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CONTINUOUS DEPENDENCE OF MILD SOLUTIONS ON INITIAL NONLOCAL DATA, OF THE NONLOCAL SEMILINEAR FUNCTIONAL-DIFFERENTIAL EVOLUTION CAUCHY PROBLEMS OF THE FIRST AND SECOND ORDER

CIĄGŁA ZALEŻNOŚĆ CAŁKOWYCH ROZWIĄZAŃ OD NIELOKALNYCH WARUNKÓW POCZĄTKOWYCH, NIELOKALNYCH SEMILINIOWYCH EWOLUCYJNYCH FUNKCJONALNO-RÓŻNICZKOWYCH ZGADNIEŃ CAUCHY'EGO PIERWSZEGO I DRUGIEGO RZĘDU

Abstract

The aim of the paper is to prove two theorems on the continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal semilinear functional-differential evolution Cauchy problems of the first and second orders. The paper is based on publications [1–10] and is a generalization of paper [5].

Keywords: evolution Cauchy problems, functional-differential problems, first and second order problems, continuous dependence of solutions, nonlocal conditions.

Streszczenie

W artykule udowodniono dwa twierdzenia o ciągłej zależności rozwiązań całkowych od nielokalnych warunków początkowych, semiliniowych funkcjonalno-różniczkowych zagadnień ewolucyjnych Cauchy'ego pierwszego i drugiego rzędu. Artykuł bazuje na publikacjach [1–10] i jest uogólnieniem publikacji [5].

Słowa kluczowe: ewolucyjne zagadnienia Cauchy'ego, funkcjonalno-różniczkowe zagadnienia, zagadnienia pierwszego i drugiego rzędu, ciągła zależność rozwiązań, warunki nielokalne.

Continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal functional-differential evolution Cauchy problem of the first order

1. Introduction to Part I

In this part of the paper, we assume that E is a Banach space with norm $\|\cdot\|$ and $-A$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \geq 0}$ on E.

Throughout this part of the paper, we use the notation:

$$I = [0, a], \text{ where } a > 0,$$

$$M = \sup\{\|T(t)\|: \ t \in I\}$$

and:

$$X = C(I, E).$$

Let p be a positive integer and $t_1, \ldots, t_p$ be given real numbers such that $0 < t_1 < \ldots < t_p \leq a$. Moreover, let $C_i (i = 1, \ldots, p)$ be given real numbers and:

$$K := \sum_{i=1}^{p} |C_i|.$$

Consider the nonlocal functional-differential evolution Cauchy problem of the first order:

$$u'(t) + Au(t) = f\big(t, u(t), u(b_1(t)), \ldots, u(b_m(t))\big), \quad t \in I \setminus \{0\}, \tag{1}$$

$$u(0) + \sum_{i=1}^{p} C_i u(t_i) = x_0, \tag{2}$$

where:

$f : I \times E^{m+1} \to E, \ b_i : I \to I \ (i = 1, 2, \ldots, m)$ and $x_0 \in E$.

In this part of the paper, we shall study a continuous dependence of the mild solution, on initial nonlocal data (2), of the nonlocal semilinear functional-differential evolution Cauchy problem (1)–(2). The definition of this solution will be given in the next section.

This part of the paper is based on publications $[1, 3–10]$ and generalizes some results from $[5]$ in this sense that, now, we consider functional-differential problems in contrast to $[5]$, where differential problems were considered.

2. Theorem about a mild solution of the nonlocal functional-differential evolution Cauchy problem of the first order

A function u belonging to X and satisfying the integral equation:

$$u(t)=T(t)x_0-T(t)\left(\sum_{i=1}^{p}C_i u(t_i)\right)+\int_0^t T(t-s)f\big(s,u(s),u\big(b_1(s)\big),\ldots,u\big(b_m(s)\big)\big)ds, \ t\in I, \quad (3)$$

is said to be a mild solution of the nonlocal Cauchy problem (1)–(2).

Theorem 2.1. Assume that:

(i) $f:I\times E^{m+1}\to E$ is continuous with respect to the first variable on I, $b_i\in C(I,I) \ (i=1,2,\ldots,m)$ and there exists constant $L>0$ such that:

$$\left\|f\big(s,z_1,\ldots,z_{m+1}\big)-f\big(s,\tilde{z}_1,\ldots,\tilde{z}_{m+1}\big)\right\|\leq L\sum_{i=1}^{m+1}\|z_i-\tilde{z}_i\| \ \text{for } s\in I, \ z_i,\tilde{z}_i\in E \,(i=1,2,\ldots,m). \quad (4)$$

(ii) $M\big((m+1)aL+K\big)<1.$

(iii) $x_0\in E.$

Then the nonlocal Cauchy problem (1)–(2) has a unique mild solution.

Proof. See $[3]$, Theorem 2 and page 13.

3. Continuous dependence of a mild solution, on initial nonlocal data (2), of the nonlocal Cauchy problem (1)–(2).

In this section, there is the main result of Part I.

Theorem 3.1. Let all the assumptions of Theorem 2.1 be satisfied. Suppose that u is the mild solution (satisfying (3)) from Theorem 2.1. Moreover, let $v\in X$, satisfying the equation:

$$v(t)=T(t)y_0-T(t)\left(\sum_{i=1}^{p}C_i v(t_i)\right)+\int_0^t T(t-s)f\big(s,v(s),v\big(b_1(s)\big),\ldots,v\big(b_m(s)\big)\big)ds, \ t\in I, \quad (5)$$

be the mild solution to the nonlocal problem:

$$v'(t)+Av(t)=f\big(t,v(t),v\big(b_1(t)\big),\ldots,v\big(b_m(t)\big)\big), \ t\in I\setminus\{0\},$$

$$v(0)+\sum_{i=1}^{p}C_i v(t_i)=y_0,$$

where $y_0\in I.$

Then for an arbitrary $e > 0$ there is $\delta > 0$ such that if:

$$\left\| x_0 - y_0 \right\| < \delta \tag{6}$$

then:

$$\left\| u - v \right\|_X < \varepsilon. \tag{7}$$

Proof. Let ε be a positive number and let:

$$\delta := \frac{1 - MK - (m+1)aML}{M}\varepsilon. \tag{8}$$

Observe that, from (3) and (5),

$$u(t) - v(t) = T(t)(x_0 - y_0) - T(t)\left(\sum_{i=1}^{p} C_i \left(u(t_i) - v(t_i) \right) \right)$$

$$+ \int_0^t T(t-s)\Big(f\big(s, u(s), u(b_1(s)), \ldots, u(b_m(s))\big) - f\big(s, v(s), v(b_1(s)), \ldots, v(b_m(s))\big) \Big)ds, \ t \in I. \tag{9}$$

Consequently, by (9) and (4),

$$\left\| u - v \right\|_X \le M \left\| x_0 - y_0 \right\| + MK \left\| u - v \right\|_X + (m+1)\, aML \left\| u - v \right\|_X.$$

From the above inequality:

$$\big(1 - MK - (m+1)aML \big)\left\| u - v \right\|_X \le M \left\| x_0 - y_0 \right\|. \tag{10}$$

By (10), (6) and (8),

$$\left\| u - v \right\|_X \le \frac{M}{1 - MK - (m+1)aML}\left\| x_0 - y_0 \right\| < \frac{M}{1 - MK - (m+1)aML}\delta = \varepsilon.$$

Therefore, (7) holds. It means that the mild solution of the nonlocal Cauchy problem (1)–(2) is continuously dependent on the initial nonlocal data (2).

The proof of Theorem 3.1 is complete.

Part II
Continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal functional-differential evolution Cauchy problem of the second order

4. Introduction to Part II

In the second part of the paper, we consider the nonlocal functional-differential evolution Cauchy problem of the second order:

$$u''(t) = Au(t) + f\big(t, u(t), u(b_1(t)), \ldots, u(b_m(t))\big), \ \ t \in I \setminus \{0\}, \tag{11}$$

$$u(0) = x_0, \tag{12}$$

$$u'(0) + \sum_{i=1}^{p} C_i u(t_i) = x_1, \tag{13}$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from the Banach space E (with norm $\|\cdot\|$) into itself, $u : I \to E$, $f : I \times E^{m+1} \to E, b_i \in C(I,I) (i = 1,2,\ldots,m),\ I = [0,a],\ a > 0,\ x_0, x_1 \in E,\ C_i \in \mathbb{R}\ (i=1,\ldots,p)$ and $t_1,\ldots,t_p$ are as in Part I.

We will use the set:

$$\tilde{E} := \{x \in E : C(t)x \text{ is of class } C^1 \text{ with respect to } t\}$$

and the sine family $\{S(t) : t \in \mathbb{R}\}$ defined by the formula:

$$S(t)x := \int_0^t C(s)x\,ds, \quad x \in E,\ t \in \mathbb{R}.$$

In this part of the paper, we shall study a continuous dependence of a mild solution, on initial nonlocal data (12)–(13), of the nonlocal Cauchy problem (11)–(13). The definition of this solution will be given in the next section.

The second part of the paper is based on papers [2, 5, 6] and generalizes some results from [5] in this sense that, now, we consider functional-differential problems in contrast to [5], where differential problems were considered.

5. Theorem about a mild solution of the nonlocal functional-differential evolution Cauchy problem of the second order

A function u belonging to $C^1(I,E)$ and satisfying the integral equation:

$$u(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} C_i u(t_i)\right) +$$
$$+ \int_0^t S(t-s) f\big(s, u(s), u(b_1(s)), \ldots, u(b_m(s))\big)ds,\ t \in I,$$
$$\tag{14}$$

is said to be a mild solution of the nonlocal Cauchy problem (11)–(13).

Theorem 5.1. Assume that:

(i) $f : I \times E^{m+1} \to E$ is continuous with respect to the first variable $t \in I,\ b_i \in C(I,I)$ $(i = 1,2,\ldots,m)$ and there exists a positive constant $L > 0$ such that:

$$\left\| f(s,z_1,\ldots,z_{m+1}) - f(s,\tilde{z}_1,\ldots,\tilde{z}_{m+1}) \right\| \le L \sum_{i=1}^{m+1} \left\| z_i - \tilde{z}_i \right\| \text{ for } s \in I, \ z_i, \tilde{z}_i \in E \ (i=1,2,\ldots,m+1), (15)$$

(ii) $C\big((m+1)aL+K\big)<1,$ where:

$$C := \big\{ \sup \left\| C(t) \right\| + \left\| S(t) \right\| + \left\| S'(t) \right\| : t \in I \big\} \text{ and } K := \sum_{i=1}^{p} |C_i|,$$

(iii) $x_0 \in \tilde{E}$ and $x_1 \in E.$

Then, the nonlocal Cauchy problem (11)–(13) has a unique mild solution.

Proof. See $[2]$, Theorem 2.1.

6. Continuous dependence of a mild solution, on initial nonlocal data (12)–(13), of the nonlocal Cauchy problem (11)–(13)

In this section, there is the main result of Part II.

Theorem 6.1. Let all the assumptions of Theorem 5.1 be satisfied. Suppose that u is the mild solution (satisfying (14)) from Theorem 5.1. Moreover, let v satisfying the equation:

$$v(t) = C(t)y_0 + S(t)y_1 - S(t)\left(\sum_{i=1}^{p} C_i v(t_i) \right) + \int_0^t S(t-s)f\big(s,v(s),v(b_1(s)),\ldots,v(b_m(s))\big)ds, \ t \in I, (16)$$

be the mild solution of the nonlocal problem:

$$v''(t) = Av(t) + f\big(t,v(t),v(b_1(t)),\ldots,v(b_m(t))\big), \quad t \in I \setminus \{0\},$$

$$v(0) = y_0,$$

$$v'(0) + \sum_{i=1}^{p} C_i v(t_i) = y_1,$$

where $y_0 \in \tilde{E}$ and $y_1 \in E.$

Then, for an arbitrary $\varepsilon > 0$ there is $\delta > 0$ such that if:

$$\left\| x_0 - y_0 \right\| < \delta, \ \left\| x_1 - y_1 \right\| < \delta \tag{17}$$

then:

$$\left\| u - v \right\|_X < \varepsilon, \tag{18}$$

where $X = C(I, E).$

Proof. Let ε be a positive number and let:

$$\delta := \frac{1 - CK - aCL(m+1)}{2C} \varepsilon. \tag{19}$$

Observe that, from (14) and (16),

$$u(t)-v(t)=C(t)(x_0-y_0)+S(t)(x_1-y_1)-S(t)\left(\sum_{i=1}^{p}C_i(u(t_i)-v(t_i))\right)+$$

$$+\int_0^t S(t-s)\Big(f\big(s,u(s),u(b_1(s)),\ldots,u(b_m(s))\big)-f\big(s,v(s),v(b_1(s)),\ldots,v(b_m(s))\big)\Big)ds,\ t\in I.\ (20)$$

Consequently, by (20) and (15),

$$\|u-v\|_X \le C\|x_0-y_0\|+C\|x_1-y_1\|+CK\|u-v\|_X+aCL(m+1)\|u-v\|_X.$$

From the above inequality:

$$\big(1-CK-aCL(m+1)\big)\|u-v\|_X \le C\big(\|x_0-y_0\|+\|x_1-y_1\|\big). \tag{21}$$

By (21), (17) and (19),

$$\|u-v\|_X \le \frac{C}{1-CK-aCL(m+1)}\big(\|x_0-y_0\|+\|x_1-y_1\|\big)<\frac{C}{1-CK-aCL(m+1)}\cdot 2\delta=\varepsilon.$$

Therefore, (18) holds. This means that the mild solution of the nonlocal Cauchy problem (11)–(13) is continuously dependent on the initial nonlocal data (12)–(13).

The proof of Theorem 6.1 is complete.

Remark

The nonlocal problems considered in the paper have a physical interpretation. For this purpose, see $[4]$.

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