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SETS WITH NO SUBSETS OF HIGHER WEAK TRUTH-TABLE DEGREE

A b s t r a c t. We consider the weak truth-table reducibility $\leq_{wtt}$ and we prove the existence of wtt-introimmune sets in Δ^0_2. This closes the gap on the existence of arithmetical r-introimmune sets for all the known reducibilities $\leq_r$ strictly contained in the Turing reducibility.

1. Introduction

The existence of sets without subsets of higher Turing degree was proved by Soare [11]. In terms of their complexity, we know by Jockusch [7] that they cannot be arithmetical, and later Simpson [10] even proved that they cannot be hyperarithmetical. A natural question is to consider reducibilities $\leq_r$ that are strictly contained in the Turing reducibility $\leq_T$ and to

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see if there are arithmetical sets without subsets of higher r-degree. The reader unfamiliar with these reducibilities can see e.g. [6, 8, 9, 12]. The approach of to consider such reducibilities $\leq_r$ and to study the existence of arithmetical sets without subsets of higher r-degree was initiated in [5], in which r-*introimmune* sets have been introduced. An infinite set A of natural numbers is r-introimmune if for every subset B of A with $|A\backslash B| = \infty$ we have $A \not\leq_r B$. Some common reducibilities strictly contained in $\leq_T$ studied in Computability Theory are the following, from the smallest to the largest: the *one-one* $\leq_1$, the *many-one* $\leq_m$, the *truth-table* $\leq_{tt}$ and the *weak truth-table* reducibility $\leq_{wtt}$. r-introimmune sets have no subsets of higher r-degree for all the reducibilities $\leq_r$ of the list. In [5] it was proved the existence of arithmetical c-introimmne sets, where $\leq_c$ is the conjunctive reducibility, a particular truth-table reducibility. More specifically, it was proved the existence of c-introimmune Δ^0_4 sets. This was improved by Ambos-Spies [1] by showing the existence of tt-introimmune Δ^0_2 sets. So, from Ambos-Spies' result we know that there are arithmetical r-introimmune sets for all the reducibilities $\leq_r$ of the above list up to $\leq_{tt}$. In this paper we close the gap by considering the *weak truth-table* reducibility $\leq_{wtt}$, and we prove the existence of arithmetical wtt-introimmune sets, in particular wtt-introimmune Δ^0_2 sets. Since we currently do not know intermediate reducibilities between $\leq_{wtt}$ and $\leq_T$, we deduce that for all the known reducibilities $\leq_r$ strictly contained in $\leq_T$ there are arithmetical r-introimmune sets.

2. Notation

Our notation is standard and we mainly refer to [9, 12]. Letter $\mathbb{N}$ denotes the set of natural numbers. We identify each subset of $\mathbb{N}$ with its characteristic function. Given any two sets $A, B \subseteq \mathbb{N}$, $A\backslash B$ denotes the set difference of A and B. We fix a computable permutation $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. A string is any function $\alpha : \{0, 1, \ldots, n\} \to \{0, 1\}$, where $n \in \mathbb{N}$. $\emptyset$ denotes the empty string. The length of a string α, in short $|\alpha|$, is the cardinality of its domain. Given two strings α and β, we write:

- $\alpha \subseteq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha(m) \leq \beta(m)$ for every $m < |\alpha|$,

- $\alpha \sqsubseteq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha(m) = \beta(m)$ for every $m < |\alpha|$,
- $\alpha \sqsubset \beta$ if $\alpha \sqsubseteq \beta$ and $\alpha \neq \beta$.

For every string β and every $m \leq |\beta|$, $\beta \upharpoonright m$ is the string $\alpha \sqsubseteq \beta$ with $|\alpha| = m$. If α is a string and $b \in \{0, 1\}$ then αb denotes the string of length $|\alpha| + 1$ such that $\alpha \sqsubset \alpha b$ and $\alpha b(|\alpha|) = b$. We fix an effective acceptable enumeration $\Phi_0, \Phi_1, \ldots$ of the Turing functionals. We fix also an effective acceptable enumeration $\varphi_0, \varphi_1, \ldots$ of the Turing-computable unary functions. Finally, given two sets $A, B \subseteq \mathbb{N}$, A is *weak truth table* reducible to B, in short $A \leq_{wtt} B$, if there exists a number $e \in \mathbb{N}$ and a total computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that:

i) $\Phi_e^B = A$,

ii) for every $x \in \mathbb{N}$, the computation of the e-th oracle Turing machine with oracle B on input x asks the oracle only numbers less than $\varphi(x)$.

In this case we say that (Φ_e, φ) *wtt*-reduces A to B. The weak truth-table reducibility is also known in literature as the *bounded Turing reducibility* $\leq_{bT}$.

3. Main result

Given any reducibility $\leq_r$ and given any set $A \subseteq \mathbb{N}$, the r-degree of A is the class $\{B \subseteq \mathbb{N} : A \equiv_r B\}$, where $A \equiv_r B$ if and only if $A \leq_r B$ and $B \leq_r A$. A set A does not have subsets of higher r-degree if $A \not\leq_r B$ for every $B \subseteq A$. So a *wtt*-introimmune set does not have subsets of higher *wtt*-degree. In this section we prove the existence of a *wtt*-introimmune set in the class Δ_2^0. Thus, for each known reducibility $\leq_r$ strictly contained in $\leq_T$ there are arithmetical r-introimmune sets. As for the arithmetical complexity we observe that for each reducibility $\leq_r$ such that $\leq_1 \Rightarrow \leq_r$ there cannot be r-introimmune sets in Σ_1^0, because such sets are immune. This follows from the fact that each 1-introimmune set is immune.

Proposition 3.1. *Each* 1*-introimmune set is immune.*

Proof. Let $A \subseteq \mathbb{N}$ be an infinite set and let us suppose that A is not immune. Then there exists an infinite recursive set $R \subseteq A$. Let $f : \mathbb{N} \to \mathbb{N}$

be a total one-one computable function such that $R = \{f(0), f(1), \ldots\}$. Let us consider the infinite set

$$R_0 = \{f(\langle 0, n\rangle) : n \in \mathbb{N}\} \subseteq R.$$

Then,

$$A \backslash R_0 \subseteq A$$

and

$$|A \backslash (A \backslash R_0)| = |R_0| = \infty.$$

It follows that A is not 1-introimmune, because $A \leq_1 A\backslash R_0$ is witnessed by the total one-one computable function $g : \mathbb{N} \to \mathbb{N}$ defined in the following way:

1. $g(x) = x$ for every $x \notin R$, and
2. $g(f(\langle n, m\rangle)) = f(\langle n + 1, m\rangle)$ for every $n, m \in \mathbb{N}$.

It is routine to check that for every $x \in \mathbb{N}$, $x \in A \Leftrightarrow g(x) \in A\backslash R_0$. □

We know of the existence of m-introimmune sets in the class Π^0_1 [3, 4]. We leave as an open question the existence of wtt-introimmune sets in Π^0_1.

Theorem 3.2. *There exists a wtt-introimmune set in* Δ^0_2.

Proof. By the finite-extension method we construct a set A satisfying the following requirements for every $a, b, e \in \mathbb{N}$:

$$P_{2e} : |A| \geq e,$$

and

$N_{2\langle a,b\rangle+1} : (\Phi_a, \varphi_b)$ does not wtt-reduce A to any $X \subseteq A$ with $|A\backslash X| = \infty$.

The satisfaction of all the requirements P_{2e} guarantees that A is infinite, while the satisfaction of all the requirements $N_{2\langle a,b\rangle+1}$ guarantees that A is wtt-introimmune. □

3.1 Strategy

Set A will be constructed by infinitely many stages $s = 0, 1, \ldots$. At every stage s we define the finite set A_s, and the final set will be

$$A = \lim_{s\to\infty} A_s,$$

with $A_s \subseteq A_{s+1}$ for every $s \geq 0$. Set A will be a subset of $\{h(n) : n \geq 0\}$, where $h : \mathbb{N} \to \mathbb{N}$ is a suitable *dominating* function.

Definition 3.3. (dominating function). A function $g : \mathbb{N} \to \mathbb{N}$ is *dominating* if for every total computable function $\varphi : \mathbb{N} \to \mathbb{N}$, $\varphi(n) < g(n)$ for almost every n.

Let $K = \{x \in \mathbb{N} : \varphi_x(x) \downarrow\}$ be the halting set, and let g be any increasing dominating K-computable function with $g(0) > 0$. Let us define the increasing sequence $(g^n(0) : n \geq 1)$ in the following way: $g^1(0) = g(0)$, and for every $n \geq 1$ $g^{n+1}(0) = g(g^n(0))$. Let us define for every $n \geq 1$

$$h(n) = g^n(0),$$

with $h(0) = 0$. Then, h is a dominating K-computable function which satisfies the following property.

Proposition 3.4. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be any total computable function. Then for almost every $n \in \mathbb{N}$, for every $m \leq n$*

$$\varphi(h(m)) < h(n+1).$$

Proof. Given any such φ, let us consider the total computable function

$$\tilde{\varphi}(n) = \max\{\varphi(u) : u \leq n\}.$$

Let n_0 be such that for every $n \geq n_0$

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)). \tag{1}$$

Then, for every $n \geq n_0$ and for every $m \leq n$

$$\varphi(h(m)) = \varphi(g^m(0)) \tag{2}$$

by definition of h, and

$$\varphi(g^m(0)) \leq \tilde{\varphi}(g^n(0)) \tag{3}$$

by $(g^n(0) : n \geq 1)$ increasing and by the definition of $\tilde{\varphi}$. Finally

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)) = g^{n+1}(0) = h(n+1) \tag{4}$$

by (1) and by the definition of h. □

3.2 Strategies to satisfy requirements

To satisfy each requirement P_{2e} we add an element to A at e opportune stages. The strategy to satisfy each requirement $N_{2\langle a,b\rangle+1}$ is essentially the method used in [1]. To satisfy $N_{2\langle a,b\rangle+1}$ means in particular to prevent (5):

$$(\exists X)[X \subseteq A \text{ and } |A \backslash X| = \infty \text{ and } \Phi_a^X = A]. \tag{5}$$

But (5) implies that there is an infinite sequence $(n_s : s \geq 0)$ of natural numbers such that

$$\Phi_a^X(h(n_s)) = A(h(n_s)) = 1 \text{ and } X(h(n_s)) = 0. \tag{6}$$

So, we wait for a stage $s+1$ at which

$$\varphi_b(h(s)) < h(s+1) \tag{7}$$

and for some $X \subseteq A_s \subseteq \{h(0), h(1), \ldots, h(s-1)\}$ it is

$$\Phi_a^X(h(0)) = A_s(h(0)), \ldots, \Phi_a^X(h(s-1)) = A_s(h(s-1)) \tag{8}$$

and

$$\Phi_a^X(h(s)) = 1. \tag{9}$$

Then, we force $\Phi_a^X(h(s))$ to be wrong by setting $A_{s+1}(h(s)) = 0$. Observe that by (7) and by $X \subseteq A_s \subseteq \{h(0), \cdots, h(s-1)\}$ the computation of $\Phi_a^X(h(s))$ depends only on number less than or equal to $h(s)$.

3.3 Formalization

We formalize the above strategies and the construction of the set A. First, we define formally the conditions under which a requirement requires attention. Then, we will give an algorithm for the construction of the set A

by defining the actions needed to satisfy all the requirements. In order to better handle some proofs later we introduce first the following notation: given any string α, let X_α be the set

$$\{h(n) : n < |\alpha| \wedge \alpha(n) = 1\}.$$

From now on, Φ^α stands for Φ^{X_α} for each string α. The algorithm with which we will construct our set $A = \bigcup_{s \geq 0} A_s$ will generate by stages infinitely many strings $\alpha_0 \sqsubset \alpha_1 \sqsubset \cdots$. The final set A will be

$$A = \lim_{s \to \infty} \alpha_s,$$

where α_s is the string obtained by the end of stage s with $|\alpha_s| = s$ and denoting $A_s = X_{\alpha_s}$.

3.3.1 Requirements requiring attention

Fix a stage $s+1$, and let α_s be the string constructed by the end of stage s.

- Requirement P_{2e} requires attention at stage $s+1$ if

$$|A_s| < e.$$

- Requirement $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ via the string α with $|\alpha| = |\alpha_s| = s$ if the following conditions hold.

  C1: $\varphi_b(h(s)) < h(s+1)$,

  C2: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$, for every $m < s$,

  C3: $\alpha \subseteq \alpha_s$,

  C4: (for every $m < s$), $[\Phi_a^\alpha(h(m)) = \alpha_s(m)]$,

  C5: $\Phi_a^{\alpha 0}(h(s)) = 1$.

We describe the meaning of each condition. Condition **C1** makes the computation of $\Phi_a^\alpha(h(s))$ depending only on numbers less than or equal to $h(s)$. Condition **C2** says that (Φ_a, φ_b) could be a *wtt*-reduction. Condition **C3** says that the set X_α is a subset of the constructed set A_s. Conditions **C4** and **C5** formalize (8) and (9), that is

$$\Phi_a^{X_\alpha}(h(0)) = A_s(h(0)), \ldots, \Phi_a^{X_\alpha}(h(s-1)) = A_s(h(s-1))$$

and

$$\Phi_a^{X_\alpha}(h(s)) = 1.$$

3.3.2 Construction of the set A

We say that a N-requirement requires attention at stage $s+1$ if it requires attention at stage $s+1$ via some string α of length s. A requirement R_n has higher priority than a requirement R_m if $n < m$. At any stage $s+1$ a requirement R_n is *active* if it is the highest priority requirement requiring attention. The algorithm to construct the set A is the following.

Algorithm

- Stage 0. Set $\alpha_0 = \emptyset$.

- Stage $s+1$. Let α_s be the string constructed by the end of stage s, and let R_n be the active requirement. If n is even, then set $\alpha_{s+1} = \alpha_s 1$, otherwise set $\alpha_{s+1} = \alpha_s 0$.

End of algorithm

Set $A = \lim_{s\to\infty} \alpha_s$. The construction of A is by the finite extension method, thus for every stage $s \geq 0$ and for every $n < |\alpha_s|$, $\alpha_s(n) = A(h(n))$. Now we have to prove that the construction is correct, that is that each requirement is met and that $A \in \Delta_2^0$.

Lemma 3.5. *Every requirement requires attention at most finitely often and is met.*

Proof. By induction on the index n of the requirement R_n. Let $n \geq 0$ be given, and let s_0 be the minimum stage such that no requirement of higher priority than R_n requires attention after s_0. Distinguish two cases on n.

- $R_n = P_{2e}$. Let us suppose that it requires attention at stage $s+1 > s_0$.

By hypothesis P_{2e} is active from stage $s+1$ onwards. At each of these consecutive stages we add one element, so in at most $t \leq e$ stages starting from $s+1$ the cardinality of A_{s+t} will be e, P_{2e} is satisfied and it will no longer require attention.

- $R_n = N_{2\langle a,b\rangle+1}$. By Proposition 3.4 we can make the following further hypothesis on s_0: for every $s \geq s_0$ and for every $m < s$,

$$\varphi_b(h(m)) < h(s). \qquad (10)$$

From (10) we get the following

Claim 3.6. *For every string α and α' of length at least s_0, if $\alpha \sqsubseteq \alpha'$, then*

$$(\forall m < |\alpha|)[\Phi_a^{\alpha}(h(m)) = \Phi_a^{\alpha'}(h(m))]. \qquad (11)$$

Proof. Let $\alpha \sqsubseteq \alpha'$ with $|\alpha| \geq s_0$. For every $m < |\alpha|$ the computation of $\Phi_a^{X_\alpha}(h(m))$ can ask the oracle only numbers less that $\varphi_b(h(m)) < h(|\alpha|)$, where

$$X_\alpha \subseteq \{h(0), h(1), \ldots, h(|\alpha|-1)\}.$$

On the other hand, $\alpha \sqsubseteq \alpha'$ means that

$$\alpha = \alpha' \upharpoonright |\alpha|,$$

that is X_α is equal to $X_{\alpha'}$ up to $h(|\alpha|-1)$. Therefore the two computations $\Phi_a^{X_\alpha}(h(m))$ and $\Phi_a^{X_{\alpha'}}(h(m))$ are equal for every $m < |\alpha|$. *End of proof of Claim 3.6.*

The proof that $N_{2\langle a,b\rangle+1}$ requires attention at most finitely often is distributed in the following three claims[1].

Claim 3.7. *If $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1 > s_0$ via α, then for every s' with $s_0 \leq s' < s$ it holds that $\alpha(s') = A(h(s'))$.*

Proof. Let α_s be the string constructed by the end of stage s. For the sake of contradiction, let s' be the minimum such that $s_0 \leq s' < s$ and $\alpha(s') \neq A(h(s'))$. By hypothesis $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1$, thus by condition **C3**

[1] Technically, the proofs of these three claims are based on [2].

$$\alpha(s') \leq A(h(s')), \tag{12}$$

that is

$$\alpha(s') = 0 \text{ and } A(h(s')) = 1. \tag{13}$$

Let us consider $\beta = \alpha \upharpoonright s'$, that is

$$\beta \sqsubseteq \alpha \text{ and } |\beta| = s'. \tag{14}$$

We prove that $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β, and this implies $\alpha_{s'+1} = \alpha_{s'}0$, that is $\alpha_{s'+1}(s') = 0$; but $\alpha_{s'+1} \sqsubseteq \alpha_s$, whence $\alpha_s(s') = 0$, that is $A(h(s')) = 0$, contradicting (13). In order to prove that $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β it is enough to check that all the conditions **C1**, **C2**, **C3**, **C4** and **C5** hold for β and $\alpha_{s'}$ at stage $s'+1$.

- **C1**: $\varphi_b(h(s')) < h(s'+1))$ holds by (10) because $s'+1 \geq s_0$.
- **C2**: $\Phi_a^\beta(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s' < |\alpha|$, because **C2** holds at stage $s+1$ w.r.t. α.
- **C3**: $\beta \subseteq \alpha_{s'}$, because $\alpha_{s'} \sqsubseteq \alpha_s$, $\alpha \subseteq \alpha_s$ and $\beta = \alpha \upharpoonright s'$.
- **C4**: $\beta \sqsubseteq \alpha$ with both the lengths of β and α at least s_0, so by Claim 1 for every $m < |\beta|$
$$\Phi_a^\beta(h(m)) = \Phi_a^\alpha(h(m)). \tag{15}$$
Moreover, for every $m < |\beta|$
$$\Phi_a^\alpha(h(m)) = A(h(m)) \tag{16}$$
because **C4** holds at stage $s+1$ w.r.t. α. Thus, by equations (15) and (16)
$$\Phi_a^\beta(h(m)) = A(h(m)) \tag{17}$$
for every $m < |\beta|$.
- **C5**: We observe first that $\beta 0 \sqsubseteq \alpha$, because by (13) it is $\alpha(s') = 0$ and by (14) it is $|\beta| = s'$. Then, by (10) the computation of $\Phi_a^{\beta 0}(h(s'))$ depends only on numbers $\leq h(s')$, which means that
$$\Phi_a^{\beta 0}(h(s')) = \Phi_a^\alpha(h(s')).$$

But by hypothesis $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$, that is at stage $s+1$ condition **C4** holds for every $m < s$, in particular for $m = s' < s$, so by the second equality of (13)

$$\Phi_a^{\alpha}(h(s')) = A(h(s')) = 1.$$

Therefore

$$\Phi_a^{\beta 0}(h(s')) = 1$$

and **C5** is satisfied. Hence, all the conditions **C1**, **C2**, **C3**, **C4** and **C5** are satisfied by β and $\alpha_{s'}$, so $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β with $|\beta| = s'$. But as before observed this causes $A(h(s')) = 0$, contradicting (13). *End of proof of Claim 3.7.*

Claim 3.8. *Let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1 > s_0$, and let α' be such that $\alpha \sqsubset \alpha'$. Then, $N_{2\langle a,b\rangle+1}$ does not require attention via α'.*

Proof. By hypothesis, at the end of stage $s+1$ is

$$A(h(s)) = 0. \tag{18}$$

Let $s' > s$, and for the sake of contradiction let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α' at stage $s'+1$. First, we note that it cannot be $\alpha 1 \sqsubseteq \alpha'$, because otherwise it would be

$$\alpha'(s) = 1$$

and by (18) $A(h(s)) = 0$, that is $\alpha_{s'}(s) = 0$, from which $\alpha' \not\subseteq \alpha_{s'}$, contradicting condition **C3** $\alpha' \subseteq \alpha_{s'}$ at stage $s'+1$. Thus it has to be

$$\alpha 0 \sqsubseteq \alpha'. \tag{19}$$

Since by hypothesis $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1$ it follows that **C5** is satisfied, that is

$$\Phi_a^{\alpha 0}(h(s)) = 1.$$

On the other hand, by (19)

$$\Phi_a^{\alpha'}(h(s)) = \Phi_a^{\alpha 0}(h(s)) = 1.$$

But at stage $s'+1$ $N_{2\langle a,b\rangle+1}$ requires attention via α', so by condition **C4** for $m = s < s'$

$$\Phi_a^{\alpha'}(h(s)) = A(h(s)),$$

that is $A(h(s)) = 1$, which contradicts (18). *End of proof of Claim 3.8.*

Claim 3.9. *For every string α of length s_0, there is at most one string α' properly extending α such that $N_{2\langle a,b\rangle+1}$ requires attention via α'.*

Proof. Let α be a string such that $|\alpha| = s_0$, and let α' and α'' be two strings properly extending α, that is

$$\alpha(m) = \alpha'(m) = \alpha''(m)$$

for every $m < s_0$. Let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α' at stage $s'+1 > s_0$ and via α'' at stage $s''+1 > s_0$. Without loss of generality let us suppose that $|\alpha'| \leq |\alpha''|$. By Claim 3.7, for every t with $s_0 \leq t < s'$ it is

$$\alpha'(t) = A(h(t)) = \alpha''(t).$$

If $|\alpha'| = |\alpha''|$, then $\alpha' = \alpha''$. Otherwise $\alpha' \sqsubset \alpha''$, but this contradicts Claim 3.8. *End of proof of Claim 3.9*

Since there are 2^{s_0} string of length s_0, by Claim 4 requirement $N_{2\langle a,b\rangle+1}$ requires attention at most 2^{s_0} times after stage s_0.

We prove now that $N_{2\langle a,b\rangle+1}$ is met. For the sake of contradiction let us suppose that $N_{2\langle a,b\rangle+1}$ is not met. This means that there exists $B \subseteq A$ such that

$$\Phi_a^B = A \tag{20}$$

and

$$|A\backslash B| = \infty. \tag{21}$$

Moreover, for every $x \in \mathbb{N}$ all the queries made in the computation $\Phi_a^B(x)$ are bounded by $\varphi_b(x)$. We proved that $N_{2\langle a,b\rangle+1}$ requires attention at most finitely often. Hence, there is a minimum stage s_0 after which $N_{2\langle a,b\rangle+1}$ does not require attention. By Proposition 3.4 and by (20) and (21) let $s+1 > s_0$ such that the following three conditions are satisfied:

$$\varphi_b(h(s)) < h(s+1), \tag{22}$$

$$\Phi_a^B(h(s) = A(h(s)) = 1 \tag{23}$$

and

$$B(h(s)) = 0. \tag{24}$$

We show that $N_{2\langle a,b\rangle+1}$ requires attention at $s+1$, which is a contradiction. By (22) at stage $s+1$ condition **C1** holds. Let us consider the string α of length s such that

$$\alpha(m) = B(h(m)) \tag{25}$$

for every $m < s$. String α satisfies all the conditions **C2**, **C3**, **C4** and **C5**:

- **C2**: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s$, because we are assuming that (Φ_a, φ_b) *wtt*-reduces A to B;
- **C3**: $\alpha \subseteq \alpha_s$ because $B \subseteq A$;
- **C4**: by (20) and (25), for every $m < s$ $\Phi_a^\alpha(h(m)) = A(h(m)) = \alpha_s(m)$;
- **C5**: by (24) and (25), for every $m \leq s$

  $$\alpha 0(m) = B(h(m)),$$

  therefore by (23)

  $$\Phi_a^{\alpha 0}(h(s)) = \Phi_a^B(h(s)) = 1.$$

Thus $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ via α, which is a contradiction. □

It remains to prove that the set A is in Δ_2^0.

Lemma 3.10. *A is in Δ_2^0.*

Proof. We show that A is Turing reducible to the halting set K. It is enough to observe that oracle K suffices to find the active requirement at any stage, hence to generate the sequence $(\alpha_s : s \geq 0)$. We describe first an algorithm that at any stage $s+1$ finds the active requirement and computes the extension α_{s+1} of α_s. Fix a stage $s+1$ and let α_s be the string obtained by the end of stage s. Enumerate and check all the requirements $R_0, R_1, \ldots$, stopping as soon as one of them satisfies the conditions under which it requires attention. For the part concerning the check, let R_n be a requirement of the above list and distinguish two cases:

- $R_n = P_{2n}$. It is decidable whether or no P_{2e} requires attention, and in this case oracle K is unnecessary.

- $R_n = N_{2\langle a,b\rangle+1}$. With oracle K compute first $h(s)$ and $h(s+1)$. Let $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ be the formula obtained by the conjunction of the formulas expressing conditions **C1**, **C2**, **C3**, **C4** and **C5** with X_α in place of α. Then, $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ if the formula

$$(\exists\alpha)[|\alpha| = |\alpha_s| \wedge F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))] \tag{26}$$

is true. In (26) the existential quantifier on the oracle variable α is bounded, and for each such α oracle K suffices to compute the relative finite set X_α. All the values $h(m)$ for $m < s$ required in the formula are also computable with K. Finally, observe that $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ is a Σ^0_1 formula, so oracle K is enough to test its truth. This shows that K suffices to generate $(\alpha_s : s \geq 0)$. To decide A, given any $x \in \mathbb{N}$ generate the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{m+1}$, where m is the minimum such that $h(m) \geq x$. If $h(m) > x$ then reject x. Otherwise, accept x if and only if $\alpha_{m+1}(m) = 1$.

This concludes the proof of Lemma 3.10 and the proof of the theorem. □

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