REPORTS ON MATHEMATICAL LOGIC
52 (2017), 83–99
doi:10.4467/20842589RM.17.005.7143

Tomasz SKURA

REFUTATIONS IN WANSING'S LOGIC

A b s t r a c t. A refutation system for Wansing's logic **W** (which is an expansion of Nelson's logic) is given. The refutation system provides an efficient decision procedure for **W**. The procedure consists in constructing for any normal form a finite syntactic tree with the property that the origin is non-valid iff some end node is non-valid. The finite model property is also established.

1. Introduction

Wansing's logic **W** (see [7]) is defined in a semantic way. It is the set of formulas valid in all Nelson models augmented by a possibility connective M. (Nelson models for the extended language will be called models.) The problem of axiomatizing this logic was dealt with in [3,4]. However, the question whether it is decidable seems to be open.

Received 23 March 2017
Keywords and phrases: Nelson's logic, nonmonotonic inference, decision procedures, refutation systems.
AMS subject classification: 03B60, 03B25, 03F03.

In this paper we prove that $\mathbf{W}$ is decidable. We give a refutation system that provides an efficient decision procedure. The procedure consists in constructing for any normal form a finite syntactic tree with the property that the origin is non-valid iff some end node is non-valid. We also establish the finite model property in a refined form, that is, we show that $\mathbf{W}$ is characterized by the class of finite tree models.

2. Preliminaries

Let FOR be the set of all formulas generated from the set $VAR = \{p, q, r, ...\}$ of propositional variables by the connectives $\rightarrow, \wedge, \vee, \sim$ (strong negation), M (possibility). If $\Psi \cup \{A, B\}$ is a finite set of formulas, we write $\Psi \longrightarrow A$ instead of $\bigwedge \Psi \rightarrow A$ and $A, \Psi \longrightarrow B$ instead of $\{A\} \cup \Psi \longrightarrow B$. ($\bigwedge \Psi = p \rightarrow p$ if $\Psi = \emptyset$.) If $A \in FOR$ then $SUB(A)$ is the set of all subformulas of A. For any $A, B \in FOR$, we define:

$A \equiv B = (A \rightarrow B) \wedge (B \rightarrow A)$
$A \Leftrightarrow B = (A \equiv B) \wedge (\sim A \equiv \sim B)$ (strong equivalence)
$\neg A = A \rightarrow \sim A$ (intuitionistic negation)

A *model* is a triple $(W, \leq, V)$, where W is a non-empty set of points (worlds), $\leq$ is a reflexive and transitive relation on $W \times W$, and V is a function assigning to every propositional variable at $x \in W$ either 1 (true) or -1 (false) or 0 (undecided), and extended to all formulas as follows. ($V(A, x) = 0$ iff $V(A, x) \neq 1$ and $V(A, x) \neq -1$.)

$V(A \wedge B, x) = 1$ iff $V(A, x) = 1$ and $V(B, x) = 1$.
$V(A \wedge B, x) = -1$ iff $V(A, x) = -1$ or $V(B, x) = -1$.
$V(A \vee B, x) = 1$ iff $V(A, x) = 1$ or $V(B, x) = 1$.
$V(A \vee B, x) = -1$ iff $V(A, x) = -1$ and $V(B, x) = -1$.
$V(A \rightarrow B, x) = 1$ iff for every $y \geq x$, if $(A, y) = 1$ then $V(B, y) = 1$.
$V(A \rightarrow B, x) = -1$ iff $V(A, x) = 1$ and $V(B, x) = -1$.
$V(\sim A, x) = 1$ iff $V(A, x) = -1$.
$V(\sim A, x) = -1$ iff $V(A, x) = 1$.
$V(\mathsf{M}A, x) = 1$ iff for some $y \geq x$, we have $V(A, y) = 1$.
$V(\mathsf{M}A, x) = -1$ iff $V(A, x) = -1$.

Also, V satisfies the following condition.

(*Persistence*) For any $A \in VAR$, both A and $\sim A$ are persistent.

(Here a formula A is said to be *persistent* iff $V(A, y) = 1$ whenever both $V(A, x) = 1$ and $x \leq y$.)

We say that a formula A is *valid in a model* $(W, \leq, V)$ iff $V(A, x) = 1$ for every $x \in W$, and A is *valid* iff A is valid in all models.

The logic **W** is the set of all valid formulas. We also write $\models A$ for "A is valid" (and $\not\models A$ for "A is non-valid"). A set Θ of formulas is said to be true (non-valid,...) iff so is every $A \in \Theta$.

We also say that A is *equivalent* to B, if $\models A \equiv B$.

The one-point tree $T = (x, (x, x))$ is especially important. The symbol **3** will denote the set of formulas valid in every model (T, V). We write $v(A)$ instead of $V(A, x)$.

Remark 2.1. *(i) The models for the language without* M *characterize Nelson's three-valued logic, now usually called* **N3** *(see e.g. [1] and [2] for more information).*

(ii) $\mathsf{M}A$ *need not be persistent in a model. That is why some intuitionistic laws are not in* **W** *(for example,* $A \to (B \to A)$*). Moreover, the Deduction Theorem does not hold. However, the variables, the negated variables (that is formulas* $\sim A$*, where* $A \in VAR$*), and the formulas of the kind* $A \to B$*, where* $A, B \in FOR$*, are persistent.*

Proposition 2.2. *It is easy to check the following.*

1. *If* $\models A$ *and* $\models A \to B$*, then* $\models B$.
2. $\models (A \to B) \to ((B \to C) \to (A \to C))$
3. $\models (B \to C) \to ((A \to B) \to (A \to C))$
4. $\models (A \to B) \to ((A \wedge C) \to (B \wedge C))$
5. $\models \sim\sim A \equiv A \quad \models \sim(A \wedge B) \equiv \sim A \vee \sim B \quad \models \sim(A \vee B) \equiv \sim A \wedge \sim B$
   $\models \sim(A \to B) \equiv A \wedge \sim B \quad \models A \wedge \sim A \to B$
6. $\models A \wedge B \to A \quad \models A \wedge B \to B$
7. $\models A \wedge B \equiv B \wedge A \quad \models A \vee B \equiv B \vee A$
8. $\models (A \to B \wedge C) \equiv (A \to B) \wedge (A \to C)$
9. $\models (A \vee B \to C) \equiv (A \to C) \wedge (B \to C)$
10. $\models (A \wedge B \to C) \equiv (A \to (B \to C))$*, where* A *is persistent.*

11. $\models A \wedge (A \to B) \equiv A \wedge B$, *where* B *is persistent.*
12. $\models\sim \mathsf{M}A \equiv\sim A$
13. $\models \mathsf{M}A \vee \neg A$
14. $\models (\mathsf{M}A \to B) \wedge \neg A \equiv \neg A$
15. $\models (B \to \mathsf{M}A) \wedge \neg A \equiv \neg A \wedge \neg B$
16. $\models (A \vee B, \Psi \longrightarrow C) \equiv (A, \Psi \longrightarrow C) \wedge (B, \Psi \longrightarrow C)$
17. If $\models A$ *then* $\models (\Psi \longrightarrow B) \equiv (A, \Psi \longrightarrow B)$.
18. If $\models A \equiv B$ *then* $\models (A, \Psi \longrightarrow C) \equiv (B, \Psi \longrightarrow C)$.
19. If $\models A$ *and* $\models B$, *then* $\models A \wedge B$.
20. $\models (A \wedge B, \Psi \longrightarrow C) \equiv (A, B, \Psi \longrightarrow C)$
21. $\models \bigwedge\{A_1, ..., A_n\} \equiv \bigwedge\{B_1, ..., B_n\}$, *where* $\{A_1, ..., A_n\} = \{B_1, ..., B_n\}$.

3. Normal Forms

Our normal form procedure is a modification of the procedure for Intuitionistic Logic described in [5].

Definition 3.1. (i) *A general form* is a formula

$$F = \Sigma \longrightarrow a$$

where

$$\Sigma = \Delta \cup \Delta^{\mathsf{M}} \cup \Gamma$$
$$\Delta = \{(a_i \to b_i) \to c_i : 1 \leq i \leq k\}$$
$$\Delta^{\mathsf{M}} = \{\mathsf{M}d_j \wedge (e_j \to \mathsf{M}d_j) : 1 \leq j \leq m\}$$

all a_i, b_i, c_i, d_j, e_j are variables,
a is a (negated) variable,
and Γ is a finite set of formulas of the kind

b or $b \to c$ or $b \to (c \to d)$ or $b \to c \vee d$

where b, c, d are (negated) variables. *The rank* $r(F)$ *of* F is $k + m$.

(ii) *A persistent general form* is

$$F^* = \Delta, \Delta^{\mathsf{M}}_{-}, \Gamma \longrightarrow a$$

where $\Delta^{\mathsf{M}}_{-} = \{e_j \to \mathsf{M}d_j : 1 \leq j \leq m\}$. *The rank* $r(F^*)$ *of* F^* is $k + m$.

Definition 3.2. *A normal form* is a general form $F = \Sigma \longrightarrow a$ satisfying the following condition.

If $b \to B \in \Gamma$ then $b \notin \Gamma$.

(The rank of F is $k + m$.)

Definition 3.3. *A special normal form* is a normal form F such that $F_0 \notin \mathbf{3}$, where

$F_0 = \Gamma \longrightarrow a$.

Proposition 3.4. *Let F be a normal form. Then*

(i) $\models F_0$ iff $F_0 \in \mathbf{3}$.

(ii) $F_0 \in \mathbf{3}$ iff either $a \in \Gamma$ or for some variable A, we have $A, \sim A \in \Gamma$.

Proof. We only prove (i). Of course, if $\models F_0$ then $F_0 \in \mathbf{3}$, so we show that if $F_0 \in \mathbf{3}$ then $\models F_0$. Suppose that $F_0 \in \mathbf{3}$ but $\not\models F_0$. Note that $a \notin \Gamma$ and for no variable A, both $A \in \Gamma$ and $\sim A \in \Gamma$. (Otherwise $\models F_0$.) Also, F is a normal form, so if $b \to B \in \Gamma$ then $b \notin \Gamma$. Let v be a valuation such that

$$v(A) = \begin{cases} 1 & \text{if } A \in \Gamma \\ -1 & \text{if } \sim A \in \Gamma \\ 0 & \text{otherwise} \end{cases} \qquad (A \in VAR)$$

Then $v(A) = 1$ for all $A \in \Gamma$ and $v(a) \neq 1$, so $v(F_0) \neq 1$. Hence $F_0 \notin \mathbf{3}$, which is a contradiction. □

For any $A \in FOR$ we construct the formula F_A in the following way.

First, for every subformula B of A, we define a unique corresponding variable p_B thus. If $B \in VAR$ then $p_B = B$, and if $B \notin VAR$ then p_B is a new variable.

Second, we define the set Δ_A as follows.

$$\begin{aligned} \Delta_A = \{&(p_C \otimes p_D) \Leftrightarrow p_{C \otimes D} : C \otimes D \in SUB(A), \otimes \in \{\to, \wedge, \vee\}\} \cup \\ &\{\oslash C \Leftrightarrow p_{\oslash C} : \oslash C \in SUB(A), \oslash \in \{\sim, \mathsf{M}\}\} \end{aligned}$$

Finally, we define: $F_A = \Delta_A \longrightarrow p_A$.

Lemma 3.5. *Let $A \in FOR$. Then $\models \Delta_A \longrightarrow (B \Leftrightarrow p_B)$ for any subformula B of A.*

Proof. See Appendix. □

Lemma 3.6. *For any formula A we have: $\models A$ iff $\models F_A$.*

Proof. (i) Assume that $\models F_A$. Let s be a substitution such that $s(p_B) = B$ ($B \in FOR$). Then $s(\Delta_A)$ consists of formulas $B \Leftrightarrow B$ (which are valid), and $s(p_A)$ is A. Of course, $\models s(F_A)$. Therefore $\models A$.

(ii) Assume that $\models A$. We have $\models \Delta_A \longrightarrow (B \Leftrightarrow p_B)$ for any $B \in SUB(A)$ (by Lemma 3.5). Hence, in particular, $\models \Delta_A \longrightarrow (A \Leftrightarrow p_A)$. So

$$\models \Delta_A \longrightarrow (A \to p_A)$$

It is easy to check that $\models B \to C$ whenever both $\models A$ and $\models B \to (A \to C)$ for any $A, B, C \in FOR$. Therefore $\models \Delta_A \longrightarrow p_A$. □

Lemma 3.7. *F_A is equivalent to $\bigwedge \Psi$ for some finite set Ψ of normal forms. Every $B \in \Psi$ has the form $\Sigma \longrightarrow p_A$ and $\models \bigwedge \Sigma \to \bigwedge \Delta_A$.*

Proof. See Appendix. □

Corollary 3.8. *For every formula A, there are normal forms $A_1, ..., A_n$ with the property that $\models A$ iff $\models A_i$ for all $1 \leq i \leq n$.*

4. Refutation System

Refutation Axioms: Every special normal form of rank 0.

Refutation Rules:

Normal form rules:

$$(R) \qquad \frac{F_1, ..., F_k, H_1, ..., H_m}{F}$$

where $F = \Delta, \Delta^{\mathsf{M}}, \Gamma \longrightarrow a$ is a special normal form of rank > 0 and

$$F_i = a_i, b_i \to c_i, \Delta_i, \Delta^{\mathsf{M}}_{-}, \Gamma \longrightarrow b_i$$
$$\Delta_i = \Delta - \{(a_i \to b_i) \to c_i\} \qquad (1 \leq i \leq k)$$
$$H_j = d_j, \Delta, \Delta^{\mathsf{M}}_j, \Gamma \longrightarrow {\sim} d_j$$
$$\Delta^{\mathsf{M}}_j = \Delta^{\mathsf{M}}_{-} - \{e_j \to \mathsf{M} d_j\} \qquad (1 \leq j \leq m)$$

$(R_i) \qquad \dfrac{G_i}{F} \qquad (1 \leq i \leq k)$

where F is a special normal form of rank > 0 and

$$G_i = c_i, a_i \to b_i, \Delta_i, \Delta^{\mathsf{M}}, \Gamma \longrightarrow a$$

Normalization rules:

$$(R^{\to}) \qquad \frac{A, B, \Psi \longrightarrow C}{A, A \to B, \Psi \longrightarrow C} \qquad \text{(where } B \text{ is persistent)}$$

$$(R^{\vee}) \qquad \frac{A, \Psi \longrightarrow C}{A \vee B, \Psi \longrightarrow C} \qquad \frac{B, \Psi \longrightarrow C}{A \vee B, \Psi \longrightarrow C}$$

$$(R^{\mathsf{M}}) \qquad \frac{\mathsf{M}B, A \to \mathsf{M}B, \Psi \longrightarrow C}{A \to \mathsf{M}B, \Psi \longrightarrow C} \qquad \frac{\neg A, \neg B, \Psi \longrightarrow C}{A \to \mathsf{M}B, \Psi \longrightarrow C}$$

We say that A is *refutable* (in symbols $\dashv A$) iff A is derivable from refutation axioms by refutation rules.

Remark 4.1. *By Proposition 3.4ii, the refutation axioms can be characterized in a syntactic way as follows. Let F be a normal form of rank 0 (so $F = F_0$). Then F is a special normal form iff both $a \notin \Gamma$ and for no vriable A we have $A, \sim A \in \Gamma$.*

Remark 4.2. *The rules R_i $(1 \leq i \leq k)$ and the normalization rules have the following property. Let E' be the premiss and let E be the conclusion of any of these rules. Then $\models E \to E'$, so these rules are refutation rules for* **W**. *In Section 6 it will be shown that R preserves non-validity as well.*

Lemma 4.3. *Every persistent general form $F^* = \Sigma^* \longrightarrow a$ of rank r is equivalent to $\bigwedge \Psi$ for some finite set Ψ of general forms of rank $\leq r$. Each $F \in \Psi$ has the form $\Sigma' \longrightarrow a$, $\models \bigwedge \Sigma' \to \bigwedge \Sigma^*$, and F^* can be obtained from F by R^{M}.*

Proof. Let $F^* = \Sigma^* \longrightarrow a$ be a persistent general form, where $\Sigma^* = \Delta \cup \Delta^{\mathsf{M}}_{-} \cup \Gamma$. By Proposition 2.2(13,17,16,15), F^* is equivalent to

$\mathsf{M}d_1 \vee \neg d_1, \Sigma^* \longrightarrow a$, which is equivalent to
$(\mathsf{M}d_1, \Sigma^* \longrightarrow a) \wedge (\neg d_1, \Sigma^* \longrightarrow a)$,

which is equivalent to $D_1 \wedge D_2$, where D_1 results from F^* by replacing $e_1 \to \mathsf{M}d_1$ by $\mathsf{M}d_1 \wedge (e_1 \to \mathsf{M}d_1)$ and D_2 results from F^* by replacing $e_1 \to \mathsf{M}d_1$ by $\neg e_1 \wedge \neg d_1$. By repeating this with the remaining formulas $e_j \to \mathsf{M}d_j$ we get that F^* is equivalent to $\bigwedge \Psi$ for some finite set of general forms of the kind $\Sigma' \longrightarrow a$, where $\Sigma' = \{A_1, ..., A_m\} \cup \Delta \cup \Gamma$ and

$$A_j \in \{\mathsf{M}d_1 \wedge (e_1 \to \mathsf{M}d_1), \neg e_1 \wedge \neg d_1\} \qquad (1 \le j \le m)$$

Each $F \in \Psi$ is of rank $\le r$, $\models \bigwedge \Sigma' \to \bigwedge \Sigma^*$, and F^* can be obtained from F by R^{M}. □

Lemma 4.4. *Every general form $F = \Sigma \longrightarrow a$ of rank r is equivalent to $E_1 \wedge ... \wedge E_n$, for some normal forms $E_1, ..., E_n$ of rank r. Each E_i has the form $\Sigma' \longrightarrow a$, $\models \bigwedge \Sigma' \to \bigwedge \Sigma$, and F can be obtained from E_i by $R^{\to}, R^{\vee}$.*

Proof. By induction on the number of $\to$-occurrences in Σ (see [5] for more details). □

Corollary 4.5. *Every persistent general form $F^* = \Sigma^* \longrightarrow a$ of rank r is equivalent to $\bigwedge \Psi$ for some finite set Ψ of normal forms of rank $\le r$. Each $F \in \Psi$ has the form $\Sigma' \longrightarrow a$, $\models \bigwedge \Sigma' \to \bigwedge \Sigma^*$, and F^* can be obtained from F by $R^{\mathsf{M}}, R^{\to}, R^{\vee}$.*

Proof. From Lemma 4.3 and Lemma 4.4. □

5. Completeness

Proposition 5.1. *Let F be a normal form of rank > 0.*

(i) If both $\models F_i$ and $\models G_i$, then $\models F$, where $1 \le i \le k$.
(ii) If $\models H_j$ then $\models F$, where $1 \le j \le m$.

Proof. (i) Let $1 \le i \le k$. Suppose that $\models F_i$ and $\models G_i$, but $\not\models F$. Then there is a model $(W, \le, V)$ with the property that for some $x \in W$, we have $V(\bigwedge \Sigma, x) = 1$ and $V(a, x) \ne 1$. Either $a_i \to b_i$ is true or is not true at x.

(Case 1) $a_i \to b_i$ is true at x. Then c_i is also true at x (for Δ is true at x). So G_i is not true at x, which means that $\not\models G_i$.

(Case 2) $a_i \to b_i$ is not true at x. Then there is $y \geq x$ such that a_i is true and b_i is not true at y. Since $\Sigma' = \Delta \cup \Delta^{\mathsf{M}}_{-} \cup \Gamma$ is persistent, each $A \in \Sigma'$ is true at y. Also, $b_i \to c_i$ is true at y (because so is $(a_i \to b_i) \to c_i$). Hence F_i is not true at y, so that $\not\models F_i$.

This is a contradiction.

(ii) Let $1 \leq j \leq m$. Suppose that $\models H_j$ but $\not\models F$. Then for some $x \in W$ and some model $(W, \leq, V)$, every $A \in \Sigma$ is true at x and a is not true at x. Hence d_j is true at some $y \geq x$. So $\sim d_j$ is false at y. Thus, H_j is not true at y, so that $\not\models H_j$, which is a contradiction. □

Theorem 5.2. *Let F be a normal form. Then either $\models F$ or $\dashv F$.*

Proof. By induction on the rank r of F.

(1) $r = 0$. Then $F = F_0$. Either $F_0 \in \mathbf{3}$ or $F_0 \notin \mathbf{3}$. If $F_0 \in \mathbf{3}$ then $\models F$ by Proposition 3.4. And if $F_0 \notin \mathbf{3}$ then F is a refutation axiom, so $\dashv F$. Thus, either $\models F$ or $\dashv F$.

(2) $r > 0$ and the theorem holds for normal forms of rank $< r$.

Consider the formulas F_i, G_i $(1 \leq i \leq k)$, and H_j $(1 \leq j \leq m)$. All G_i are general forms of rank $< r$, and all F_i, H_j are persistent general forms of rank $< r$. Hence, by Lemma 4.4 and Corollary 4.5, each of them is equivalent to a conjunction of normal forms of rank $< r$, which, by the induction hypothesis, are valid or refutable. So, by Lemma 4.4 and Corollary 4.5, we get

$\models F_i$ or $\dashv F_i$
$\models G_i$ or $\dashv G_i$ $\quad (1 \leq i \leq k)$
$\models H_j$ or $\dashv H_j$ $\quad (1 \leq j \leq m)$

Note that if $\dashv G_i$ for some i, then $\dashv F$ by R_i, so we assume that $\models G_i$ for all i. Also, if $F_0 \in \mathbf{3}$ then $\models F$ (by Proposition 3.4). Thus, we may assume that $F_0 \notin \mathbf{3}$, so F is a special normal form.

(Case 1) All F_i, H_j are refutable. Then $\dashv F$ by R.

(Case 2.1) Some F_i is valid. Then $\models F$ by Proposition 5.1i (because $\models G_i$).

(Case 2.2) Some H_j is valid. Then $\models F$ by Proposition 5.1ii.

Therefore either $\models F$ or $\dashv F$. □

6. Refutation Trees

Refutations in an axiomatic refutation system are derivations and they can be presented as finite trees as follows.

Definition 6.1. *A refutation tree for a formula* E is a finite tree of formulas satisfying the following conditions.

(i) The origin is E.
(ii) If F is an end node, then F is a refutation axiom.
(iii) If $E_1, ..., E_n$ are the immediate successors of a node F, then F is obtained from $E_1, ..., E_n$ by a refutation rule.

We now turn syntactic refutation trees into semantic countermodels by adapting the techniques introduced in [5].

First, for every refutation tree $RT(E)$, we construct a finite reflexive transitive tree $T(E)$ by deleting the nodes obtained by the normalization rules and R_i. More formally, let $N(E)$ be the number of nodes in $RT(E)$.

(1) $N(E) = 1$. Then E is a refutaton axiom, so E is a special normal form F. We put:
The origin $x(F) = F$ and $T(F)$ is F viewed as a reflexive transitive point.

(2) $N(E) > 1$ and every refutation tree with fewer nodes has its corresponding finite reflexive transitive tree.

(2.1) E is obtained from its immediate successors by R, so E is a special normal form F and the immediate successors are $F_1, ..., F_k, H_1, ..., H_m$. Also, the finite reflexive transitive trees $T(F_1), ..., T(H_m)$ have been constructed. Then $T(F)$ is the finite reflexive transitive tree with origin $x(F) = F$ and $x(F_1), ..., x(H_m)$ (with their trees) are the immediate successors of $x(F)$.

(2.2) E is obtained form its immediate successor E' by R_i, where $1 \leq i \leq k$, or by a normalization rule. Then $x(E) = x(E')$ and $T(E) = T(E')$.

Remark 6.2. *Every node in* $T(E)$ *is a special normal form* F *(so* $F_0 \notin \mathbf{3}$*).*

Second, we define a valuation V by assigning either 1 or 0 or -1 to a propositional variable A at a node F as follows.

$$V(A,F) = \begin{cases} 1 & \text{if } A \in \Gamma \\ -1 & \text{if } {\sim} A \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

Remark 6.3. *By inspecting the refutation rules, we can see that if $F = \Sigma \longrightarrow a$ is a node in $RT(E)$ and $\Sigma' \longrightarrow a'$ is a successor of F, then $\Pi \subseteq \Sigma'$, where Π is the set all (negated) variables in Γ. So* persistence *is satisfied. Also, for no variable A we have $A, {\sim} A \in \Gamma$. (Otherwise $F_0 \in \mathbf{3}$.) Thus, V is indeed a valuation.*

Finally, we show that $(T(E), V)$ is a countermodel for E.

Theorem 6.4. *Let $RT(E)$ be a refutation tree for $E = \Sigma \longrightarrow a$. Then $V(\Sigma, x(E)) = 1$ and $V(a, x(E)) \neq 1$.*

Proof. By induction on the number $N(E)$ of nodes in $RT(E)$.

(1) $N(E) = 1$. Then E is a refutation axiom, so E is a special normal form $F = F_0 = \Gamma \longrightarrow a$ and $F \notin \mathbf{3}$. We have $x(E) = F$.

If $b \to B \in \Gamma$, where b is a (negated) variable, then $b \notin \Gamma$ (because F is a normal form), so that $V(b, F) \neq 1$. So every $b \to B \in \Gamma$ is true at F. Hence Γ is true at F. Moreover $a \notin \Gamma$. (Otherwise $F \in \mathbf{3}$.) Hence $V(\Gamma, F) = 1$ and $V(a, F) \neq 1$.

(2) $N(E) > 1$ and the theorem holds for refutation trees with fewer nodes.

(Case 1) E is obtained from its immediate successors by R, so E is a special normal form F (so $F_0 \notin \mathbf{3}$) and the immediate successors are $F_1, ..., H_m$. Then $x(F) = F$. Further, for each $1 \leq i \leq k$ the countermodel corresponding to $RT(F_i)$ is $(T(F_i), V_i)$, where $T(F_i)$ is the subtree of $T(F)$ generated by $x(F_i)$ and V_i is V restricted to $T(F_i)$; and for each $1 \leq j \leq m$ the countermodel corresponding to $RT(H_j)$ is $(T(H_j), V_j)$, where $T(H_j)$ is the subtree of $T(F)$ generated by $x(H_j)$ and V_j is V restricted to $T(H_j)$. Since the number of nodes in $RT(F_i)$ (in $RT(H_j)$) is $< N(E)$, by the induction hypothesis we have:

For every $1 \leq i \leq k$, $\{a_i, b_i \to c_i\} \cup \Delta_i \cup \Delta^{\mathsf{M}}_{-} \cup \Gamma$ is true at $x(F_i)$ (so at every $y \geq x(F_i)$ because this set is persistent) and b_i is not true at $x(F_i)$.

For every $1 \leq j \leq m$, $\{d_j\} \cup \Delta \cup \Delta^{\mathsf{M}}_j \cup \Gamma$ is true at every $y \geq x(H_j)$. Hence $(a_i \to b_i) \to c_i$ is true at $x(F_i)$. (Otherwise $a_i \to b_i$ is true and c_i is not true at some $y \geq x(F_i)$, so b_i is true at y, so c_i is true at y, which is impossible.) Thus Δ is true at each $y > F$. Also, $a_i \to b_i$ is not true at $x(F_i)$, so $(a_i \to b_i) \to c_i$ is true at F $(1 \leq i \leq k)$. Further, $\mathsf{M}d_j$ is true at F for all $1 \leq j \leq m$, so Δ^{M} is true at F (because Δ^{M}_{-} is true at every $y > F$). Finally, Γ is true at F and a is not true at F by the definition of V (see (1) above). Therefore Σ is true and a is not true at F.

(Case 2) $E = \Sigma \longrightarrow a$ is obtained from its immediate successor $E' = \Sigma' \longrightarrow a$ by R_i or by a normalization rule. Then $x(E) = x(E')$ and $T(E) = T(E')$. By the induction hypothesis, Σ' is true and a is not true at $x(E')$. Since $\models \bigwedge \Sigma' \to \bigwedge \Sigma$, we have that Σ is true and a is not true at $x(E)$. □

Corollary 6.5. *The rule R preserves non-validity.*

Proof. Assume that $\not\models F_1, ..., \not\models H_m$. Then, by Lemma 4.4 and Corollary 4.5, there are finite sets $\Psi_1, ..., \Psi_{k+m}$ of normal forms such that all $\bigwedge \Psi_1, ..., \bigwedge \Psi_{k+m}$ are non-valid. Hence for every $1 \leq i \leq k+m$, there is $E_i \in \Psi_i$ such that E_i is non-valid. By Theorem 5.2, we have $\dashv E_i$ for all i, so $\dashv F_1, ..., \dashv H_m$ (by Lemma 4.4 and Corollary 4.5). Hence $\dashv F$ by R. So F is not valid in $(T(F), V)$ by Theorem 6.4. Therefore F is non-valid. □

Corollary 6.6. *(Soundness) Let F be a normal form. If $\dashv F$ then $F \notin \mathbf{W}$.*

Proof. Because the refutation axioms are non-valid and the refutation rules preserve non-validity (see Remark 4.2 and Corollary 6.5). □

7. Applications

7.1. A Decision Procedure

Our decision procedure is based on the following fact.

Proposition 7.1. *Let F be a normal form of rank > 0 and let Ψ be the set of all premisses of the rule R (that is, $\Psi = \{F_1, ..., H_m\}$). Then $\{F\} \cup \Theta$ is non-valid iff either $\Psi \cup \Theta$ is non-valid or $\{G_1\} \cup \Theta$ or ... or $\{G_k\} \cup \Theta$ is non-valid, where Θ is a set of formulas.*

Proof. By Proposition 5.1 and the fact the refutation rules $R, R_1, ..., R_k$ preserve non-validity. $\Box$

Our decision procedure can be described as follows.

Start with the origin $\{F\}$. The immediate successors of $\{F\}$ are

$\Psi_0, \Psi_1, ..., \Psi_k$

where $\Psi_0 = \{F_1, ..., F_k, H_1, ..., H_m\}$, $\Psi_i = \{G_i\}$ $(1 \leq i \leq k)$. (Of course, each A in every Ψ_i $(0 \leq i \leq k)$ is of rank $< r(F)$.)

Now, using Lemma 4.4 and Corollary 4.5, normalize all Ψ_i getting the sets $NF(\Psi_i)$ of sets of normal forms of rank $< r(F)$ $(0 \leq i \leq k)$.

Next, for each node in $NF(\Psi_i)$ $(0 \leq i \leq k)$, write its immediate successors by employing Proposition 7.1.

As a result, we get a finite tree consisting of finite sets of formulas with the following property.

If $\Upsilon_1, ..., \Upsilon_n$ are the immediate successors of a node Υ, then Υ is non-valid iff some Υ_i is non-valid.

(Also, the origin is $\{F\}$, and the end nodes are finite sets of normal forms of rank 0.)

Therefore F is non-valid iff some end node is non-valid.

7.2. The Finite Model Property

Theorem 7.2. *Let A be a formula such that $\not\models A$. Then A is not valid in some finite tree model.*

Proof. Assume that $\not\models A$. Then, by Lemma 3.6, $\not\models F_A$. By Lemma 3.7, there is a non-valid normal form $F = \Sigma \longrightarrow p_A$ such that $\models \bigwedge \Sigma \to \bigwedge \Delta_A$. Hence, by Theorem 5.2, $\dashv F$. By Theorem 6.4, we have $V(\bigwedge \Sigma, x) = 1$ and $V(p_A, x) \neq 1$ (where $x = x(F)$), so $V(\bigwedge \Delta_A, x) = 1$, so $V(A \equiv p_A, x) = 1$ (by Lemma 3.5), so that $V(A, x) \neq 1$. Therefore A is not valid in $(T(F), V)$, which is a finite tree model. $\Box$

8. A Simpler Refutation System

Since **W** has the finite model property, we can simplify the refutation rule R as follows.

$$(R') \qquad \frac{F_1, ..., F_k}{F}$$

where F is a Special Normal Form. Here by a *Special Normal Form* we mean a normal form F such that $F_0 \notin \mathbf{3}$ and each $H_j \notin \mathbf{3}$ $(1 \leq j \leq m)$.

This is justified by the following.

Lemma 8.1. *For every* $1 \leq j \leq m$*, we have*
$\models H_j$ *iff* $H_j \in \mathbf{3}$.

Proof. We only show that if $H_j \in \mathbf{3}$ then $\models H_j$. Suppose that $H_j \in \mathbf{3}$ but $\not\models H_j$. Then for some point x in some finite model $(W, \leq, V)$, we have $V(H_j, x) \neq 1$. So there is $y \geq x$ such that $V(\Sigma_j, x) = 1$ and $V({\sim} d_j) \neq 1$, where $\Sigma_j = \{d_j\} \cup \Delta \cup \Delta_j^{\mathsf{M}} \cup \Gamma$. Hence $V(\Sigma_j, w) = 1$ for all $w \geq y$, Σ_j being persistent. Since W is finite, there is an end point $z \geq y$ with the property that $V(\Sigma_j, z) = 1$. Consider the one-point reflexive tree $(\{z\}, (z, z))$. Let $v(B) = V(B, z)$ $(B \in VAR)$. Then $v(A) = V(A, z)$ for all $A \in FOR$. Hence $v(\Sigma_j) = 1$ and $v(d_j) = -1 \neq 1$. Therefore $H_j \notin \mathbf{3}$, which is a contradiction. □

In the decision procedure for F we first check the formulas $F_0, H_1, ..., H_m$. If some of these formulas is in $\mathbf{3}$, then $\models F$ (by Propositions 3.4, 5.1 and Lemma 8.1). If each of them is not in $\mathbf{3}$, then we proceed as in Section 7.1.

9. Appendix

We now prove Lemmas 3.5 and 3.7. The proofs are simple but pretty tedious. Since the Deduction Theorem does not hold here, we establish the relevant facts about valid formulas in a semantic way.

Proof of Lemma 3.5.

By induction on the complexity of B.

(1) $B \in VAR$. Then the lemma is true.

(2) $B \notin VAR$ and the lemma is true for simpler subformulas. We only consider the cases where $B = C \to D$ and $B = \mathsf{M}C$.

(Case 1) $B = C \to D$. By the induction hypothesis, we have:

$$(\dagger) \qquad \models \Delta_A \longrightarrow (C \Leftrightarrow p_C) \qquad \models \Delta_A \longrightarrow (D \Leftrightarrow p_D)$$

Also, $\models \Delta_A \longrightarrow ((p_C \to p_D) \Leftrightarrow p_{C\to D})$ by the definition od Δ_A. Then

$$\models \Delta_A \longrightarrow ((C \to D) \Leftrightarrow p_{C\to D})$$

Indeed, otherwise for some model $(W, \leq, V)$ and some $x \in W$ we have:
$V(\Delta_A, x) = 1$ and $V(B \Leftrightarrow p_B, x) \neq 1$.

(Case 1.1) $V(B \to p_B, x) \neq 1$. Then B is true and p_B is not true at some $y \geq x$. Since Δ_A is persistent, Δ_A is true at y, so $(p_C \to p_D) \to p_B$ is also true at y. Hence $p_C \to p_D$ is not true at y. So there is $w \geq y$ such that p_C is true and p_D is not true at w. Thus C is true and D is not true at w (by †), so B is not true at w. But B is persistent and is true at y, so B is also true at w. This is a contradiction.

(Case 1.2) $V(p_B \to B, x) \neq 1$. Similar to Case 1.

(Case 1.3) $V({\sim} B \equiv {\sim} p_B, x) \neq 1$. (Recall that $\models {\sim}(C \to D) \equiv C \wedge {\sim} D$.)

(Case 2) $B = \mathsf{M}C$. By the induction hypothesis, $\models \Delta_A \longrightarrow (C \Leftrightarrow p_C)$. Also, $\models \Delta_A \longrightarrow ((\mathsf{M}p_C \Leftrightarrow p_{\mathsf{M}C})$ by the definition od Δ_A. Then

$$\models \Delta_A \longrightarrow (\mathsf{M}C \Leftrightarrow p_{\mathsf{M}C})$$

Indeed, otherwise Δ_A is true and $B \Leftrightarrow p_B$ is not true at some x in some model. We only consider the cases where $\mathsf{M}C \to p_{\mathsf{M}C}$ is not true at x and where ${\sim}\mathsf{M}C \to {\sim} p_{\mathsf{M}C}$ is not true at x (the other cases being similar).

(Case 2.1) $\mathsf{M}C \to p_{\mathsf{M}C}$ is not true at x. Then there is $y \geq x$ such that $\mathsf{M}C$ is true and $p_{\mathsf{M}C}$ is not true at y (so $\mathsf{M}p_C$ is not true at y for Δ_A is true at y). Hence C is true at some $w \geq y$, so p_C is true at w (because Δ_A is true at w), so $\mathsf{M}p_C$ is true at y, which is a contradiction.

(Case 2.2) ${\sim} \mathsf{M}C \to {\sim} p_{\mathsf{M}C}$ is not true at x. Then ${\sim}\mathsf{M}C$ is true and ${\sim}p_{\mathsf{M}C}$ is not true at some x in some model. Hence ${\sim}\mathsf{M}p_C$ is not true at x. By Proposition 2.2(12), ${\sim}C$ is true at x (so ${\sim}p_C$ is true at x) and ${\sim}p_C$ is not true at x. This is a contradiction.

Proof of Lemma 3.7

Let Δ_1 be the set of all M-free formulas in Δ_A and let $\Delta_2 = \Delta_A - \Delta_1$.

First, we transform Δ_1 into $\Delta \cup \Gamma$ (see Definition 3.1) by using the following valid equivalences.

Replace $(b \to c) \Leftrightarrow d$ by
$((b \to c) \equiv d) \wedge (b \to ({\sim} c \to {\sim} d)) \wedge ({\sim} d \to b) \wedge ({\sim} d \to {\sim} c)$
Replace $(b \wedge c) \Leftrightarrow d$ by
$(b \to (c \to d)) \wedge (d \to b) \wedge (d \to c) \wedge$
$({\sim} d \to {\sim} b \vee {\sim} c) \wedge ({\sim} b \to d) \wedge ({\sim} c \to d)$
Replace $(b \vee c) \Leftrightarrow d$ by
$(d \to b \vee c) \wedge (b \to d) \wedge (c \to d) \wedge$
$({\sim} b \to ({\sim} c \to d)) \wedge (d \to {\sim} b) \wedge (d \to {\sim} c)$
Replace ${\sim} b \Leftrightarrow c$ by $({\sim} b \equiv c) \wedge (b \equiv {\sim} c)$

The resulting formula $\Delta, \Gamma, \Delta_2 \longrightarrow p_A$ is equivalent to $F_A = \Delta_A \longrightarrow p_A$ (by Proposition 2.2).

Second, we deal with Δ_2 as follows. Every member of Δ_2 has the form $\mathsf{M}b \Leftrightarrow c$, that is, $(\mathsf{M}b \equiv c) \wedge ({\sim} \mathsf{M}b \equiv {\sim} c)$. By Proposition 2.2(12), ${\sim} \mathsf{M}b \equiv {\sim} c$ is equivalent to ${\sim} b \equiv {\sim} c$, so we can eliminate all such formulas and get $F' = \Delta, \Gamma', \Delta_2' \longrightarrow p_A$, which is equivalent to F_A. (Here Δ_2' consists of formulas $\mathsf{M}b \equiv c$.)

We eliminate each formula $\mathsf{M}b \equiv c$ in Δ_2' thus. By Proposition 2.2(13,16), F' is equivalent to $F_1' \wedge F_2'$, where
$F_1 = \mathsf{M}b, \Delta, \Gamma', \Delta_2' \longrightarrow p_A$
$F_2 = \neg b, \Delta, \Gamma', \Delta_2' \longrightarrow p_A$
Now, $(\mathsf{M}b \equiv c) \wedge \mathsf{M}b$ is equivalent to $(c \to \mathsf{M}b) \wedge \mathsf{M}b \wedge c$ and $(\mathsf{M}b \equiv c) \wedge \neg b$ is equivalent to $\neg b \wedge \neg c$ (by Proposition 2.2(11,14,15)). The conjunct $(c \to \mathsf{M}b) \wedge \mathsf{M}b$ will make up Δ^{M}, and the conjunct c is added to Γ' to compose a new Γ. By eliminating all formulas in Δ_2 in this way, we get general forms $\Sigma^1 \longrightarrow p_A, ..., \Sigma^n \longrightarrow p_A$ with the property that F_A is equivalent to $(\Sigma^1 \longrightarrow p_A) \wedge ... \wedge (\Sigma^n \longrightarrow p_A)$ and $\models \bigwedge \Sigma^i \to \bigwedge \Sigma_A$. Finally, (by using Proposition 2.2(11, 16)) for every $\Sigma^i \longrightarrow p_A$ we obtain an equivalent conjunction of normal forms with the desired property.

Acknowledgments

I am grateful to the Humboldt Foundation for supporting this research in 2016 in Bochum, Germany. I also thank Heinrich Wansing for his hospitality and useful discussions.

References

[1] Y. Gurevich, Intuitionistic logic with strong negation, *Studia Logica* **36** (1977), 49–59.

[2] S.P. Odintsov, Constructive Negations and Paraconsistency, Springer, Dordrecht, 2008.

[3] H. Omori, An axiomatization of Wansing's expansion of Nelson's logic, *Reports on Mathematical Logic* **50** (2015), 41–51.

[4] H. Omori, A note on Wansing's expansion of Nelson's logic – a correction to "An axiomatization of Wansing's expansion of Nelson's logic", *Reports on Mathematical Logic* **51** (2016), 133–144.

[5] T. Skura, Refutation systems in propositional logic, *Handbook of Philosophical Logic* **16** (2011), 115–157.

[6] T. Skura, Refutation Methods in Modal Propositional Logic, Semper, Warszawa, 2013.

[7] H. Wansing, Semantics-based nonmonotonic inference, *Notre Dame Journal of Formal Logic* **36** (1995), 44–54.

Institute of Philosophy
University of Zielona Góra
Wojska Polskiego 71A, 65-762 Zielona Góra

T.Skura@ifil.uz.zgora.pl