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ON PCF SPACES WHICH ARE NOT FRÉCHET-URYSOHN

A b s t r a c t. By means of a forcing argument, it was shown by Pereira that if CH holds then there is a separable PCF space of height $\omega_1 + 1$ which is not Fréchet-Urysohn. In this paper, we give a direct proof of Pereira's theorem by means of a forcing-free argument, and we extend his result to PCF spaces of any height $\delta + 1$ where $\delta < \omega_2$ with $\mathrm{cf}(\delta) = \omega_1$.

1. Introduction

An important series of results getting cardinal bounds on the behaviour of the power function at singular cardinals was obtained by Shelah in the late 1980s by studying the reduced products of cardinals below the concerned

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singular cardinal. This led to the so called theory of possible cofinalities, which contains many important and unexpected results in cardinal arithmetic and which also found interesting applications in algebra and set-theoretic topology (see [1],[5] and [11]).

If A is a set of regular cardinals, $\mathrm{PCF}(A)$ is defined to be

$$\{\mathrm{cf}(\Pi A/D) : D \text{ is an ultrafilter on } A\}.$$

It is well-known that, assuming A is an interval of regular cardinals that is *progressive*, i.e. it satisfies that $|A| < \min(A)$, the PCF operator is a closure operation on subsets of $\mathrm{PCF}(A)$, and hence we have a natural topology associated with it, by letting $C \subseteq \mathrm{PCF}(A)$ be closed iff $\mathrm{PCF}(C) = C$. The resulting topological space is compact, Hausdorff and scattered (and hence 0-dimensional). It is known that, under Stone duality, the notion of a compact, Hausdorff, scattered space corresponds to the notion of a superatomic Boolean algebra (i.e. a Boolean algebra in which every subalgebra is atomic).

By an *LCS space* we mean a locally compact, Hausdorff and scattered space. Recall that for an LCS space X and an ordinal α, the *αth Cantor-Bendixson level* of X is defined by $I_\alpha(X)$ = the set of isolated points of $X \setminus \bigcup\{I_\beta(X) : \beta < \alpha\}$. We define the *height of* X as $\mathrm{ht}(X)$ = the least ordinal α such that $I_\alpha(X) = \emptyset$. We refer the reader to the survey papers [2] and [8] for a wide list of results on the existence of various types of LCS spaces.

The following notion, which permits us to construct in a direct way LCS spaces from partial orders, is a useful tool in the study of the PCF operator.

Definition 1.1. Assume that $T = \bigcup\{T^{(\alpha)} : \alpha < \eta\}$ for some non-zero ordinal η where each $T^{(\alpha)}$ is a non-empty set and $T^{(\alpha)} \cap T^{(\beta)} = \emptyset$ for $\alpha < \beta < \eta$. Assume that for every $x \in T$, b_x is a subset of T such that the following conditions hold:

1. If $x \in T^{(\gamma)}$, then $b_x \cap \bigcup\{T^{(\zeta)} : \gamma \leq \zeta < \eta\} = \{x\}$ and $b_x \cap T^{(\zeta)}$ is infinite for every $\zeta < \gamma$.

2. If $x \in b_y$ then $b_x \subseteq b_y$.

3. If $x, y \in T$, there are finitely many elements $z_1, \ldots, z_n \in T$ such that $b_x \cap b_y = b_{z_1} \cup \cdots \cup b_{z_n}$.

For $x, y \in T$, we put $x \preceq y$ iff $x \in b_y$. Clearly, $\preceq$ is a partial order on T. Then, we will say that $\mathcal{T} = (T, \preceq)$ is an *LCS poset*, whose *associated space* $X = X(\mathcal{T})$ is defined as follows. The underlying set of $X(\mathcal{T})$ is T. And for every $x \in T$, we define a basic neighbourhood of x in X as a set of the form $b_x \setminus (b_{x_1} \cup \cdots \cup b_{x_n})$ where $n < \omega$ and $x_1, \ldots, x_n \prec x$. Clearly, b_x is a compact neighbourhood of x for every $x \in X$. Then, we have that X is a locally compact, Hausdorff and scattered space such that $\mathrm{ht}(X) = \eta$ and $I_\alpha(X) = T^{(\alpha)}$ for every $\alpha < \eta$. If Y is a subset of T, we denote by $\overline{Y}$ the closure of Y in X. Also, we write $\mathrm{ht}(\mathcal{T}) = \mathrm{ht}(X)$ and $I_\alpha(\mathcal{T}) = I_\alpha(X)$ for every ordinal α.

We want to remark that our definition of an LCS poset is equivalent to the definition of an SBA ordering given in [7] and to the definition of a Bonnet partial order given in [9]. However, our definition will be more adequate to carry out the constructions of the desired spaces.

Now, we introduce the refinement of the notion of an LCS poset due to Magidor and Foreman, in which some conditions are added in order to reflect the fundamental properties of the PCF operator on $\{\aleph_n : n \geq 1\}$.

Definition 1.2. A *PCF structure* is an LCS poset $(\theta + 1, \preceq)$ where θ is an infinite ordinal such that the following conditions are satisfied:

(PCF1) If $\nu \prec \mu$ then $\nu \in \mu$.

(PCF2) $\overline{\omega} = \theta + 1$.

(PCF3) If $I \subseteq \theta + 1$ is an interval, then $\overline{I}$ is also an interval.

(PCF4) $\xi \prec \theta$ for every $\xi \in \theta$.

(PCF5) For each $\nu \in \theta$ of uncountable cofinality there is a club C_ν ν such that $\overline{C_\nu} \subseteq \nu + 1$.

We say that a space X is a *PCF space*, if there is a PCF structure $\mathcal{T}$ such that $X = X(\mathcal{T})$. Note that, by condition (PCF4), every PCF space is compact. Also, we have that the Boolean algebra associated with a PCF space is a well-generated Boolean algebra in the sense defined in [4].

It is known that in ZFC there is no PCF structure of size $\geq \omega_4$, and that this result implies Shelah's remarkable theorem that $2^{\aleph_\omega} < \aleph_{\omega_4}$ if $\aleph_\omega$ is a strong limit (see [5] and [11]). On the other hand, it was shown in [6] that for every ordinal $\eta < \omega_3$ it is relatively consistent with ZFC that there is a PCF space X such that $\mathrm{ht}(X) = \eta + 1$ and $|I_\alpha(X)| = \omega$ for each $\alpha < \eta$.

Recall that if X is a space and $Y \subseteq X$, the *sequential closure* of Y in X is defined as $\lim(Y) = \{x \in X :$ there is a sequence $(y_n)_n$ contained in Y such that $\lim_n y_n = x\}$. We say that X is *sequential*, if for every $Y \subseteq X$ with $\lim(Y) = Y$ we have that Y is closed. It was shown by Todorcevic that every PCF space is sequential, and a generalization of this result was shown in [9, Section 2]. Also, recall that a space X is *Fréchet-Urysohn*, if for every $Y \subseteq X$ we have $\lim(Y) = \overline{Y}$. Clearly, every Fréchet-Urysohn space is sequential. By using a forcing argument based on constructions carried out previously in [3] and [10], it was shown by Pereira in [9, Theorem 2] that under CH there is a PCF space of height $\omega_1 + 1$ which is not Fréchet-Urysohn. This result gives a partial answer to the question on the possible sequential ranks of PCF spaces (see [7] and [9]).

Then, in this paper we will give a simpler and more direct proof of Pereira's theorem by means of a forcing-free argument and we will extend his result from PCF spaces of height $\omega_1 + 1$ to PCF spaces of any height $\delta + 1$ where $\delta < \omega_2$ with $\mathrm{cf}(\delta) = \omega_1$. This last result is in a certain sense best possible, because it is known that under CH there is no PCF space of height $\omega_2 + 1$ (see [2]). However, we do not know whether it is relatively consistent with ZFC that there is a PCF space of height $\omega_2 + 1$ which is not Fréchet-Urysohn.

2. Proof of the theorems

First, we prove in a direct way the result shown in [9, Theorem 2].

Theorem 2.1. *If CH holds, there is a PCF space of height* $\omega_1 + 1$ *which is not Fréchet-Urysohn.*

Proof. As in [9], we will construct a PCF structure $\mathcal{T} = (\omega_1 + 1, \preceq)$ of height $\omega_1 + 1$ such that in $X(\mathcal{T})$ no sequence contained in ω converges to ω_1. Therefore, we will have that $\omega_1 \in \overline{\omega}$ but $\omega_1 \notin \lim(\omega)$, and thus $X(\mathcal{T})$ is not Fréchet-Urysohn. We fix an enumeration $\{d_\beta : 0 < \beta < \omega_1\}$ of the infinite subsets of ω. We put $I_\alpha = \{\omega \cdot \alpha + n : n < \omega\}$ for every $\alpha < \omega_1$. We write $S_\alpha = \bigcup\{I_\beta : \beta \leq \alpha\}$, and we write $T = \omega_1 + 1$. For every $x \in \omega_1$, we put $\pi(x) = \alpha$ if $x \in I_\alpha$.

The required PCF structure will be a partial order obtained from sets $b_x \subseteq T$ for $x \in T$. So, proceeding by transfinite induction on α, we construct for every $\alpha < \omega_1$ and every $x \in I_\alpha$ a subset b_x of S_α such that the following conditions hold:

(1) $b_x \cap I_\alpha = \{x\}$ and $b_x \cap I_\beta$ is infinite for every $\beta < \alpha$.

(2) If $x \in b_y$ then $b_x \subseteq b_y$.

(3) If $x, y \in S_\alpha$, there are finitely many elements $z_1, \ldots, z_n \in S_\alpha$ such that $b_x \cap b_y = b_{z_1} \cup \cdots \cup b_{z_n}$.

(4) If $x, y \in I_\beta$ for some $\beta \leq \alpha$ with $x \neq y$, then $b_x \cap b_y$ is a finite subset of ω.

(5) If $z \in I_\gamma$ and $\gamma \leq \beta \leq \alpha$, then $\{y \in I_\beta : b_y \cap b_z \neq \emptyset\}$ is finite.

(6) For every $0 < \beta \leq \alpha$ there is an element $x \in S_\alpha$ such that $b_x \cap d_\beta$ is infinite.

From conditions (1) and (5) we deduce that for every $x_1, \ldots, x_n \in S_\alpha$, $\omega \setminus (b_{x_1} \cup \cdots \cup b_{x_n})$ is an infinite set. For this, note that if $\alpha = 0$ we are done by condition (1). So, assume that $\alpha > 0$. First, we infer from condition (5) that there is an element $z \in I_\alpha$ such that $b_z \cap (b_{x_1} \cup \cdots \cup b_{x_n}) = \emptyset$. And now, by condition (1), we have that $b_z \cap \omega = b_z \cap I_0$ is infinite, and so $\omega \setminus (b_{x_1} \cup \cdots \cup b_{x_n})$ is infinite.

We put $b_x = \{x\}$ for every $x \in \omega$. Now, assume that $0 < \alpha < \omega_1$ and b_x has been defined for every $x \in \bigcup\{I_\beta : \beta < \alpha\}$. Our aim is to define the sets b_y for $y \in I_\alpha$. We put $Z = \bigcup\{I_\beta : \beta < \alpha\}$. Without loss of generality, we may assume that there is no element $x \in Z$ such that $b_x \cap d_\alpha$ is infinite. Let $\{x_m : m < \omega\}$ be an enumeration of Z. First, we assume that α is a limit ordinal. Let $\{\alpha_n : n < \omega\}$ be a strictly increasing sequence of ordinals converging to α. We construct an infinite subset $U = \{u_n : n < \omega\}$ of Z and an infinite subset $V = \{v_n : n < \omega\}$ of U such that the following conditions hold:

(a) $\bigcup\{b_{u_n} : n < \omega\} = Z$,

(b) if $u_n \in V$ then $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \emptyset$,

(c) if $u_n \in V$ then $\pi(u_n) > \alpha_n$.

Assume that $n \geq 0$ and we have picked the elements $u_0, \ldots, u_{n-1}$. If $n = 2k$ for some $k \geq 0$, we define u_n as the first element u in the enumeration $\{x_m : m < \omega\}$ such that $u \notin \bigcup\{b_{u_m} : m < n\}$. Now, suppose that $n = 2k+1$

for some $k \geq 0$. By condition (5), there is an element $u_n \in Z$ such that $\pi(u_n) > \max\{\alpha_n, \pi(u_0), \ldots, \pi(u_{n-1})\}$ and $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \emptyset$. Then, we define $U = \{u_n : n < \omega\}$ and $V = \{v_k : k < \omega\}$ where $v_k = u_{2k+1}$ for $k < \omega$. Clearly, conditions $(a) - (c)$ hold.

Put $y_n = \omega \cdot \alpha + n$ for $n < \omega$. First, assume that there are infinitely many elements v in V such that $b_v \cap d_\alpha \neq \emptyset$. Let a be an infinite subset of ω such that $b_{v_k} \cap d_\alpha \neq \emptyset$ for every $k \in a$ and $\omega \setminus a$ is infinite. Let $\{a_k : 0 < k < \omega\}$ be a partition of $\omega \setminus a$ into infinite subsets. Then, we define $b_{y_0} = \{y_0\} \cup \bigcup\{b_{v_n} : n \in a\}$, and for $k > 0$ we put $b_{y_k} = \{y_k\} \cup \bigcup\{b_{v_n} : n \in a_k\}$. And if $\{v \in V : b_v \cap d_\alpha \neq \emptyset\}$ is finite, we consider a partition $\{a_k : k < \omega\}$ of ω into infinite subsets and then we define $b_{y_0} = \{y_0\} \cup \bigcup\{b_{v_n} : n \in a_0\} \cup d_\alpha$ and for $0 < k < \omega$ we define $b_{y_k} = \{y_k\} \cup \bigcup\{b_{v_n} : n \in a_k\}$. We can check that conditions $(1) - (6)$ hold in both cases. For this, note that conditions (1), (2) and (6) are obvious, and conditions (3), (4) and (5) follow from conditions (a) and (b) and the assumption that there is no element $x \in Z$ such that $b_x \cap d_\alpha$ is infinite.

Now, assume that $\alpha = \gamma + 1$ is a successor ordinal. We construct an infinite subset $U = \{u_n : n < \omega\}$ of Z and an infinite subset $V = \{v_n : n < \omega\}$ of $U \cap I_\gamma$ such that the following holds:

(a) $\bigcup\{b_{u_n} : n < \omega\} = Z$,

(b) if $u_n \in V$ then $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \emptyset$.

If $n = 2k$ for some $k \geq 0$, we pick u_n as above. And if $n = 2k + 1$ for some $k \geq 0$, we apply condition (5) to find an element $u_n \in I_\gamma$ such that $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \emptyset$. Then, we put $U = \{u_n : n < \omega\}$ and $V = \{u_{2n+1} : n < \omega\}$. And now we define b_y for $y \in I_\alpha$ proceeding as in the preceding paragraph. So, conditions $(1) - (6)$ hold.

Finally, we put $b_{\omega_1} = \omega_1 + 1$. Let $\preceq$ be the partial order obtained from the sets b_x for $x \in \omega_1 + 1$. We have that $\mathcal{T} = (T, \preceq)$ is a PCF structure. Conditions (PCF1), (PCF2) and (PCF4) clearly hold, condition (PCF5) is also obvious because $T = \omega_1 + 1$, and condition (PCF3) follows from the fact that for every infinite interval I contained in $\omega_1 + 1$ there is an ordinal $\alpha < \omega_1$ such that $I_\alpha \setminus I$ is finite. Also, it is easy to see that $I_\alpha(\mathcal{T}) = I_\alpha$ for every $\alpha < \omega_1$ and $I_{\omega_1}(\mathcal{T}) = \{\omega_1\}$. And, by condition (6), we have that $\omega_1 \notin \lim(\omega)$. So, $(T, \preceq)$ is the required PCF structure. $\square$

Now, we extend Pereira's theorem to PCF spaces of height $< \omega_2$. So, our aim is to prove the following result.

Theorem 2.2. *If CH holds, then for every ordinal $\delta < \omega_2$ with cf(δ) = ω_1 there is a PCF space of height $\delta + 1$ which is not Fréchet-Urysohn.*

Proof. We assume that $\omega_1 < \delta < \omega_2$. We will construct a PCF structure $\mathcal{T} = (\delta + 1, \preceq)$ of height $\delta + 1$ such that in $X(\mathcal{T})$ no sequence contained in ω converges to δ. Therefore, we will have that $\delta \in \overline{\omega}$ but $\delta \notin \lim(\omega)$, and thus $X(\mathcal{T})$ is not Fréchet-Urysohn. We put $J_0 = \omega$ and $J_\alpha = \{\alpha + n : n < \omega\}$ for any limit ordinal $\alpha < \delta$. For every $x \in \delta$, we put $\pi(x) = \alpha$ if $x \in J_\alpha$.

Let $\{\delta_\xi : \xi < \omega_1\}$ be a strictly increasing sequence of ordinals cofinal in δ such that $\delta_0 = 0$ and δ_ξ is a limit for $0 < \xi < \omega_1$. By the construction carried out in the proof of Theorem 2.1, there is an LCS poset $\mathcal{T}' = (T', \preceq')$ satisfying the following conditions:

(i) $T' = \bigcup\{J_{\delta_\xi} : \xi < \omega_1\}$,

(ii) $\mathrm{ht}(\mathcal{T}') = \omega_1$, $I_\zeta(\mathcal{T}') = J_{\delta_\zeta}$ for every $\zeta < \omega_1$ and $I_{\omega_1}(\mathcal{T}') = \emptyset$,

(iii) $\preceq'$ is the partial order obtained from sets $b_x \subseteq T'$ for $x \in T'$, which satisfy conditions (1) – (6) in the proof of Theorem 2.1 replacing I_α with J_{δ_α} for $\alpha < \omega_1$.

Let $\{\alpha_\zeta : \zeta < \omega_1\}$ be an enumeration without repetitions of the limit ordinals of $\delta \setminus \{\delta_\xi : \xi < \omega_1\}$. In order to find the desired PCF structure of height $\delta + 1$, we construct by transfinite induction on $\xi < \omega_1$ an LCS poset $\mathcal{T}_\xi = (T_\xi, \preceq_\xi)$ such that the following conditions hold:

(1) $T_\xi = T' \cup \bigcup\{J_{\alpha_\mu} : \mu < \xi\}$.

(2) If $\langle \beta_\zeta : \zeta < \omega_1 \rangle$ is the strictly increasing enumeration of $\{\delta_\nu : \nu < \omega_1\} \cup \{\alpha_\nu : \nu < \xi\}$, then $I_\zeta(\mathcal{T}_\xi) = J_{\beta_\zeta}$ for every $\zeta < \omega_1$ and $I_{\omega_1}(\mathcal{T}_\xi) = \emptyset$.

Also, $\preceq_\xi$ will be the partial order obtained from sets $b_x^{(\xi)} \subseteq T_\xi$ for $x \in T_\xi$, which will be constructed satisfying the following conditions:

(3) If $x \in J_{\beta_\gamma}$, then $b_x^{(\xi)} \cap \bigcup\{J_{\beta_\zeta} : \gamma \leq \zeta < \omega_1\} = \{x\}$ and for each $\zeta < \gamma$ the set $b_x^{(\xi)} \cap J_{\beta_\zeta}$ is infinite.

(4) If $x \in b_y^{(\xi)}$ then $b_x^{(\xi)} \subseteq b_y^{(\xi)}$.

(5) If $\mu < \xi$ and $x \in T_\mu$, then $b_x^{(\mu)} \subseteq b_x^{(\xi)}$ and $b_x^{(\mu)} \cap J_{\beta_\nu} = b_x^{(\xi)} \cap J_{\beta_\nu}$ for each $\nu < \omega_1$ such that $J_{\beta_\nu} \subseteq T_\mu$.

(6) If $x, y \in T_\xi$, there are finitely many elements $z_1, \dots, z_n \in T_\xi$ such that $b_x^{(\xi)} \cap b_y^{(\xi)} = b_{z_1}^{(\xi)} \cup \dots \cup b_{z_n}^{(\xi)}$.

(7) If $x, y \in T_\xi$ with $x \neq y$ and $\pi(x) = \pi(y)$, then $b_x^{(\xi)} \cap b_y^{(\xi)}$ is a finite subset of ω.

(8) If $\mu < \xi$, $x, y \in T_\mu$ and $b_x^{(\mu)} \cap b_y^{(\mu)} = b_{z_1}^{(\mu)} \cup \dots \cup b_{z_n}^{(\mu)}$, then $b_x^{(\xi)} \cap b_y^{(\xi)} = b_{z_1}^{(\xi)} \cup \dots \cup b_{z_n}^{(\xi)}$.

(9) If $x \in J_{\beta_\gamma}$ and $\gamma \leq \zeta < \omega_1$, then $\{y \in J_{\beta_\zeta} : b_y^{(\xi)} \cap b_x^{(\xi)} \neq \emptyset\}$ is finite.

We define $T_0 = T'$ and $b_x^{(0)} = b_x$ for every $x \in T'$. Now, assume that $\xi = \mu + 1$ is a successor ordinal. We define $T_\xi = T_\mu \cup J_{\alpha_\mu}$. Let $\langle \beta_\zeta : \zeta < \omega_1 \rangle$ be the strictly increasing enumeration of $\{\delta_\nu : \nu < \omega_1\} \cup \{\alpha_\nu : \nu < \mu\}$. Let γ be the first ordinal ζ such that $\alpha_\mu < \beta_\zeta$. We put $x_k = \beta_\gamma + k$ for $k < \omega$. And we consider a partition $\{Y_k : k < \omega\}$ of J_{α_μ} into infinite subsets. First, we assume that γ is a successor ordinal $\eta + 1$. Our aim is to define the sets $b_x^{(\xi)}$ for $x \in T_\xi$. Fix $k < \omega$. We construct an infinite subset $U_k = \{u_n : n < \omega\}$ of $b_{x_k}^{(\mu)} \setminus \{x_k\}$ and an infinite subset $V_k = \{v_n : n < \omega\}$ of $U_k \cap J_{\beta_\eta}$ such that the following conditions hold:

(a) $\bigcup\{b_{u_n}^{(\mu)} : n < \omega\} = b_{x_k}^{(\mu)} \setminus \{x_k\}$,

(b) if $u_n \in V_k$ then $b_{u_n}^{(\mu)} \cap \bigcup\{b_{u_m}^{(\mu)} : m < n\} = \emptyset$.

For this, let $\{z_m : m < \omega\}$ be an enumeration of $b_{x_k}^{(\mu)} \setminus \{x_k\}$. Assume that $n \geq 0$ and we have picked the elements $u_0, \dots, u_{n-1}$. Suppose that $n = 2i$ for some $i \geq 0$. Note that the set $b_{x_k}^{(\mu)} \setminus \bigcup\{b_{u_m}^{(\mu)} : m < n\}$ is infinite by condition (3) for μ. Then, we define u_n as the first element u in the enumeration $\{z_m : m < \omega\}$ such that $u \notin \bigcup\{b_{u_m}^{(\mu)} : m < n\}$. Now, suppose that $n = 2i + 1$ for some $i \geq 0$. By condition (3) for μ, the set $b_{x_k}^{(\mu)} \cap J_{\beta_\eta}$ is infinite. So, by condition (9) for μ, there is an element $u_n \in b_{x_k}^{(\mu)} \cap J_{\beta_\eta}$ such that $b_{u_n}^{(\mu)} \cap \bigcup\{b_{u_m}^{(\mu)} : m < n\} = \emptyset$. Then, we define $U_k = \{u_n : n < \omega\}$ and $V_k = \{v_n : n < \omega\}$ where $v_n = u_{2n+1}$ for $n < \omega$. Clearly, conditions (a) and (b) hold. Now, let $\{y_n : n < \omega\}$ be an enumeration without repetitions of Y_k. And let $\{a_n : n < \omega\}$ be a partition of ω into infinite subsets. Then, we define $b_{y_n}^{(\xi)} = \{y_n\} \cup \bigcup\{b_{v_m}^{(\mu)} : m \in a_n\}$ for $n < \omega$. Also, if $x \in \bigcup\{J_{\beta_\zeta} : \zeta \geq \gamma\}$ we define $b_x^{(\xi)} = b_x^{(\mu)} \cup \bigcup\{Y_k : x_k \in b_x^{(\mu)}, k < \omega\}$. So, in particular we have $b_{x_k}^{(\xi)} = b_{x_k}^{(\mu)} \cup Y_k$ for every $k < \omega$. Finally, we put $b_x^{(\xi)} = b_x^{(\mu)}$ for every $x \in \bigcup\{J_{\beta_\zeta} : \zeta < \gamma\}$.

We can check that conditions $(1)-(9)$ hold for ξ. We prove conditions (8) and (9). The rest of the conditions are easier to verify. In order to check condition (8), suppose that $\mu < \xi$, $s, t \in T_\mu$ and $b_s^{(\mu)} \cap b_t^{(\mu)} = b_{z_1}^{(\mu)} \cup \cdots \cup b_{z_n}^{(\mu)}$. We assume that $s, t \in \bigcup\{J_{\beta_\zeta} : \zeta \geq \gamma\}$. Otherwise, the argument is simpler. First, we show that $b_{z_i}^{(\xi)} \subseteq b_s^{(\xi)} \cap b_t^{(\xi)}$ for $i \in \{1, \ldots, n\}$. If $z_i \in \bigcup\{J_{\beta_\zeta} : \zeta < \gamma\}$, we have $b_{z_i}^{(\xi)} = b_{z_i}^{(\mu)}$, and so we are done. Suppose that $z_i \in \bigcup\{J_{\beta_\zeta} : \zeta \geq \gamma\}$. Let $y \in b_{z_i}^{(\xi)}$. As $b_{z_i}^{(\xi)} \setminus b_{z_i}^{(\mu)} \subseteq J_{\alpha_\mu}$, we may assume that $y \in J_{\alpha_\mu}$. Hence, there is a $k \in \omega$ such that $y \in Y_k$. Since $y \in b_{z_i}^{(\xi)} \cap Y_k$, we infer that $x_k \in b_{z_i}^{(\mu)}$, thus $x_k \in b_s^{(\mu)} \cap b_t^{(\mu)}$, and so $y \in b_s^{(\xi)} \cap b_t^{(\xi)}$. Now, we prove that $b_s^{(\xi)} \cap b_t^{(\xi)} \subseteq b_{z_1}^{(\xi)} \cup \cdots \cup b_{z_n}^{(\xi)}$. So, assume that $y \in b_s^{(\xi)} \cap b_t^{(\xi)} \cap J_{\alpha_\mu}$. Let $k \in \omega$ be such that $y \in Y_k$. It follows that $x_k \in b_s^{(\mu)} \cap b_t^{(\mu)}$. Hence, there is an $i \in \{1, \ldots, n\}$ such that $x_k \in b_{z_i}^{(\mu)}$, and thus $y \in b_{z_i}^{(\xi)}$.

Now, in order to verify condition (9), we prove that if $\beta_\nu < \alpha_\mu$ and $s \in J_{\beta_\nu}$, then $\{y \in J_{\alpha_\mu} : b_y^{(\xi)} \cap b_s^{(\xi)} \neq \emptyset\}$ is finite. The rest of the cases are easier to verify. Note that since $\beta_\nu < \alpha_\mu$, we have $b_s^{(\xi)} = b_s^{(\mu)}$. By condition (9) for μ, we deduce that $Z = \{x \in J_{\beta_\gamma} : b_x^{(\mu)} \cap b_s^{(\xi)} \neq \emptyset\}$ is finite. Also, by condition (6) for μ, we have that for each $x_k \in Z$ there are finitely many elements $y_1, \ldots, y_n \in T_\mu$ such that $b_{x_k}^{(\mu)} \cap b_s^{(\xi)} = b_{y_1}^{(\mu)} \cup \cdots \cup b_{y_n}^{(\mu)}$. Then, by using conditions (a) and (b), we infer that for each $x_k \in Z$ the set $\{v \in V_k : b_v^{(\mu)} \cap b_s^{(\xi)} \neq \emptyset\}$ is finite. So, it follows that $\{y \in J_{\alpha_\mu} : b_y^{(\xi)} \cap b_s^{(\xi)} \neq \emptyset\}$ is finite too.

Also, if γ is a limit ordinal, we consider a strictly increasing sequence of ordinals $\{\gamma_n : n < \omega\}$ converging to γ and then, by means of an argument similar to the one given above, for every $k < \omega$ we can construct an infinite subset $U_k = \{u_n : n < \omega\}$ of $b_{x_k}^{(\mu)} \setminus \{x_k\}$ and an infinite subset $V_k = \{v_n : n < \omega\}$ of U_k such that the following conditions hold:

(a) $\bigcup\{b_{u_n}^{(\mu)} : n < \omega\} = b_{x_k}^{(\mu)} \setminus \{x_k\}$,

(b) if $u_n \in V_k$ then $b_{u_n}^{(\mu)} \cap \bigcup\{b_{u_m}^{(\mu)} : m < n\} = \emptyset$,

(c) if $u_n \in V_k$ then $\pi(u_n) > \max\{\beta_{\gamma_n}, \pi(u_0), \ldots, \pi(u_{n-1})\}$.

Since γ is the first ordinal ζ such that $\alpha_\mu < \beta_\zeta$, we have that $\sup\{\beta_{\gamma_n} : n < \omega\} \leq \alpha_\mu$. Thus, we can define the sets $b_x^{(\xi)}$ for $x \in T_\xi$ proceeding as above.

Next, assume that ξ is a limit ordinal. We define $T_\xi = \bigcup\{T_\mu : \mu < \xi\}$. Assume that $x \in T_\xi$. Let ζ be the least ordinal $\mu < \xi$ such that $x \in T_\mu$. Then, we define $b_x^{(\xi)} = \bigcup\{b_x^{(\eta)} : \zeta \leq \eta < \xi\}$. We can verify that conditions $(1)-(9)$ hold for ξ.

Note that $\delta = \bigcup\{T_\xi : \xi < \omega_1\}$. Then, in order to define the desired PCF structure $(T, \preceq)$, first we define the LCS poset $(T^*, \preceq^*)$ as follows. We put $T^* = \delta + 1$. Assume that $x \in \delta$. Let γ be the least ordinal $\xi < \omega_1$ such that $x \in T_\xi$. We define $b^*_x = \bigcup\{b^{(\xi)}_x : \gamma \leq \xi < \omega_1\}$. And we put $b^*_\delta = \delta + 1$. From conditions (4) and (5), we infer that $x \in b^*_y$ implies $b^*_x \subseteq b^*_y$ for every $x, y \in T^*$. And from conditions (5) and (8), we deduce that if $\xi < \omega_1$, $x, y \in T_\xi$ and $b^{(\xi)}_x \cap b^{(\xi)}_y = b^{(\xi)}_{z_1} \cup \cdots \cup b^{(\xi)}_{z_n}$, then $b^*_x \cap b^*_y = b^*_{z_1} \cup \cdots \cup b^*_{z_n}$. Now, if $x, y \in \delta+1$, we put $x \preceq^* y$ iff $x \in b^*_y$. Then, $\mathcal{T}^* = (T^*, \preceq^*)$ is an LCS poset on $\delta + 1$ with $\mathrm{ht}(\mathcal{T}^*) = \delta + 1$, $I_\alpha(\mathcal{T}^*) = J_{\omega\cdot\alpha}$ for $\alpha < \delta$ and $I_\delta(\mathcal{T}^*) = \{\delta\}$. And since $\delta \notin \lim(\omega)$, we have that $X(\mathcal{T}^*)$ is not Fréchet-Urysohn. Also, it is easy to see that $\mathcal{T}^*$ satisfies conditions PCF(1)$-$PCF(4). Then, in order to obtain the desired PCF structure $(T, \preceq)$ on $\delta+1$, we use an idea given by Ruyle in [10, Page 45] (see also [6, Lemma 2.9]). We put $T = \delta + 1$. Since $\delta < \omega_2$, we can easily construct a sequence $\langle C_\xi : \xi < \delta \text{ with } \mathrm{cf}(\xi) = \omega_1\rangle$ such that the following holds:

1. Each C_ξ is a club subset of ξ such that every element of C_ξ has cofinality ω.
2. $C_\xi \cap C_\eta = \emptyset$ for $\xi < \eta < \delta$ with $\mathrm{cf}(\xi) = \mathrm{cf}(\eta) = \omega_1$.

Then, we define a bijection h from $\delta + 1$ to itself. For each $\xi < \delta$ with $\mathrm{cf}(\xi) = \omega_1$ and for each $\mu \in C_\xi$, we pick an element $\nu \in J_\mu$ with $\nu \prec^* \xi$, and then we put $h(\mu) = \nu$ and $h(\nu) = \mu$. For any other points, let h be the identity. Now, we define the partial order $\preceq$ on $\delta + 1$ by letting $x \preceq y$ iff $h(x) \preceq^* h(y)$. It is straightforward to check that $(T, \preceq)$ is the required PCF structure. □

Now, by using Theorem 2.2, we can extend the observation given at the beginning of [9, Section 3], and so we have that for every ordinal $\delta < \omega_2$ with $\mathrm{cf}(\delta) = \omega_1$, the PCF axioms listed above are not sufficient to prove that if $\aleph_\omega^\omega > \aleph_\delta \cdot 2^\omega$ then there is a countable sequence $\langle \aleph_{n_k} : k < \omega\rangle$ where each n_k is a natural number such that

$$\mathrm{tcf}(\prod_k \aleph_{n_k}, <_{Fin}) = \aleph_{\delta+1}.$$

However, by the results shown in [9, Section 2], we know that for every limit ordinal $\delta < \omega_4$, if $\aleph_\omega^\omega > \aleph_\delta \cdot 2^\omega$ then for every ordinal $\beta < \delta$ there is a countable sequence $\langle \aleph_{\alpha_k+1} : k < \omega \rangle$ in $[\aleph_{\beta+1}, \aleph_\delta)_{REG}$ such that

$$\mathrm{tcf}(\prod_k \aleph_{\alpha_k+1}, <_{Fin}) = \aleph_{\delta+1}.$$

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