REPORTS ON MATHEMATICAL LOGIC
53 (2018), 79–96
doi: 10.4467/20842589RM.18.005.8838

Peter SCHUSTER and Daniel WESSEL

A GENERAL EXTENSION THEOREM FOR DIRECTED-COMPLETE PARTIAL ORDERS

A b s t r a c t. The typical indirect proof of an abstract extension theorem, by the Kuratowski-Zorn lemma, is based on a one-step extension argument. While Bell has observed this in case of the axiom of choice, for subfunctions of a given relation, we now consider such extension patterns on arbitrary directed-complete partial orders. By postulating the existence of so-called total elements rather than maximal ones, we can single out an immediate consequence of the Kuratowski-Zorn lemma from which quite a few abstract extension theorems can be deduced more directly, apart from certain definitions by cases. Applications include Baer's criterion for a module to be injective. Last but not least, our general extension theorem is equivalent to a suitable form of the Kuratowski-Zorn lemma over constructive set theory.

Received 27 June 2017
Keywords and phrases: Extension theorems; Kuratowski-Zorn lemma; transfinite methods.
AMS subject classification: 03E25, 03F65.

1. Introduction

An invocation of the Kuratowski-Zorn lemma (KZL) [11, 19] often takes place within an indirect proof of a universal statement. Supposing towards a contradiction that there be any counterexample, the maximal counterexample provided by KZL helps—by what Bell calls "one-step extension argument" [5]—to the desired contradiction. Crucially though, this one-step argument does not depend on maximality, and in fact a more general method is hovering in the background, which a priori is not limited to hypothetical counterexamples only. An alternative approach thus seems desirable which at once is more affirmative inasmuch as it really focuses on the one-step argument, but still is in the spirit of KZL. To this end, we distill a general extension theorem (GET) for directed-complete partial orders, the intended meaning being that the poset under consideration consists of partial extensions of which one is to be proved total. The principal hypothesis of GET encodes the one-step argument which can also be found in proofs of specific extension theorems such as the ones going back to Hahn and Banach: that every partial extension can be extended by any potential element of its domain—which, of course, is impossible for any maximal extension. As compared with the typical indirect proof by KZL of such an extension theorem, GET allows for a fairly direct proof relative to a certain type of definition by cases. This is possible because GET already postulates the existence of a total extension rather than a maximal one.

In this note we proceed as follows. First, in Section 2, we explain the concept of an extension pattern on a partially ordered class, and we provide several elementary examples. Then, in Section 3, we phrase our general extension principle GET and prove it equivalent over constructive set theory **CZF** both to a certain variant of the Hausdorff Maximal Principle as well as to a form of KZL which is suitable for **CZF**. In Section 4 we focus on a specific application from module theory, obtaining a proof of Baer's criterion by means of GET. Finally, we obtain from GET a classically equivalent induction principle in Section 5.

On method and foundations

We work in constructive set theory **CZF** [3, 4] which is based on intuitionistic logic. Due to the choice of this setting, sometimes certain assumptions have to be made explicit which otherwise would be trivial in classical set theory **ZF**. For instance, a set S is *discrete* if

$$\forall x, y \in S\,(\, x = y \vee x \neq y\,).$$

A subset T of a set S is *detachable* if

$$\forall x \in S\,(\, x \in T \vee x \notin T\,).$$

These are instances of the *restricted principle of excluded middle* (REM)

$$\varphi \vee \neg\varphi$$

where φ is a *bounded* formula, i.e., one in which all quantifiers occur only in one of the forms $\exists x \in y$ or $\forall x \in y$. We make use of class notation and terminology [4], notably when it comes to phrasing KZL over **CZF**. A class is said to be *predicative* [4] if it can be defined by a bounded formula. Every class occurring in this note will be predicative, and we denote classes by script letters. Given a class $\mathcal{E}$, a *partial order* of $\mathcal{E}$ is a subclass $\leqslant$ of $\mathcal{E} \times \mathcal{E}$ that satisfies the usual axioms of a partial order: reflexivity, transitivity, antisymmetry. For instance, if S is a set, then the class $\mathcal{P}(S)$ of all subsets of S is partially ordered by the subset relation. Note that **CZF** does not postulate the axiom of power set! However, recall that **CZF** has *exponentiation*, i.e., if S and T are sets, then so is the class of all functions $f : S \to T$. Due to exponentiation, in most of the applications we have in mind the classes in question actually do form sets. Furthermore, if S is a set, then the class of all *finite* subsets of S forms a set as well. Here, a set S is *finite* if there is $n \geqslant 0$ and a surjective function $\{\,1, \ldots, n\,\} \to S$. In the following, whenever we write that a certain principle is "classically implied" by another, then we mean that REM is adopted in order to prove the implication in question. Last but not least, we point out that **CZF** with *unrestricted* excluded middle proves the same theorems as **ZF** does [4, Corollary 4.2.8].

2. Extension

Typically, when attempting to prove an extension theorem, e.g., in case of the Hahn-Banach theorem, an application of KZL takes the following form. Given a map $f : Y \to Z$ and $Y \subseteq X$, we are asked for an extension of f, i.e., another map $g : X \to Z$ for which $f(x) = g(x)$ for all $x \in Y$. When structural properties need to be preserved, this rarely is a trivial task, but a way out is provided by KZL. So one considers the collection E of intermediate extensions of f, and proceeds by showing E to be closed under unions of directed subsets. Due to KZL, the directed-complete partially ordered set E then has a maximal element, and a "one-step extension argument" [5] helps to the desired conclusion that the maximal element indeed has domain X. This one-step argument is captured by what Bell calls the *extension principle* for a family E of partial functions on a set X:

$$\forall x \in X \; \forall f \in E \; \exists g \in E \; (\; f \subseteq g \;\wedge\; x \in \mathrm{dom}(g) \;).$$

Under this extension principle, if f is a maximal element of E, then in fact $\mathrm{dom}(f) = X$, which is to say that f is total.

Example 2.1. Let R be a relation with $\mathrm{dom}(R) = X$, and let E be the set of subfunctions of R. If the domain $\mathrm{dom}(f)$ is a detachable subset of X whenever $f \in E$, then a simple definition by cases allows for an extension principle for E [5]. For let $f \in E$ and $x \in X$. If $x \in \mathrm{dom}(f)$, then an extension g of f by x is trivially given by $g = f$; and in case of $x \notin \mathrm{dom}(f)$ we may set $g = f \cup \{\, (x, y) \,\}$ for any y such that $(x, y) \in R$. On the other hand, KZL applies to E and thus gives a subfunction of R with the same domain. This is how AC (appropriately formulated and with classical reasoning) may be deduced from KZL with the aid of an extension principle.

If EP denotes the assertion that for every relation R there is an extension principle for the set of subfunctions of R, then KZL+EP entails AC in **CZF**. As is well known [3, 4, 8, 9], AC implies REM. While KZL is "constructively neutral" [5], it is EP which implies REM [5].

The situation one encounters may change, but the overall strategy with KZL remains quite the same, if only there is a one-step argument at hand.[1]

[1] While to our knowledge a one-step argument was made explicit first by Szpilrajn [17], more recent explicit occurrences include [12], of course on top of [5].

We are thus led to rephrase Bell's extension principle in a somewhat more general fashion.

Definition 2.2. Let $(\mathcal{E}, \leqslant)$ be a partially ordered class. An *extension pattern* $(\mathcal{X}, \Vdash)$ on $\mathcal{E}$ is given by a class $\mathcal{X}$ together with a class relation $\Vdash \subseteq \mathcal{X} \times \mathcal{E}$ satisfying the *extension property*

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, \exists e' \in \mathcal{E} \left(e \leqslant e' \ \wedge \ x \Vdash e' \right).$$

An element e of $\mathcal{E}$ is said to be *total* if

$$\forall x \in \mathcal{X} \, (x \Vdash e).$$

We use *extension data* as a name for the elements of $\mathcal{X}$.

The intended meaning of an extension pattern is best explained in analogy with Bell's principle for a family of functions. Where the latter keeps track of the domain of a function, general extension data $x \in \mathcal{X}$ are related to elements $e \in \mathcal{E}$ in a similar manner but by means of an arbitrary relation $\Vdash$.[2]

Lemma 2.3. *Let $\mathcal{E}$ be a partially ordered class. If $e \in \mathcal{E}$ is maximal, then e is total for every extension pattern $(\mathcal{X}, \Vdash)$ on $\mathcal{E}$.*

Proof. Let $e \in \mathcal{E}$ be maximal and let $x \in \mathcal{X}$. Then there is $e' \in \mathcal{E}$ with $e \leqslant e'$ and $x \Vdash e'$. Since e is maximal, we actually have $e = e'$, hence $x \Vdash e$. Therefore $x \Vdash e$ for all $x \in \mathcal{X}$ which means that e is total. □

Example 2.4. Let E be a set of partial functions on a set X. This E is a poset, naturally ordered by inclusion. Consider every element x of X as extension data and define $\Vdash$ by

$$x \Vdash e \quad \text{iff} \quad x \in \text{dom}(e).$$

Bell's extension principle says that this is an extension pattern. However, for this to go through constructively, in general we need $\text{dom}(e)$ to be a detachable subset of X for every $e \in E$. In fact, this is the prime example of an extension pattern, and once encounters it under various circumstances once one sets out to capture specific one-step principles in terms of extension patterns.

[2] Our choice of notation follows the one for Sambin's *Basic Pairs* [15] by which is meant a relation $\Vdash$ between sets X and S.

Example 2.5 ("Trivial Pattern")**.** Let $\mathcal{E}$ be a partially ordered class. By stipulating $\mathcal{X} = \emptyset$ we get an extension pattern, all elements $e \in \mathcal{E}$ for which trivially are total. In particular, if $\mathcal{E}$ is a partially ordered class with extension pattern, then the collection of all total elements need not form a set. Furthermore, total elements need not be maximal.

Example 2.6 ("Maximal Pattern")**.** Let $\mathcal{E}$ be a partially ordered class with decidable partial order. This $\mathcal{E}$ works as a class of extension data for itself by way of a definition by cases:

$$x \Vdash e \quad \text{iff} \quad (x \geqslant e \to x = e) \qquad \text{and} \qquad e' = \begin{cases} x & \text{if } x \geqslant e \\ e & \text{otherwise} \end{cases}$$

for all $x, e \in \mathcal{E}$. We then indeed have the extension property

$$e \leqslant e' \quad \text{and} \quad x \Vdash e'.$$

In fact, if $x \geqslant e$, then $e' = x$ and thus $x \Vdash e'$; if $x \ngeqslant e$, then $e' = e$ and thus again $x \Vdash e'$. With respect to the relation $\Vdash$, the total elements for this pattern are precisely the maximal ones.

Example 2.7. Recall that a partially ordered class $\mathcal{E}$ is said to be *directed* if every pair of elements has an upper bound, i.e.,

$$\forall x \in \mathcal{E}\, \forall y \in \mathcal{E}\, \exists e \in \mathcal{E}\, \big(x \leqslant e \wedge y \leqslant e \big).$$

Consider this as extension property for the pattern on $\mathcal{E}$ which is defined by again taking $\mathcal{E}$ to be a class of extension data for itself and stipulating

$$x \Vdash e \quad \text{iff} \quad x \leqslant e.$$

An element e of $\mathcal{E}$ is total for this pattern if and only if e is the greatest element of $\mathcal{E}$.

3. Equivalence

Let $(\mathcal{E}, \leqslant)$ be a partially ordered class. From now on, by a *directed subset* in $\mathcal{E}$ we understand an inhabited subset D of $\mathcal{E}$ such that every pair of elements of D has an upper bound in D. We say that $\mathcal{E}$ is a *directed-complete* partially ordered class, for short a *dcpo*, if $\mathcal{E}$ is such that every directed subset D of $\mathcal{E}$ has a *least upper bound* $\sup D \in \mathcal{E}$ [4, 2]. Most dcpo's under consideration in this context are made of certain subsets of a fixed set, ordered by inclusion, for which suprema of directed families simply are unions. A dcpo $\mathcal{E}$ is said to be *set-generated* if there is a subset G of $\mathcal{E}$ such that, for every $e \in \mathcal{E}$,

$$G_e = \{\, g \in G : g \leqslant e \,\}$$

is a directed set with

$$\sup G_e = e.$$

Remark 3.1. A directed-complete partially ordered *set* E is set-generated, of course: take $G = E$. Conversely, if we admit the axiom of power set, if G is a generating set for a (class) dcpo $\mathcal{E}$, then the class $\mathcal{P}(G)$ of all subsets of G is a set, and so is the class $\mathcal{D}(G)$ of all directed subsets of G. Then, still with the power set axiom, being the surjective image of a function

$$\mathcal{D}(G) \to \mathcal{E}, \quad D \mapsto \sup D,$$

we see that a set-generated dcpo $\mathcal{E}$ is a set.

One of the suitable forms of KZL over constructive set theory reads as follows [1].

KZL Every inhabited set-generated dcpo has a maximal element.

An extension pattern is hidden in many an indirect proof of an extension theorem by KZL, which provides a maximal element that in fact proves total. With our *general extension theorem* we extract the essence of this method.

GET Every inhabited set-generated dcpo with extension pattern has a total element.

Proposition 3.2. KZL *implies* GET.

Proof. Lemma 2.3. □

By contrast, total elements for an extension pattern need not be maximal, as was seen above with the trivial pattern in Example 2.5. The notions of totality and maximality hence do not necessarily coincide—unlike maximality, totality may depend on the pattern.

More often than not, there is an explicit method available, the application of which provides an extension e' of a given element e of $\mathcal{E}$ by arbitrary extension data $x \in \mathcal{X}$. We denote this extension e' by $f(x,e)$; indeed, we then have a class function

$$f : \mathcal{X} \times \mathcal{E} \to \mathcal{E}$$

satisfying the condition

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, \big(\, e \leqslant f(x,e) \,\wedge\, x \Vdash f(x,e) \,\big).$$

We say that an extension pattern as such is *functional*. Whether or not a pattern is functional solely depends on how extension data relate to elements $e \in \mathcal{E}$, i.e., f is not to be considered as an addendum to the definition of extension pattern, even though we could have demanded it in the first place.[3] It rather is a requirement on how X and $\Vdash$ capture the one-step argument.

fGET Every inhabited set-generated dcpo with functional extension pattern has a total element.

Clearly, GET implies fGET. It is straightforward to show that these principles are classically equivalent. In fact, by means of fGET, every set-generated dcpo has a total element for the maximal pattern from Example 2.6, which pattern indeed is functional, provided the partial order is decidable. The total element in question then is a maximal one.

As it turns out, with a more refined argument we are able to show that GET and fGET are equivalent even over constructive set theory. To this end, we make use of the following variant of the Hausdorff Maximal Principle for directed (rather than linearly ordered) subsets of a partially ordered set.

[3] It is interesting to note that fGET applies such as to convert every extension pattern into a functional one. However, in order to code additional information into extension data requires a certain definition by cases. Below we give an argument that shows GET and fGET constructively equivalent.

MDP Every inhabited partially ordered set has a maximal directed subset.

Proposition 3.3. fGET *implies* MDP.

Proof. Let $(E, \leqslant)$ be an inhabited partially ordered set. Consider the partially ordered class $\mathcal{E}$ of all directed subsets of E, ordered by inclusion (notice that this $\mathcal{E}$ refers to a bounded formula). We claim that $\mathcal{E}$ is an inhabited set-generated dcpo. A generating set for $\mathcal{E}$ is given by the set G of all *finite* directed subsets of E. Indeed, if $D \in \mathcal{E}$, then

$$\{ D_0 \in G : D_0 \subseteq D \}$$

is a set and it is directed: if D_0 and D_1 are finite directed subsets of D, then (being finite) they have a greatest element $x \in D_0$ and $y \in D_1$, respectively. Then, since D is directed, there is $z \in D$ such that $x, y \leqslant z$ and we have $D_0 \cup D_1 \cup \{ z \} \in G$ as well as $D_0 \cup D_1 \cup \{ z \} \subseteq D$. Moreover, $D = \bigcup \{ D_0 \in G : D_0 \subseteq D \}$, since G contains every singleton subset of E, in particular. Next, if $\mathcal{D}$ is a directed subset of $\mathcal{E}$, then $\bigcup \mathcal{D} \in \mathcal{E}$, whence $\mathcal{E}$ is directed-complete.

Now we describe an extension pattern on $\mathcal{E}$ with corresponding relation being the one from Example 2.6. For the class of extension data we take $\mathcal{E}$ itself and stipulate, for $C, D \in \mathcal{E}$,

$$C \Vdash D \quad \equiv \quad (C \supseteq D \rightarrow C = D).$$

Furthermore, there is a class function

$$f : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{P}(E), \quad f(C, D) = D \cup \{ z \in C : C \supseteq D \} .$$

In order to have an extension pattern, we need to verify $\mathrm{ran}(f) \subseteq \mathcal{E}$ as well as that for all $C, D \in \mathcal{E}$ we have

$$D \subseteq f(C, D) \qquad \text{and} \qquad C \Vdash f(C, D).$$

As regards the range of f, we need to show that $f(C, D)$ is directed. If $x, y \in f(C, D)$, then we can distinguish several cases. We may have both $x, y \in D$ in which case nothing needs to be checked, since D is directed. But if, say, $x \in \{ z \in C : C \supseteq D \}$, then $C \supseteq D$. This implies $x, y \in C$, and C is directed. Next, by the very definition of f we have $D \subseteq f(C, D)$. So it remains to show $C \Vdash f(C, D)$. To this end, suppose that $C \supseteq f(C, D)$.

Since $f(C, D) \supseteq D$, we get $C \supseteq D$ and it follows from the definition of f that $f(C, D) = C$, as required. By way of fGET, there is an element $D \in \mathcal{E}$ which is total for the pattern just defined. This D is a maximal directed subset of E. □

Proposition 3.4. MDP *implies* KZL.

Proof. Let $\mathcal{E}$ be an inhabited set-generated dcpo and let G be a generating subset of $\mathcal{E}$. This G is inhabited since $\mathcal{E}$ is, and we restrict the partial order on $\mathcal{E}$ to G. According to MDP, there is a maximal directed subset D of G. Since $\mathcal{E}$ is a dcpo, this D has a least upper bound $\sup D \in \mathcal{E}$. We claim that $\sup D$ is a maximal element of $\mathcal{E}$. Indeed, if $e \in \mathcal{E}$ is such that $\sup D \leqslant e$, then we have an inclusion of sets

$$D \subseteq \{ g \in G : g \leqslant e \} .$$

This is because of $D \subseteq G$ and since for every $g \in D$ we have

$$g \leqslant \sup D \leqslant e.$$

Now take into account that $\{ g \in G : g \leqslant e \}$ is directed by our very assumption on G being a generating set for $\mathcal{E}$. Therefore, by maximality of D among directed subsets of G, we get

$$D = \{ g \in G : g \leqslant e \} ,$$

whence

$$\sup D = \sup \{ g \in G : g \leqslant e \} = e,$$

as required. □

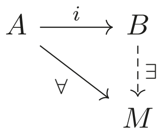
Remark 3.5. All the classes considered in this note are supposed to be predicative. This makes possible the above restriction of the order on $\mathcal{E}$ to the subset G by *bounded separation* [3, 4], and thus to obtain a partially ordered *set*.

Remark 3.6. Under the assumption of the axiom of power set—which would not have to be assumed, e.g., if classical logic were adopted due to the presence of exponentiation [3, 4]—all of the above proofs go through without further ado if set-generated (class) dcpo's are replaced by dcpo's which are sets. A similar remark applies to our treatment of induction principles in Section 5 below.

Corollary 3.7. KZL, GET, fGET *and* MDP *are equivalent over* **CZF**.

4. Application

We place ourselves in **CZF** + REM in order to make possible a certain argument by cases. Let R be a ring. In the following, ideals of R and R-modules are understood to be left ideals and left R-modules, respectively. Recall that an R-module M is *injective* if every R-homomorphism $A \to M$ can be extended along injective R-homomorphisms $i : A \to B$.

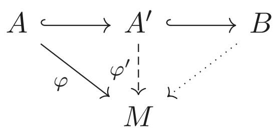


By means of Baer's criterion, injectivity of a module M can be tested by considering R-homomorphisms $I \to M$ on ideals I of R only.

Baer's Criterion. *Let R be a ring. An R-module M is injective already if every R-homomorphism $I \to M$, defined on an ideal I of R, extends onto R.*

Proposition 4.1. GET *implies Baer's criterion.*

Proof. Let M be an R-module with the property that every R-homomorphism $I \to M$, defined on an ideal I of R, extends onto R. Let $\varphi : A \to M$ and let $A \to B$ be an injective R-homomorphism; we assume that the latter is the inclusion mapping and A therefore is a submodule of B. As in the proof by Zorn's Lemma (see, e.g., [12, 18]), we consider the set E of partial extensions of φ, ordered by inclusion.



Of course, E is readily shown to be directed-complete. The one-step extension principle, which helps to show that a maximal intermediary extension of φ is total, now encodes in an extension pattern for GET as follows. We have a set X of extension data an element of which is a triple

$$(\psi, x, \nu) \in E \times B \times \mathrm{Hom}_R(R, M),$$

subject to the condition

$$\nu\big|_{I(\psi,x)} = \nu(\psi, x), \tag{$*$}$$

where

$$I(\psi, x) = (\mathrm{dom}(\psi) : x) = \{\, r \in R : rx \in \mathrm{dom}(\psi) \,\}$$

and

$$\nu(\psi, x) : I(\psi, x) \to M, \quad r \mapsto \psi(rx).$$

The relation $\Vdash \subseteq X \times E$ is then defined by

$$(\psi, x, \nu) \Vdash \psi' \qquad \text{iff} \qquad \psi \neq \psi' \quad \text{or} \quad x \in \mathrm{dom}(\psi').$$

In case of $(\psi, x, \nu) \nVdash \psi'$, we have $\psi = \psi'$ and $x \notin \mathrm{dom}(\psi')$, and the one-step extension χ of ψ' by (ψ, x, ν) can be constructed as follows (e.g., [18]):

$$\chi : \mathrm{dom}(\psi') + Rx \to M, \quad y + rx \mapsto \psi'(y) + \nu(r).$$

Then, if ψ is total for this pattern, we can directly verify that it is defined everywhere on the R-module B. For if $x \in B$, we merely need to consider the R-homomorphism $\nu(\psi, x)$ defined on the ideal $I(\psi, x)$. Due to the assumption on M, this $\nu(\psi, x)$ is extended by some $\nu \in \mathrm{Hom}_R(R, M)$. We then have $(\psi, x, \nu) \in X$ and $(\psi, x, \nu) \Vdash \psi$ by totality, from which we infer $x \in \mathrm{dom}(\psi)$. □

The extension pattern with which we have deduced Baer's criterion was defined so as to be functional. Alternatively, we could simply have set

$$X = B \qquad \text{and} \qquad x \Vdash \psi \quad \text{iff} \quad x \in \mathrm{dom}(\psi)$$

for all $x \in B$ and $\psi \in E$, with E as before. If we then had proceeded as before, we would have had to choose ν in order to perform the extension step. The above use of more complex data—here, triples (ψ, x, ν) rather than only elements x—offers a way around this inasmuch as the choice of ν is anticipated.

5. Induction

Back to **CZF**. A subclass $\mathcal{F}$ of a dcpo $\mathcal{E}$ is *closed* if the supremum of every directed subset of $\mathcal{F}$ again belongs to $\mathcal{F}$, in which case $\mathcal{F}$ is a dcpo itself.

If $\mathcal{E}$ has a functional extension pattern $(\mathcal{X}, \Vdash, f)$ and $\mathcal{F}$ moreover is such that $f(x,e) \in \mathcal{F}$ whenever $x \in \mathcal{X}$ and $e \in \mathcal{F}$, then the extension pattern restricts and $\mathcal{F}$ has a total element by fGET. The *relative* version of our extension theorem then reads as follows.

rfGET Let $\mathcal{E}$ be an inhabited set-generated dcpo with functional extension pattern $(\mathcal{X}, \Vdash, f)$ and let $\mathcal{F}$ be a closed subclass of $\mathcal{E}$ such that

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, (\, e \in \mathcal{F} \;\to\; f(x,e) \in \mathcal{F}\,). \tag{†}$$

If $\mathcal{F}$ is inhabited and set-generated, then $\mathcal{F}$ has a total element.

This is an equivalent form of fGET, of course, because every dcpo may be considered a closed subclass of itself.

Conversely, a subclass $\mathcal{O}$ of $\mathcal{E}$ is said to be *open* if it cannot contain the supremum of a directed subset D of $\mathcal{E}$ unless it meets this D in at least one element, i.e., if

$$\sup D \in \mathcal{O} \;\to\; D \between \mathcal{O}$$

for every directed subset D of $\mathcal{E}$. Here, we write $D \between \mathcal{O}$ if the intersection $D \cap \mathcal{O}$ is inhabited.[4] Classically speaking, it is easy to see that the notions of closed and open subset of a dcpo are complementary to each other. Dualising the relative version of GET results in a principle for induction on functional extension patterns.

ifGET Let $\mathcal{E}$ be a set-generated dcpo with functional extension pattern $(\mathcal{X}, \Vdash, f)$ and let $\mathcal{O}$ be an open subclass of $\mathcal{E}$ such that

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, (\, f(x,e) \in \mathcal{O} \;\to\; e \in \mathcal{O}\,). \tag{‡}$$

If $\mathcal{O}$ contains all total elements, then $\mathcal{E} = \mathcal{O}$.

Respective forms hold also for extension patterns $(\mathcal{X}, \Vdash)$ which need not be functional. In order to state them in an analogous way, let us write $\boldsymbol{f}(x,e)$ for the subclass of $\mathcal{E}$ consisting of all x*-extensions* of a given element $e \in \mathcal{E}$, i.e.,

$$\boldsymbol{f}(x,e) = \left\{\, e' \in \mathcal{E} : e \leqslant e' \wedge x \Vdash e' \,\right\}.$$

In case of a functional pattern, we have $f(x,e) \in \boldsymbol{f}(x,e)$, of course. But the latter moreover includes every total element which might be above e. The relative version of GET for closed subclasses is immediate.

[4] We have adopted this notation from Giovanni Sambin.

rGET Let $\mathcal{E}$ be an inhabited set-generated dcpo with extension pattern $(\mathcal{X}, \Vdash)$ and let $\mathcal{F}$ be a closed subclass of $\mathcal{E}$ such that

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, (\, e \in \mathcal{F} \;\to\; \boldsymbol{f}(x,e) \subseteq \mathcal{F}\,). \tag{$\dagger'$}$$

If $\mathcal{F}$ is inhabited and set-generated, then $\mathcal{F}$ has a total element.

iGET Let $\mathcal{E}$ be a set-generated dcpo with extension pattern $(\mathcal{X}, \Vdash)$ and let $\mathcal{O}$ be an open subclass of $\mathcal{E}$ such that

$$\forall x \in \mathcal{X}\, \forall e \in \mathcal{E}\, (\, \boldsymbol{f}(x,e) \between \mathcal{O} \;\to\; e \in \mathcal{O}\,). \tag{$\ddagger'$}$$

If $\mathcal{O}$ contains all total elements, then $\mathcal{E} = \mathcal{O}$.

We proceed by showing how these principles relate to each other.

Proposition 5.1.

1. iGET *implies* GET*.*

2. ifGET *implies* fGET*.*

3. rGET *classically implies* iGET*.*

4. rfGET *classically implies* ifGET*.*

Proof.

1. Given an extension pattern $(\mathcal{X}, \Vdash)$ on $\mathcal{E}$, we consider the subclass

$$\mathcal{O} = \big\{\, e \in \mathcal{E} : \exists e' \in \mathcal{E}\, (\, e \leqslant e' \wedge \forall x \in \mathcal{X}\, (\, x \Vdash e'\,)\,)\,\big\}$$

of *totally extendable* elements. This $\mathcal{O}$ contains all total elements. Since $\mathcal{O}$ is downwards monotone, i.e.,

$$\forall e, e' \in \mathcal{E}\, (\, e \leqslant e' \;\wedge\; e' \in \mathcal{O} \;\to\; e \in \mathcal{O}\,)$$

it is open (recall that directed subsets are to be inhabited) and it satisfies ($\ddagger'$). Thus $\mathcal{E} = \mathcal{O}$, whence every element of $\mathcal{E}$ is totally extendable.

2. Employ a similar argument as for the preceding item.

3. Given an extension pattern $(\mathcal{X}, \Vdash)$ on a set-generated dcpo $\mathcal{E}$, let $\mathcal{O}$ be an open subclass of $\mathcal{E}$ containing all total elements and such that $(\ddagger')$ holds. Working classically, suppose that there is $e \in \mathcal{E}$ such that $e \notin \mathcal{O}$. Consider $\mathcal{F} = \mathcal{E} - \mathcal{O}$. This $\mathcal{F}$ is an inhabited (set-generated, and even a set by Remark 3.1) closed subclass of $\mathcal{E}$ for which $(\dagger')$ holds. Therefore, $\mathcal{F}$ has a total element, however all of which should belong to $\mathcal{O}$.

4. Employ a similar argument as for the preceding item. □

We do not know whether there are constructive proofs for the above classical implications, nor if it can be shown that iGET implies ifGET over **CZF**. Anyway, keep in mind that GET and rGET, as well as fGET and rfGET, are mere reformulations of each other, respectively.

Along with Corollary 3.7 and Proposition 5.1, we see that all principles considered in this note are classically equivalent:

Corollary 5.2. KZL, GET, rGET, iGET, fGET, rfGET, ifGET *and* MDP *are equivalent over* **CZF** + REM.

Let us briefly compare our induction principles on extension patterns with Raoult's principle of *Open Induction* [14]. A subclass $\mathcal{P}$ of a partially ordered class $(\mathcal{E}, \leqslant)$ is said to be *progressive* if

$$\forall e \in \mathcal{E}\,(\,\forall e' \in \mathcal{E}\,(\,e' > e \rightarrow e' \in \mathcal{P}\,)\ \rightarrow\ e \in \mathcal{P}\,),$$

where $e' > e$ is understood to be the conjunction of $e \leqslant e'$ and $e \neq e'$. Here is a version of Open Induction for set-generated dcpo's.

OI Let $\mathcal{E}$ be a set-generated dcpo. If $\mathcal{P}$ is an open and progressive subclass of $\mathcal{E}$, then $\mathcal{P} = \mathcal{E}$.

For instance, let $(\mathcal{X}, \Vdash)$ be an extension pattern on $\mathcal{E}$, and let $\mathcal{O} \subseteq \mathcal{E}$ be open, contain every total element, and satisfy $(\ddagger')$. Suppose that totality is a decidable property on $\mathcal{E}$ insofar as that for every element $e \in \mathcal{E}$, either e is total or else there is certain data $x \in \mathcal{X}$ for which $x \nVdash e$. Now, if $e \in \mathcal{E}$ is such that $e' \in \mathcal{O}$ whenever $e' > e$, we also have $e \in O$. For either is e total, by which $e \in \mathcal{O}$ is immediate, or there is $x \in \mathcal{X}$ with $x \nVdash e$. In case of the latter, by extension there is $e' \in \mathcal{E}$ such that $e < e'$ and $x \Vdash e'$. It follows that $\boldsymbol{f}(x, e) \between \mathcal{O}$. As $\mathcal{O}$ is supposed to satisfy $(\ddagger')$, we get $e \in \mathcal{O}$. This shows how OI implies iGET under the above proviso that one can tell for each $e \in E$ whether e is total or has a witness to the opposite. Compare [10, 13, 16].

Acknowledgements

We thank the anonymous referee for valuable remarks that helped to improve the present note. We are further indebted to the anonymous referee of an earlier version of this paper, for pointing out to us a useful argument towards Proposition 3.3.

The research that has led to this paper was carried out within the projects “Abstract Mathematics for Actual Computation: Hilbert’s Program in the 21st Century” (ID 48138) and “A New Dawn of Intuitionism: Mathematical and Philosophical Advances” (ID 60842) funded by the John Templeton Foundation[5]; the project “Categorical localisation: methods and foundations” (CATLOC) funded by the University of Verona within the programme “Ricerca di Base 2015”; as well as within the project “Dipartimenti di Eccellenza 2018-2022” of the Italian Ministry of Education, Universities and Research (MIUR). Further research was undertaken when Schuster was visiting the Munich Center for Mathematical Philosophy upon kind invitation by Hannes Leitgeb and with a research fellowship “Erneuter Aufenthalt” by the Alexander-von-Humboldt Foundation. The final version of this note was prepared when both authors were visiting the Hausdorff Research Institute for Mathematics during the Trimester Program “Types, Sets and Constructions”.

References

[1] P. Aczel, Zorn’s Lemma in CZF, unpublished, 2002.

[2] P. Aczel, Aspects of general topology in constructive set theory, *Annals of Pure and Applied Logic* **137**:1–3 (2006), 3–29.

[3] P. Aczel and M. Rathjen, Notes on constructive set theory. Technical report, Institut Mittag-Leffler, 2000/01. Report No. 40.

[5] The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

[4] P. Aczel and M. Rathjen, Constructive set theory, book draft, 2010.

[5] J.L. Bell, Zorn's lemma and complete Boolean algebras in intuitionistic type theories, *The Journal of Symbolic Logic* **62**:4 (1997), 1265–1279.

[6] U. Berger, A computational interpretation of open induction, In: F. Titsworth, editor, Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, IEEE Computer Society, 2004, pp. 326–334.

[7] T. Coquand, Constructive topology and combinatorics, In: Constructivity in Computer Science (Summer Symposium San Antonio, TX, June 19–22, 1991 Proceeding), vol. 613 of Lecture Notes in Computer Science, Springer, Berlin and Heidelberg, 1992, pp. 159–164.

[8] R. Diaconescu, Axiom of choice and complementation, *Proceedings of the American Mathematical Society* **51**:1 (1975), 176–178.

[9] N.D. Goodman and J. Myhill, Choice implies excluded middle, *Mathematical Logic Quarterly* **24**:461 (1978), 25–30.

[10] M. Hendtlass and P. Schuster, A direct proof of Wiener's theorem, In: S.B. Cooper, A. Dawar and B. Löwe, editors, How the World Computes. Turing Centenary Conference and Eighth Conference on Computability in Europe, vol. 7318 of Lecture Notes in Computer Science, Springer, Berlin and Heidelberg, 2012. Proceedings, CiE 2012, Cambridge, UK, June 2012, pp. 294–303,

[11] K. Kuratowski, Une méthode d'élimination des nombres transfinis des raisonnements mathématiques, *Fundamenta Mathematicae* **3**:1 (1922), 76–108.

[12] D.G. Northcott, An Introduction to Homological Algebra, Cambridge University Press, 1960.

[13] H. Perdry, Strongly Noetherian rings and constructive ideal theory, *Journal of Symbolic Computation* **37**:4 (2004), 511–535.

[14] J.-C. Raoult, Proving open properties by induction, *Information Processing Letters* **29**:1 (1988), 19–23.

[15] G. Sambin, Some points in formal topology, *Theoretical Computer Science* **305**:1-3 (2003), 347–408.

[16] P. Schuster, Induction in algebra: a first case study, In: 27th Annual ACM/IEEE Symposium on Logic in Computer Science, IEEE Computer Society Publications, 2012. Proceedings, LICS 2012, Dubrovnik, Croatia, pp. 581–585.

[17] E. Szpilrajn, Sur l'extension de l'ordre partiel, *Fundamenta Mathematicae* **16**:1 (1930), 386–389.

[18] P. Vámos and D.W. Sharpe, Injective modules, Cambridge Tracts in Mathematics and Mathematical Physics 62, Cambridge University Press, 1972.

[19] M. Zorn, A remark on method in transfinite algebra, *Bulletin of the American Mathematical Society* **41**:10 (1935), 667–670.

Dipartimento di Informatica
Università degli Studi di Verona
Strada le Grazie 15, 37134 Verona, Italy
peter.schuster@univr.it
daniel.wessel@univr.it