Ludwik Byszewski orcid.org/0000-0001-7227-8990
ludwik.byszewski@pk.edu.pl
Institute of Mathematics, Faculty of Physics Mathematics and Computer Science, Cracow University of Technology

Nonlinear second-order delay differential equation

Nieliniowe równanie różniczkowe rzędu drugiego z opóźnieniem

Abstract

The aim of this paper is to prove the theorem on the existence and uniqueness of the classical solution of the initial-boundary value problem for a nonlinear second-order delay differential equation. For this purpose, we apply the Banach contraction principle and the Bielecki norm. The paper is based on publications [1–7] and is a generalisation of publication [6].

Keywords: second-order delay equation, initial-boundary value problem, Banach contraction principle, Bielecki norm

Streszczenie

W artykule udowodniono twierdzenie o istnieniu i jednoznaczności klasycznego rozwiązania zagadnienia początkowo-brzegowego dla nieliniowego równania różniczkowego rzędu drugiego z opóźnieniem. W tym celu stosowane jest twierdzenie Banacha o punkcie stałym i norma Bieleckiego. Artykuł bazuje na publikacjach [1–7] i jest uogólnieniem publikacji [6].

Słowa kluczowe: równanie rzędu drugiego z opóźnieniem, zagadnienie początkowo-brzegowe, twierdzenie Banacha o punkcie stałym, norma Bieleckiego.

1. Preliminaries

Is this paper, we study the following problem

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0, T], \; T > 0, \tag{1.1}$$

$$x_0 = \phi, \quad x'(T) = \beta x'(0), \quad \beta > 1, \tag{1.2}$$

where $f : [0, T] \times C([-\tau, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $\phi \in C([-\tau, 0], \mathbb{R})$, $\tau > 0$, are given functions.

Therefore, for any function $x : [-\tau, T] \to \mathbb{R}$ and any $t \in [0, T]$, we denote by x_t the function $x_t : [-\tau, 0] \to \mathbb{R}$ defined by the formula $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$.

It is easy to see that the condition $x_0 = \phi$ means that $x(s) = \phi(s)$, $s \in [-\tau, 0]$.

Moreover, for $\phi \in C([-\tau, 0], \mathbb{R})$ we use the norm

$$\|\phi\|_0 = \sup_{-\tau \leq s \leq 0} |\phi(s)|.$$

2. Theorem on the existence and uniqueness of the classical solution

Let $C^* := C([-\tau, T], \mathbb{R}) \cap C^2([0, T], \mathbb{R})$.

Definition 2.1. The function $x \in C^*$ is said to be a solution of problem $(1.1) - (1.2)$ if x satisfies equation (1.1) and conditions (1.2).

Now, we will prove the following lemma:

Lemma 2.1. Function $x \in C^*$ is a solution of problem $(1.1)–(1.2)$, where $f \in C([0, T] \times C([-\tau, 0], \mathbb{R}) \times \mathbb{R}, \mathbb{R})$ if and only if x is a solution of the following integral equation:

$$x(t) = \begin{cases} \phi(t), \; t \in [-\tau, 0], \\ \phi(0) + \dfrac{t}{\beta - 1} \displaystyle\int_0^T f(s, x_s, x'(s)) ds + \int_0^t (t - s) f(s, x_s, x'(s)) ds, \; t \in [0, T]. \end{cases}$$

Proof. If $x \in C^*$ is a solution of $(1.1) - (1.2)$ then we have

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0, T]. \tag{2.1}$$

Integration by parts gives

$$x(t) = x(0) + tx'(0) + \int_0^t (t-s)x''(s)ds. \tag{2.2}$$

Differentiating (2.2), we get

$$x'(t) = x'(0) + \int_0^t x''(s)ds.$$

Thus,

$$x'(T) = x'(0) + \int_0^T x''(s)ds.$$

Applying the boundary condition we obtain

$$x'(0) + \int_0^T x''(s)ds = \beta x'(0).$$

Thus,

$$x'(0) = \frac{1}{\beta-1}\int_0^T x''(s)ds. \tag{2.3}$$

Equation (2.2), together with (2.1) and (2.3), imply

$$x(t) = \phi(0) + \frac{t}{\beta-1}\int_0^T f(s,x_s,x'(s))ds + \int_0^t (t-s)f(s,x_s,x'(s))ds. \tag{2.4}$$

Conversely, if x is a solution of equation (2.4) then direct differentiation of (2.4) gives

$$x'(t) = \frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds + \int_0^t f(s,x_s,x'(s))ds,$$

$$x''(t) = f(t,x_t,x'(t)), \quad t\in[0,T].$$

Thus,

$$x'(0) = \frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds,$$

$$x'(T) = \frac{1}{\beta - 1} \int_0^T f(s, x_s, x'(s)) ds + \int_0^T f(s, x_s, x'(s)) ds = \frac{\beta}{\beta - 1} \int_0^T f(s, x_s, x'(s)) ds,$$

which gives

$$x'(T) = \beta x'(0).$$

The proof of Lemma 2.1 is complete.

Now, using Lemma 2.1 and the Banach contraction theorem, we shall prove the existence and uniqueness of the solution for problem $(1.1) - (1.2)$.

Theorem 2.1. Assume that $f \in C([0,T] \times C([-\tau, 0], \mathbb{R}) \times \mathbb{R}, \mathbb{R})$ and there exists $m \in L^1([0,T], \mathbb{R}_+)$ such that

$$\left| f(t, u, z) - f(t, \tilde{u}, \tilde{z}) \right| \le m(t)(\|u - \tilde{u}\|_0 + |z - \tilde{z}|) \tag{2.5}$$

for all $t \in [0,T]$, $u, \tilde{u} \in C([-\tau, 0], \mathbb{R})$, $z, \tilde{z} \in \mathbb{R}$ and

$$M(T) < \frac{\ln \beta}{T + 1}, \tag{2.6}$$

where $M(t) := \int_0^t m(r) dr$.

Then problem $(1.1) - (1.2)$ has a unique solution $x \in C^*$.

Proof: for $x \in C^1([0,T], \mathbb{R})$ let

$$\|x\|_1 := \max_{s \in [0,T]} \left\{ e^{-\gamma M(s)} \left(\max_{r \in [0,s]} |x(r)| + |x'(s)| \right) \right\},$$

where

$$T + 1 < \gamma < \frac{\ln \beta}{M(T)}. \tag{2.7}$$

Define an operator

$$F : C^1([0,T], \mathbb{R}) \to C^1([0,T], \mathbb{R})$$

by the formula

$$(Fx)(t) = \phi(0) + \frac{t}{\beta-1} \int_0^T f(s, x_s, x'(s)) ds + \int_0^t (t-s) f(s, x_s, x'(s)) ds,$$

where $x_s(r) = x(s+r) = \phi(s+r)$ for $s+r \leq 0$.

For any $x, y \in C^1([0,T], \mathbb{R})$ and $t \in [0,T]$, by (2.5), we have

$$\left| (Fx)(t) - (Fy)(t) \right| \leq \frac{t}{\beta-1} \int_0^T \left| f(s, x_s, x'(s)) - f(s, y_s, y'(s)) \right| ds$$

$$+ \int_0^t (t-s) \left| f(s, x_s, x'(s)) - f(s, y_s, y'(s)) \right| ds$$

$$\leq \frac{t}{\beta-1} \int_0^T m(s) \left(\|x_s - y_s\|_0 + |x'(s) - y'(s)| \right) ds$$

$$+ \int_0^t (t-s) m(s) \left(\|x_s - y_s\|_0 + |x'(s) - y'(s)| \right) ds$$

$$\leq \frac{T}{\beta-1} \int_0^T m(s) \left(\|x_s - y_s\|_0 + |x'(s) - y'(s)| \right) ds$$

$$+ T \int_0^t m(s) \left(\|x_s - y_s\|_0 + |x'(s) - y'(s)| \right) ds.$$

Observe that (see [6])

$$\|x_s - y_s\|_0 = \max\{|x(r) - y(r)|, \ r \in [0,s]\} \ \text{ if } s \in [0, \tau]$$

and

$$\|x_s - y_s\|_0 \leq \max\{|x(r) - y(r)|, \ r \in [0,s]\} \ \text{ if } s \in (\tau, T].$$

Therefore,

$$\left|(Fx)(t)-(Fy)(t)\right|$$

$$\leq \frac{T}{\beta-1}\int_0^T m(s)e^{\gamma M(s)}e^{-\gamma M(s)}\left(\max_{r\in[0,s]}\left|x(r)-y(r)\right|+\left|x'(s)-y'(s)\right|\right)ds$$

$$+T\int_0^t m(s)\,e^{\gamma M(s)}e^{-\gamma M(s)}\left(\max_{r\in[0,s]}\left|x(r)-y(r)\right|+\left|x'(s)-y'(s)\right|\right)ds$$

$$\leq \frac{T}{\beta-1}\left\|x-y\right\|_1\int_0^T m(s)e^{\gamma M(s)}ds+T\left\|x-y\right\|_1\int_0^t e^{\gamma M(s)}ds$$

$$=\frac{T}{\beta-1}\left\|x-y\right\|_1\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^T+T\left\|x-y\right\|_1\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^t$$

$$=\frac{T}{\gamma}\left\|x-y\right\|_1\left(\frac{e^{\gamma M(T)}-\beta}{\beta-1}+e^{\gamma M(t)}\right).$$

It follows from (2.7) that

$$e^{\gamma M(T)}-\beta<0.$$

Consequently,

$$\left|(Fx)(t)-(Fy)(t)\right|\leq\frac{T}{\gamma}\left\|x-y\right\|_1 e^{\gamma M(t)}. \tag{2.8}$$

Observe that

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$$(Fx)'(t)=\frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds+\int_0^t f(s,x_s,x'(s))ds.$$

Thus,

$$\left|(Fx)'(t)-(Fy)'(t)\right|\leq\frac{1}{\gamma}\left\|x-y\right\|_1\,e^{\gamma M(t)}. \tag{2.9}$$

From (2.8), (2.9) and from the definition of the norm $\left\| \cdot \right\|_1$, we have

$$\left\| Fx - Fy \right\|_1 \le \frac{T+1}{\gamma} \left\| x - y \right\|_1 . \tag{2.10}$$

By (2.10) and (2.7), F is a contractive operator. Consequently, by the Banach fixed point theorem, the proof of Theorem 2.1 is complete.

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