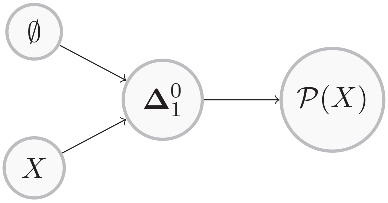

Figure 2: The Wadge hierarchy on a non-discrete space whose open sets are clopen (Lemma 2.2).

Assume now that X is not T_0, so that in particular $\mathcal{P}(X) \neq \mathbf{\Delta}^0_1(X)$, and let $A \in \mathcal{P}(X), B \in \mathcal{P}(X) \setminus \mathbf{\Delta}^0_1(X)$. Since $\mathbf{\Sigma}^0_1(X) = \mathbf{\Delta}^0_1(X)$, there exist no $J \subseteq I$ such that $B = \bigcup_{i \in J} D_i$; consequently there is $i \in I$ for which there exist $b \in D_i \cap B, b' \in D_i \setminus B$. If $f : X \to X$ is defined by letting

$$f(x) = \begin{cases} b & \text{if} \quad x \in A, \\ b' & \text{if} \quad x \in X \setminus A, \end{cases}$$

then f is continuous and witnesses $A \leq^X_W B$. $\square$

The following fact will be used several times.

Lemma 2.3. *Let X be a second countable, T_0 space, and let $\rho : Y \subseteq \mathbb{N}^\mathbb{N} \to X$ be an admissible representation. Then, for every $A, B \in \mathcal{P}(X), 1 \leq \alpha < \omega_1$:*

(1) $\rho^{-1}(A) \in D_\alpha(\mathbf{\Sigma}^0_1)(Y) \Leftrightarrow A \in D_\alpha(\mathbf{\Sigma}^0_1)(X)$,

(2) if $A \preceq^X_{TP} B \in D_\alpha(\mathbf{\Sigma}^0_1)(X)$, *then* $A \in D_\alpha(\mathbf{\Sigma}^0_1)(X)$,

(3) $\rho^{-1}(A) \in D_\alpha(\mathbf{\Pi}^0_1)(Y) \Leftrightarrow A \in D_\alpha(\mathbf{\Pi}^0_1)(X)$,

(4) if $A \preceq^X_{TP} B \in D_\alpha(\mathbf{\Pi}^0_1)(X)$, *then* $A \in D_\alpha(\mathbf{\Pi}^0_1)(X)$.

Proof. Part (1) is an instance of [1, Theorem 68], and part (2) an instance of [5, Theorem 4].

However, an inspection of the proof as given in [5] shows that the argument goes through when $\mathbf{\Sigma}^0_1$ is replaced by $\mathbf{\Pi}^0_1$ as well, which gives (3). So, similarly to [5, Theorem 4], part (4) follows from the definition of $\preceq^X_{TP}$. $\square$

Theorem 2.4. *Let $X = (X, \mathcal{T})$ be a second countable, T_0 space. Then there are three possibilities:*

(0) there is no topology τ on X such that $\leq_W^\tau = \preceq_{TP}^{\mathcal{T}}$; or

(1) there is a unique topology τ on X such that $\leq_W^\tau = \preceq_{TP}^{\mathcal{T}}$, namely $\tau = \mathcal{T}$; or

(2) there are exactly two topologies τ on X such that $\leq_W^\tau = \preceq_{TP}^{\mathcal{T}}$. In this case, X must have more than one point and $\mathcal{T}$ must be an Alexandrov topology. Moreover:

- *if X is not a singleton and $\mathcal{T}$ is the discrete topology, then case (2) occurs and the two topologies τ satisfying the equality are $\mathcal{T}$ and the trivial topology;*
- *if $\mathcal{T}$ is not discrete and case (2) occurs, then the two topologies τ satisfying the equality are $\mathcal{T}$ and $\mathbf{\Pi}_1^0(\mathcal{T})$.*

Proof. Suppose τ is a topology on X such that

$$\leq_W^\tau = \preceq_{TP}^{\mathcal{T}} \tag{3}$$

in order to show that $\leq_W^{\mathcal{T}} = \preceq_{TP}^{\mathcal{T}}$.

Notice that both the $\mathcal{T}$-open sets and the τ-open sets form an initial segment of $\preceq_{TP}^{\mathcal{T}}$: for the $\mathcal{T}$-open sets, see Lemma 2.3(2); for the τ-open sets, this is just the definition of continuity of a function and equality (3). Recall from Lemma 2.1(1,2) that $\{\emptyset\}, \{X\}$ are the two bottom $\equiv_W^\tau$-degrees, which precede every other $\equiv_W^\tau$-degree, and moreover any clopen set is the continuous preimage of every set that is not empty nor the entire space (in particular, proper, non-empty clopen sets are self-dual).

The case when $\{\emptyset\}, \{X\}$ are the only $\equiv_W^\tau$-degrees is equivalent to X being a singleton; in this situation there is only one topology on X, so that $\tau = \mathcal{T}$. For the rest of the proof it can then be assumed that X has more than one point.

If there are exactly three $\equiv_W^\tau$-degrees, say $\{\emptyset\}, \{X\}, \mathcal{D}$, then all non-empty, proper subsets of X are pairwise $\equiv_W^\tau$-equivalent so, as $\equiv_W^\tau = \approxeq_{TP}^{\mathcal{T}}$, either none of them is $\mathcal{T}$-open or they are all $\mathcal{T}$-open, again by Lemma 2.3(2). The first alternative does not apply, as $\mathcal{T}$ is T_0; consequently $\mathcal{T}$ is discrete, and $\leq_W^{\mathcal{T}} = \preceq_{TP}^{\mathcal{T}}$. In this case, the equation $\leq_W^\tau = \preceq_{TP}^{\mathcal{T}}$ is satisfied also if τ is the trivial topology, since all functions $X \to X$ are continuous for the trivial as well as the discrete topology on X. On the other hand, if τ is not discrete nor trivial, then $\leq_W^\tau \neq \preceq_{TP}^{\mathcal{T}}$, since given $A, B \in \mathcal{P}(X) \setminus \{\emptyset, X\}$ with $A \in \mathbf{\Sigma}_1^0(\tau), B \notin \mathbf{\Sigma}_1^0(\tau)$, then $A \not\equiv_W^\tau B$.

Conversely, if $\mathcal{T}$ is discrete then $\approxeq_{TP}^{\mathcal{T}}$, which is the same as $\equiv_W^\tau$, has at most three equivalence classes. So it can be assumed for the rest of the proof that there are more than three $\equiv_W^\tau$-degrees and consequently $\mathcal{T}, \tau$ are not discrete.

Fix an admissible representation $\rho : Y \subseteq \mathbb{N}^{\mathbb{N}} \to X$. Using Lemma 2.3 and the fact that admissible representations are surjective, for all $A \in \mathcal{P}(X)$,

$$\begin{aligned} \rho^{-1}(A) \in \mathbf{\Delta}_1^0(Y) \setminus \{\emptyset, Y\} \quad &\Leftrightarrow \quad A \in \mathbf{\Delta}_1^0(\mathcal{T}) \setminus \{\emptyset, X\}, \\ \rho^{-1}(A) \in \mathbf{\Sigma}_1^0(Y) \setminus \mathbf{\Delta}_1^0(Y) \quad &\Leftrightarrow \quad A \in \mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T}). \end{aligned}$$

Since $\mathcal{T}$ is T_0 and not discrete, by Lemma 2.2 it follows that $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T}) \neq \emptyset$. Consequently, $\{\emptyset\}$, $\{X\}$, $\mathbf{\Delta}_1^0(\mathcal{T}) \setminus \{\emptyset, X\}$ (if non-empty), $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$, and $\mathbf{\Pi}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$ are distinct $\approxeq_{TP}^{\mathcal{T}}$-classes, they constitute an initial segment of $\preceq_{TP}^{\mathcal{T}}$, they are ordered like the corresponding Wadge degrees in Y, and by Lemma 2.1(3) they $\preceq_{TP}^{\mathcal{T}}$-precede all other $\approxeq_{TP}^{\mathcal{T}}$-classes.

Claim. Either $\tau = \mathcal{T}$, or $\tau = \mathbf{\Pi}_1^0(\mathcal{T})$.

Proof of the claim. Recalling that τ-open sets form an initial segment with respect to $\preceq_{TP}^{\mathcal{T}}$, it is enough to observe the following facts:

- τ is not trivial, as there are more than three $\equiv_W^\tau$-degrees.
- $\tau \neq \mathbf{\Delta}_1^0(\mathcal{T})$: otherwise $\mathbf{\Sigma}_1^0(\tau) = \mathbf{\Delta}_1^0(\tau)$ and all non-empty, proper subsets of X would be τ-self-dual by Lemma 2.2, which is not the case as the elements of $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$ are not τ-self-dual since $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T}), \mathbf{\Pi}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$ are distinct $\equiv_W^\tau$-degrees.
- τ cannot contain both $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$ and $\mathbf{\Pi}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$: otherwise $\mathbf{\Sigma}_1^0(\mathcal{T}) \subseteq \mathbf{\Delta}_1^0(\tau)$, and all members of $\mathbf{\Sigma}_1^0(\mathcal{T}) \setminus \mathbf{\Delta}_1^0(\mathcal{T})$ would be τ-self-dual, which is again not the case.

$\square$

To finish the proof, notice that if $\mathbf{\Pi}^0_1(\mathcal{T})$ is a topology, then it induces the same Wadge hierarchy as $\mathcal{T}$, since these topologies have the same continuous functions. □

All cases dealt with in Theorem 2.4 can occur: for cases (0) and (1), see Theorem 3.4; for case (2) concerning a non-discrete space, see Example 5.2.

Theorem 2.4 leads to ask whether there is a nice characterisation of those second countable, T_0 spaces X such that $\leq^X_W = \preceq^X_{TP}$. Though the question remains open, in the rest of the paper some specific cases are discussed.

Since for any second countable, T_0 space X the relation $\leq^X_W \subseteq \preceq^X_{TP}$ always holds, to show equality it is enough to establish the implication $A \preceq^X_{TP} B \Rightarrow A \leq^X_W B$.

The following facts will be used repeatedly.

Lemma 2.5. *Let X be any second countable, T_0 space such that $\leq^X_W = \preceq^X_{TP}$. Then*

- $\forall A \in \mathbf{\Sigma}^0_1(X)\ \forall B \in \mathcal{P}(X) \setminus \mathbf{\Pi}^0_1(X)\ A \leq^X_W B$,
- $\forall A \in \mathbf{\Pi}^0_1(X)\ \forall B \in \mathcal{P}(X) \setminus \mathbf{\Sigma}^0_1(X)\ A \leq^X_W B$,
- *there is no triple of pairwise $\leq^X_W$-incomparable sets in $D_2(\mathbf{\Sigma}^0_1)(X)$.*

Proof. By Lemmas 2.1 and 2.3. □

3. The case of Hausdorff spaces

This section characterises the second countable, Hausdorff spaces X such that $\leq^X_W = \preceq^X_{TP}$ as the zero-dimensional metrisable spaces (Theorem 3.4). Observe also the following.

Lemma 3.1. *If X is a second countable, T_0, zero-dimensional space, then X is metrisable.*

Proof. First, notice that X is Hausdorff. Indeed, let x, y be distinct points in X. Since X is T_0, there exists an open set U containing exactly one of the points x, y. By zero-dimensionality, there exists a clopen set

$V \subseteq U$ such that $V \cap \{x, y\} = U \cap \{x, y\}$. So the clopen sets $V, X \setminus V$ are disjoint neighbourhoods of the two points.

Similarly, one establishes that X is T_3. Let $x \in X$ and let $C \subseteq X$ be closed, with $x \notin C$. Again by zero-dimensionality, there is a clopen neighbourhood V of x such that $V \cap C = \emptyset$. Then $V, X \setminus V$ are disjoint neighbourhoods of x, C, respectively.

Consequently, by Urysohn's metrisation theorem, X is metrisable. □

Proposition 3.2. *Let X be a second countable, T_0 space.*

1. *If X is zero-dimensional, then $\leq_W^X = \preceq_{TP}^X$*
2. *If X is metrisable, non-zero-dimensional, then $\leq_W^X \neq \preceq_{TP}^X$*

Proof. (1) If X is zero-dimensional, then by Lemma 3.1 it is metrisable. So the result is [5, Proposition 3].

(2) As [6] exhibits an antichain for $\leq_W^X$ of size the continuum whose members are sets in $D_2(\mathbf{\Sigma}_1^0)(X)$, to conclude it is enough to apply Lemma 2.5. □

Notice that for *Borel representable* spaces, that is spaces admitting an admissible representation whose domain is a Borel subset of $\mathbb{N}^\mathbb{N}$, Proposition 3.2(2) is [5, Corollary 2].

Proposition 3.3. *Let X be a second countable, Hausdorff space. If X is not T_3, then $\leq_W^X \neq \preceq_{TP}^X$.*

Proof. Let $C \in \mathbf{\Pi}_1^0(X), x \in X \setminus C$ without disjoint neighbourhoods. In particular, $\{x\}$ is not clopen. Using Lemma 2.5, it is enough to show $C \not\leq_W^X \{x\}$. Suppose, towards a contradiction, that $f : X \to X$ is a continuous function reducing C to $\{x\}$, and let U, V be disjoint neighbourhoods of $x, f(x)$ respectively. Then $f^{-1}(U), f^{-1}(V)$ are disjoint neighbourhoods of C, x, respectively. □

Theorem 3.4. *If X is a second countable, Hausdorff space, then $\leq_W^X = \preceq_{TP}^X$ if and only if X is zero-dimensional.*

Proof. For the forward implication, apply Proposition 3.3 and Urysohn's metrisability theorem to obtain that X is metrisable. Then use Proposition 3.2(2) for zero-dimensionality.

The backward direction is in Proposition 3.2(1). □

4. An application to the conciliatory hierarchy

This section deals with second countable spaces that are T_0 but not T_1. It shows that for such a space X the equality $\leq_W^X = \preceq_{TP}^X$ implies that X is at most countable and it carries an Alexandrov topology. As a consequence, [3, Question 1] gets a negative answer.

Proposition 4.1. *Let X be a second countable, T_0, non-T_1 space such that $\leq_W^X = \preceq_{TP}^X$. Then X carries an Alexandrov topology.*

Proof. Let $\{W_h\}_{h \in H}$ be a family of open subsets of X, and let $A = \bigcap_{h \in H} W_h$ in order to prove that A is open. As X is not T_1, let $x \in X$ be such that $\{x\} \subset \overline{\{x\}}$. By T_0, there is a closed non-empty $C \subseteq \overline{\{x\}} \setminus \{x\}$. Using Lemma 2.5, it is now enough to show that $C \not\leq_W^X A$.

Suppose, towards a contradiction, that $f : X \to X$ is a continuous function such that $C = f^{-1}(A)$. This implies that $\forall h \in H\, C \subseteq f^{-1}(W_h)$; as every $f^{-1}(W_h)$ is open, it follows that $\forall h \in H\, x \in f^{-1}(W_h)$, so $x \in f^{-1}(A)$, a contradiction. $\square$

Lemma 4.2. *Let X be a second countable, T_0 space carrying an Alexandrov topology. Then X is at most countable.*

Proof. Let $\mathcal{B}$ be an at most countable basis for X. For every $x \in X$, let $U_x = \bigcap\{U \in \mathcal{B} \mid x \in U\}$. So U_x is open, and it is the smallest open set containing x, so that $U_x \in \mathcal{B}$. As X is T_0, the function $x \mapsto U_x$ is injective, so X is at most countable. $\square$

Corollary 4.3. *Let X be a second countable, T_0, non-T_1 space such that $\leq_W^X = \preceq_{TP}^X$. Then X is at most countable.*

Proof. By Proposition 4.1 and Lemma 4.2. $\square$

From Corollary 4.3, one obtains in particular a negative answer to [3, Question 1].

Corollary 4.4. *There is no topology on $\mathbb{N}^{\leq\omega}$ such that the conciliatory preorder $\leq_c$ is the Wadge preorder induced by that topology.*

Proof. By [3], the relation $\leq_c$ is the Tang-Pequignot preorder with respect to the prefix topology on $\mathbb{N}^{\leq\omega}$. Since this topology is not T_1, the result follows from Corollary 4.3 and Theorem 2.4. $\square$

5. Some examples and questions

The main general question raised by Theorem 2.4 is the following.

Question 5.1. Is there a nice characterisation of those second countable, T_0 spaces X such that $\leq_W^X = \preceq_{TP}^X$?

For Hausdorff spaces, the answer to this question is given in Theorem 3.4. In this section, some basic examples of non-Hausdorff spaces are discussed, together with some related questions.

Example 5.2. Let the Sierpiński space $S = \{a, b\}$ be a doubleton endowed with the topology $\mathcal{T} = \{\emptyset, \{a\}, S\}$, so S is a second countable, T_0, non-T_1 space. Then $\leq_W^S = \preceq_{TP}^S$.

Indeed, $\{a\}, \{b\}$ are $\leq_W^S$-incomparable. Also, they are $\preceq_{TP}^S$-incomparable, since for any admissible representation $\rho : Y \subseteq \mathbb{N}^\mathbb{N} \to S$, one has $\rho^{-1}(\{a\}) \in \mathbf{\Sigma}_1^0(\mathbb{N}^\mathbb{N}) \setminus \mathbf{\Pi}_1^0(\mathbb{N}^\mathbb{N}), \rho^{-1}(\{b\}) \in \mathbf{\Pi}_1^0(\mathbb{N}^\mathbb{N}) \setminus \mathbf{\Sigma}_1^0(\mathbb{N}^\mathbb{N})$.

Example 5.3. Let X be a countable space with the cofinite topology. Then X is second countable, T_1, non-Hausdorff.

Let $\mathcal{A} = \mathcal{P}(X) \setminus (\mathbf{\Sigma}_1^0(X) \cup \mathbf{\Pi}_1^0(X))$. Then X has five Wadge degrees: $\{\emptyset\}, \{X\}, \mathbf{\Sigma}_1^0(X) \setminus \{\emptyset, X\}, \mathbf{\Pi}_1^0(X) \setminus \{\emptyset, X\}, \mathcal{A}$.

Indeed, if $A \in \mathbf{\Sigma}_1^0(X), B \in \mathcal{P}(X) \setminus \mathbf{\Pi}_1^0(X)$, any $f : X \to X$ such that $f|_A$ is finite-to-1 and $f^{-1}(B) = A$ is continuous and witnesses $A \leq_W^X B$. Similarly, given $A, B \in \mathcal{A}$, since A, B are infinite and coinfinite, any finite-to-1 function $f : X \to X$ such that $f^{-1}(B) = A$ is continuous and witnesses $A \leq_W^X B$.

It follows that $\leq_W^X = \preceq_{TP}^X$.

Up to homeomorphism, the topology of Example 5.3 is the Zariski topology on a countable commutative field K. This suggests the following.

Question 5.4. Given a countable commutative field K, does the relation $\leq_W^{K^n} = \preceq_{TP}^{K^n}$ hold, when K^n is endowed with the Zariski topology?

To understand uncountable spaces, the following seems a promising first step, being an uncountable analog of the space in Example 5.3.

Question 5.5. Let $\mathcal{T}$ be the compact complement topology on $\mathbb{R}$. Does $\leq_W^{\mathcal{T}} = \preceq_{TP}^{\mathcal{T}}$ hold?

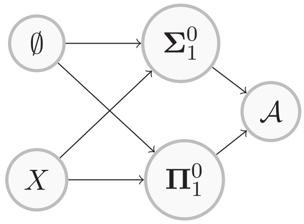

Figure 3: The Wadge hierarchy on a countable space with the cofinite topology (Example 5.3).

Observe that the spaces of Questions 5.4 and 5.5 are in fact second countable, T_1, non-Hausdorff, and they are hyperconnected.

To end the paper, notice that, as a consequence of Proposition 4.3, the condition $\leq^X_W = \preceq^X_{TP}$ is not stable under very basic topological operations.

Example 5.6. Let X be the disjoint sum of the Baire space and the Sierpiński space. Then $\leq^X_W \neq \preceq^X_{TP}$.

This example shows that the class of spaces for which the two hierarchies coincide is not closed under disjoint sum.

Example 5.7. Let X be the product of the Baire space and the Sierpiński space. Then $\leq^X_W \neq \preceq^X_{TP}$.

This example shows that the class of spaces for which the two hierarchy coincide is not closed under product.

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