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UNIVERSAL OVERCONVERGENCE AND OSTROWSKI GAPS FOR HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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Abstract. We study the universal overconvergence and its relations with Ostrowski gaps for holomorphic functions of several complex variables.

1. Introduction, main results. For $a \in \mathbb{C}^N$ and $r > 0$ let $\mathbb{B}(a, r) \subset \mathbb{C}^N$ denote the Euclidean ball centered at a with radius r. Set $\mathbb{B}(r) := \mathbb{B}(0, r)$.

If $\varphi : A \longrightarrow \mathbb{C}$, then $\|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}$.

Throughout the paper Ω stands for an open subset of $\mathbb{C}^N$, $\Omega \subsetneq \mathbb{C}^N$. For $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$ let

$$f(a + z) = T_a f(z) = \sum_{j=0}^{\infty} Q_j^{(f,a)}(z), \quad z \in \mathbb{B}(R(a)),$$

be the Taylor development of f into the series of homogeneous polynomials, where $\mathbb{B}(a, R(a))$ is the maximal ball centered at a and contained in Ω. Put

$$S_n^{(f,a)}(z) := \sum_{j=0}^{n} Q_j^{(f,a)}(z - a), \quad z \in \mathbb{C}^N,\ n \in \mathbb{N}_0.$$

We will study functions $f \in \mathcal{O}(\Omega)$ for which the series $T_a f$ is universally overconvergent in the sense of the following definition.

Definition 1.1. Let $\mathfrak{K}$ be a family of compact sets $K \subset \mathbb{C}^N \setminus \overline{\Omega}$.

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(a) We say that $T_a f$ is *universally overconvergent at a with respect to the family* $\mathfrak{K}$ ($f \in \mathfrak{O}(\Omega, \mathfrak{K}, a)$) if for all $K \in \mathfrak{K}$ and $g \in \mathcal{O}(K)$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ with $\|S_{n_k}^{(f,a)} - g\|_K \underset{k\to+\infty}{\longrightarrow} 0$.

(b) We say that $T_a f$ is *universally overconvergent with respect to the family* $\mathfrak{K}$ ($f \in \mathfrak{O}(\Omega, \mathfrak{K})$) if for all $K \in \mathfrak{K}$ and $g \in \mathcal{O}(K)$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for each compact $L \subset \Omega$ we have $\sup_{a\in L} \|S_{n_k}^{(f,a)} - g\|_K \underset{k\to+\infty}{\longrightarrow} 0$.

Obviously, $\mathfrak{O}(\Omega, \mathfrak{K}) \subset \bigcap_{a\in\Omega} \mathfrak{O}(\Omega, \mathfrak{K}, a)$. The problem of the overconvergence is strictly related to the notion of Ostrowski gaps.

Definition 1.2. Let $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ be strictly increasing sequences of natural numbers such that $p_k < q_k \le p_{k+1}$, $k \in \mathbb{N}$, and $\lim_{k\to+\infty} q_k/p_k = +\infty$. We say that the series $T_a f$ has *ordinary Ostrowski gaps* $(p_k, q_k]$, if:

- $Q_j^{(f,a)} \equiv 0$ for $j \in \{p_k + 1, \dots, q_k\}$, $k \in \mathbb{N}$.

We say that the series $T_a f$ has *Ostrowski gaps* $(p_k, q_k]$ if

- $\varepsilon_k(a) := \max\left\{\|Q_j^{(f,a)}\|_{\mathbb{B}(1)}^{1/j} : j \in \{p_k + 1, \dots, q_k\}\right\} \underset{k\to+\infty}{\longrightarrow} 0$.

Class of real cones will play a special role.

Definition 1.3. We say that a set $C \subset \mathbb{C}^N$ is a *real cone with vertex at* z_0 if $z_0 + t(C - z_0) \subset C$ for all $t \ge 0$.

Given a family $\mathfrak{K}$ of compact sets $K \subset \mathbb{C}^N \setminus \overline{\Omega}$, let $\mathfrak{C}(\Omega, \mathfrak{K})$ denote the family of all closed non-pluripolar real cones $C \subset \mathbb{C}^N \setminus \overline{\Omega}$ such that if C has vertex at z_0, then $C \cap \overline{\mathbb{B}}(z_0, r_k) \in \mathfrak{K}$ for a sequence $0 < r_k \longrightarrow +\infty$.

Remark 1.4. The following two categories of real cones are important in applications:

(a) Convex closed real cones with non-empty interior in $\mathbb{C}^N$. If C is such a cone with vertex at z_0, then obviously C is non-pluripolar and $C \cap \overline{\mathbb{B}}(x_0, r)$ is polynomially convex for every $r > 0$.

(b) Closed real cones $C \subset \mathbb{R}^N = \mathbb{R}^N + i0 \subset \mathbb{C}^N$ with non-empty interior in $\mathbb{R}^N$. If C is such a cone with vertex at x_0, then C is non-pluripolar (cf. [**7**]) and $C \cap \overline{\mathbb{B}}(x_0, r)$ is polynomially convex for every $r > 0$ (cf. [**2**], Lemma 5.4.1).

The main results of the paper are the following theorems.

The first result characterizes a class of pairs $(\Omega, \mathfrak{K})$ for which $\mathfrak{O}(\Omega, \mathfrak{K}, a) \neq \varnothing$ for every $a \in \Omega$.

Theorem 1.5. *Assume that $\Omega \subset \mathbb{C}^n$ is polynomially convex and let $G \subset \mathbb{C}^N \setminus \overline{\Omega}$ be an open polynomially convex set such that*

(*) *for all polynomially convex compact sets $K \subset G$ and $L \subset \Omega$, the compact set $K \cup L$ is also polynomially convex.*

Denote by $\mathfrak{K}(G)$ the family of all polynomially convex compact sets $K \subset G$. Then $\mathfrak{O}(\Omega, \mathfrak{K}(G), a) \neq \varnothing$ for every $a \in \Omega$.

For the classical case $N = 1$ see [**3**]. A similar result for harmonic functions in $\mathbb{R}^N$ was proved in [**1**].

The next result describes a class of open sets Ω for which the assumptions of Theorem 1.5 are satisfied with $G = \mathbb{C}^N \setminus \overline{\Omega}$.

THEOREM 1.6. *Let $\varphi \in \mathcal{O}(\mathbb{C}^n)$ be non-constant and let*

$$\Omega := \{z \in \mathbb{C}^N : \operatorname{Re} \varphi(z) < 0\} \text{ or } \Omega := \{z \in \mathbb{C}^N : -1 < \operatorname{Re} \varphi(z) < 0\}.$$

Then Ω satisfies the assumptions of Theorem 1.5 with $G = \mathbb{C}^N \setminus \overline{\Omega}$.

It is natural to ask when $\mathfrak{O}(\Omega, \mathfrak{K}) = \mathfrak{O}(\Omega, \mathfrak{K}, a)$ for every $a \in \Omega$. A partial answer will be given by the following result.

THEOREM 1.7. *Under the assumptions of Theorem 1.5 assume additionally that Ω is connected. Let $\mathfrak{K}_0(G)$ denote the family of all $K \in \mathfrak{K}(G)$ such that*

(**) *there exists a real cone $C \in \mathfrak{C}(\Omega, \mathfrak{K}(G))$ for which $C \cap K = \varnothing$ and $K \cup (C \cap \overline{\mathbb{B}}(z_0, r_k))$ is polynomially convex for a sequence $r_k \longrightarrow +\infty$.*

Then for any $a \in \Omega$, $f \in \mathfrak{O}(\Omega, \mathfrak{K}(G), a)$, $K \in \mathfrak{K}_0(G)$, and $g \in \mathcal{O}(K)$ there exist sequences $\{p_s\}_{s=1}^\infty$, $\{q_s\}_{s=1}^\infty$ with $p_s < q_s \le p_{s+1}$, $q_s/p_s \longrightarrow +\infty$ such that:

— *for each $b \in \Omega$ the series $T_b f$ has Ostrowski gaps $(p_s, q_s]$,*

— *for each compact $L \subset \Omega$ we have $\sup_{b \in L} \|S_{p_s}^{(f,b)} - g\|_K \underset{s \to +\infty}{\longrightarrow} 0$.*

In particular, $\mathfrak{O}(\Omega, \mathfrak{K}(G), a) \subset \mathfrak{O}(\Omega, \mathfrak{K}_0(G))$ for any $a \in \Omega$.

A similar result for harmonic functions in $\mathbb{R}^N$ was proved in [**4**], [**5**].

THEOREM 1.8. *Let φ be a non-constant affine function, let*

$$\Omega := \{z \in \mathbb{C}^N : \operatorname{Re} \varphi(z) < 0\} \text{ or } \Omega := \{z \in \mathbb{C}^N : -1 < \operatorname{Re} \varphi(z) < 0\},$$

and let $\mathfrak{K}$ denote the family of all polynomially convex compact sets $K \subset \mathbb{C}^N \setminus \overline{\Omega}$. Then every compact $K \in \mathfrak{K}$ satisfies condition (**) of Theorem 1.7. Consequently, for any $a \in \Omega$, $f \in \mathfrak{O}(\Omega, \mathfrak{K}, a)$, $K \in \mathfrak{K}$, and $g \in \mathcal{O}(K)$ there exist sequences $\{p_s\}_{s=1}^\infty$, $\{q_s\}_{s=1}^\infty$ with $p_s < q_s \le p_{s+1}$, $q_s/p_s \longrightarrow +\infty$ such that:

— *for each $b \in \Omega$ the series $T_b f$ has Ostrowski gaps $(p_s, q_s]$,*

— *for each compact $L \subset \Omega$ we have $\sup_{b \in L} \|S_{p_s}^{(f,b)} - g\|_K \underset{s \to +\infty}{\longrightarrow} 0$.*

In particular, $\mathfrak{O}(\Omega, \mathfrak{K}, a) = \mathfrak{O}(\Omega, \mathfrak{K})$ for any $a \in \Omega$.

Notice that in Theorem 1.8 the domain Ω is either a half-space or a *layer* between two parallel hyperplanes.

The proof of Theorem 1.7 will be based on the following result (which is of independent interest).

THEOREM 1.9. *Assume that Ω is connected, $f \in \mathcal{O}(\Omega)$, $a \in \Omega$, and $T_a f$ has ordinary Ostrowski gaps $(m_k, n_k]$. Then for every compact $L \subset \Omega$ and for every $R > 0$ we have*

$$\limsup_{k\to+\infty} \sup_{b\in L} \|S_{m_k}^{(f,b)} - S_{m_k}^{(f,a)}\|_{\mathbb{B}(R)}^{1/n_k} < 1.$$

In particular,

$$\lim_{k\to+\infty} \sup_{b\in L} \|S_{m_k}^{(f,b)} - S_{m_k}^{(f,a)}\|_{\mathbb{B}(R)}^{1/m_k} = 0.$$

The theorem is a slight generalization of Theorem 2(2^o) from [**9**]. In the case of harmonic functions in $\mathbb{R}^N$ a weaker result was proved in [**5**].

2. Auxiliary results.

REMARK 2.1. (a) If C is a non-pluripolar real cone, then $V_C \equiv 0$, where V_C denotes the global extremal function (cf. Example 2.16 in [**8**]).

(b) The series $T_a f$ has Ostrowski gaps $(m_k, n_k]$ iff the function

$$f_1(z) := \sum_{j\in I} Q_j^{(f,a)}(z-a),$$

defined in a neighborhood of a, extends to an entire function, where $I := \bigcup_{k=1}^{\infty}\{p_k+1,\dots,q_k\}$. Then $f = f_0 + f_1$, where $f_0 \in \mathcal{O}(\Omega)$ and the series $T_a f_0 = \sum_{j\in\mathbb{N}_0\setminus I} Q_j^{(f,a)}$ has ordinary Ostrowski gaps $(m_k, n_k]$.

(c) If $g \in \mathcal{O}(\mathbb{C}^N)$, then for any compact $L \subset \mathbb{C}^N$ and $R > 0$, by the Cauchy inequalities, we get

$$\limsup_{n\to+\infty} \sup_{b\in L} \|S_n^{(g,b)} - g\|_{\mathbb{B}(R)}^{1/n} < 1.$$

(d) Assume that $f \in \mathfrak{O}(\Omega, \mathfrak{K}, a)$ and C is a real cone with vertex at z_0 such that $C \cap \overline{\mathbb{B}}(z_0, r_k) \in \mathfrak{K}$, $k \in \mathbb{N}$, where $r_k \longrightarrow +\infty$. Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\|S_{n_k}^{(f,a)}\|_{C\cap\mathbb{B}(z_0,r_k)} \le 1/k$, $k \in \mathbb{N}$.

(e) Suppose that $f \in \bigcap_{a\in\Omega} \mathfrak{O}(\Omega, \mathfrak{K}, a)$, where $\mathfrak{K}$ is the family of *all* polynomially convex compacts $K \subset \mathbb{C}^N \setminus \overline{\Omega}$. Assume, additionally, that Ω is fat, i.e. $\Omega = \operatorname{int} \overline{\Omega}$. Then Ω is the domain of existence of f (i.e. every connected component D of Ω is the domain of existence of $f|_D$).

Indeed, suppose that there exists a point $a \in \Omega$ such that $T_a f$ converges locally uniformly in a ball $\mathbb{B}(a, r) \not\subset \Omega$ to a function $\widetilde{f}$. Take a ball $\overline{\mathbb{B}}(z_0, \varepsilon) \subset \mathbb{B}(a, r) \setminus \overline{\Omega}$. Then $\overline{\mathbb{B}}(z_0, \varepsilon) \in \mathfrak{K}$. Since $f \in \mathfrak{O}(\Omega, \mathfrak{K}, a)$, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that $S_{n_k}^{(f,a)} \longrightarrow \widetilde{f} + 1$ uniformly on $\overline{\mathbb{B}}(z_0, \varepsilon)$. On the other hand, $S_{n_k}^{(f,a)} \longrightarrow \widetilde{f}$ uniformly on $\overline{\mathbb{B}}(z_0, \varepsilon)$; a contradiction.

The following theorem characterizes holomorphic functions with Ostrowski gaps.

THEOREM 2.2 (cf. [**8**]). (a) *Assume that T_af has Ostrowski gaps $(m_k, n_k]$. Let $\widetilde{D}$ be the set of all points $b \in \mathbb{C}^N$ such that the sequence $\{S_{n_k}^{(f,a)}\}_{k=1}^{\infty}$ is uniformly convergent in a neighborhood of b. Let D denote the connected component of $\widetilde{D}$ that contains the point a. Then:*

- *D is the domain of existence of the function $z \longmapsto T_af(z-a)$,*
- *D is polynomially convex,*
- *$\limsup_{k\to+\infty} \|f - S_{n_k}^{(f,a)}\|_K^{1/n_k} < 1$ for every compact $K \subset D$.*

(b) *For every polynomially convex open set $\Omega \subset \mathbb{C}^N$ and a point $a \in \Omega$ there exists a function $f \in \mathcal{O}(\Omega)$ such that:*

- *T_af has Ostrowski gaps $(m_k, n_k]$ (for some sequences $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$),*
- *Ω is the domain of existence of f,*
- *$S_{n_k}^{(f,a)} \longrightarrow f$ locally uniformly in Ω.*

REMARK 2.3. Assume that Ω is connected, $f \in \mathcal{O}(\Omega)$, $a \in \Omega$, and T_af has Ostrowski gaps $(m_k, n_k]$. Let D be as in Theorem 2.2(a). Then $\Omega \subset D$.

Remark 2.1(a) and Theorem 3 from [**8**] give (see also [**6**] for the case $N = 1$)

THEOREM 2.4. *Assume that $f \in \mathcal{O}(\Omega)$, $a \in \Omega$, C is a non-pluripolar real cone, and*

$$\limsup_{k\to+\infty} |S_{n_k}^{(f,a)}(z)|^{1/n_k} \le 1, \quad z \in C.$$

Then there exists a subsequence $\{n_{k_\ell}\}_{\ell=1}^{\infty}$ such that T_af has Ostrowski gaps $(\lfloor \frac{n_{k_\ell}}{\ell} \rfloor, n_{k_\ell}]$.

The following theorem proves that the Ostrowski gaps at one point propagate to other points.

THEOREM 2.5 (cf. [**9**], Theorem 2(1^o)). *Assume that Ω is connected, $f \in \mathcal{O}(\Omega)$, $a \in \Omega$, and T_af has Ostrowski gaps $(m_k, n_k]$. Let $\{k_\ell\}_{\ell=1}^{\infty}$ be a subsequence such that*

$$\frac{n_{k_\ell}}{m_{k_\ell}} > \ell^2, \quad \ell \in \mathbb{N}.$$

Then for every $b \in \Omega$ the series T_bf has Ostrowski gaps $(m_{k_\ell}, \lfloor \frac{n_{k_\ell}}{\ell} \rfloor]$.

Notice that the gaps $(m_{k_\ell}, \lfloor \frac{n_{k_\ell}}{\ell} \rfloor]$ are independent of b.

EXAMPLE 2.6. In the context of Theorem 2.5 notice that it may happen that T_af has ordinary Ostrowski gaps $(m_k, n_k]$, but for any subsequence

$\{k_\ell\}_{\ell=1}^\infty$ there exists a $b \in \Omega$ such that the series $T_b f$ has no Ostrowski gaps $(m_{k_\ell}, n_{k_\ell}]$. Consider the following example. Let

$$f(z) := \sum_{k=2}^{\infty} z^{k!}, \quad z \in \mathbb{D},$$

where $\mathbb{D}$ denotes the unit disc. The above series has ordinary Ostrowski gaps $(m_k, n_k] := ((k-1)!, k!-1]$. Take an arbitrary subsequence $\{k_\ell\}_{\ell=1}^\infty$. Then for every $b \in (0,1)$ the series $T_b f$ has no Ostrowski gaps $(m_{k_\ell}, n_{k_\ell}]$.

Indeed, since $m_k < \frac{k!}{2} < n_k$, for every $b \in (0,1)$ we obtain

$$\begin{aligned}\varepsilon_\ell(b) &= \max\{\|Q_j^{(f,b)}\|_{\mathbb{D}}^{1/j} \colon m_{k_\ell} < j \le n_{k_\ell}\} = \max\{(\tfrac{1}{j!}|f^{(j)}(b)|)^{1/j} \colon m_{k_\ell} < j \le n_{k_\ell}\} \\ &= \max\Big\{\Big(\sum_{n=2}^{\infty}\binom{n!}{j} b^{n!-j}\Big)^{1/j} \colon m_{k_\ell} < j \le n_{k_\ell}\Big\} \overset{s:=k_\ell}{\ge} \Big(\sum_{n=2}^{\infty}\binom{n!}{n!-\frac{s!}{2}} b^{n!-\frac{s!}{2}}\Big)^{2/s!} \\ &\ge \Big(\binom{s!}{\frac{s!}{2}} b^{\frac{s!}{2}}\Big)^{2/s!} = b\Big(\binom{s!}{\frac{s!}{2}}\Big)^{2/s!} \ge 3b, \quad \ell \gg 1,\end{aligned}$$

where the last inequality follows from Stirling's formula:

$$\Big(\binom{2p}{p}\Big)^{1/p} = \Big(\frac{(2p)!}{(p!)^2}\Big)^{1/p} \approx \Big(\frac{\sqrt{2\pi 2p}\,(\frac{2p}{e})^{2p}}{(\sqrt{2\pi p}\,(\frac{p}{e})^p)^2}\Big)^{1/p} = 4\Big(\frac{1}{\sqrt{\pi p}}\Big)^{1/p} \underset{p\to+\infty}{\longrightarrow} 4.$$

Notice that the same argument shows that for arbitrary sequences $\{p_\ell\}_{\ell=1}^\infty$, $\{q_\ell\}_{\ell=1}^\infty$ with $p_\ell < \frac{k_\ell!}{2} \le q_\ell < p_{\ell+1}$, $q_\ell/p_\ell \longrightarrow +\infty$, the series $T_b f$ has no Ostrowski gaps $(p_\ell, q_\ell]$ for every $b \in (0,1)$.

Observe that in the situation of Theorem 2.5, we have

$$q_\ell = \Big\lfloor \frac{n_{k_\ell}}{\ell} \Big\rfloor = \Big\lfloor \frac{k_\ell! - 1}{\ell} \Big\rfloor = \frac{k_\ell!}{\ell} - 1 < \frac{k_\ell!}{\ell} \le \frac{k_\ell!}{2}.$$

3. Proofs.

Proof of Theorem 1.5. The main idea of the proof is taken from [**1**].

We may assume that $a = 0$. Fix an $r > 0$ such that $\overline{\mathbb{B}}(r) \subset \Omega$.

Let $\mathcal{P}_m(\mathbb{C}^N)$ stand for the space of all complex polynomials of degree $\le m$ of N-variables and let $\mathcal{P}(\mathbb{C}^N) := \bigcup_{m=0}^\infty \mathcal{P}_m(\mathbb{C}^N)$.

Lemma 3.1. *Under the assumptions of Theorem 1.5, for any $\varepsilon, R > 0$, $\mu \in \mathbb{N}$, $h \in \mathcal{O}(\mathbb{C}^N)$, $K \in \mathfrak{K}(G)$, and a polynomially convex compact $L \subset \Omega$ with $\mathbb{B}(r) \subset L$, there exists a $P \in \mathcal{P}_m(\mathbb{C}^N)$ with $m > \mu$ such that:*

$$\|P - h\|_K < \varepsilon, \tag{1}$$

$$\|P\|_L < \varepsilon, \tag{2}$$

$$\Big\|\sum_{j=0}^{\mu} |Q_j^{(P,0)}|\Big\|_{\overline{\mathbb{B}}(R)} < \varepsilon. \tag{3}$$

PROOF. In view of (*) the set $K \cup L$ is polynomially convex. Fix an $0 < \eta < \varepsilon$ such that $\eta \sum_{j=0}^{\mu} (\frac{R}{r})^j < \varepsilon$. By the Oka–Weil theorem (cf. [**10**], Theorem 1.5.1), there exists a polynomial $P \in \mathcal{P}_m(\mathbb{C}^N)$ such that $\|P - h\|_K < \eta$ and $\|P\|_L < \eta$. We may assume that $m > \mu$. Thus we have (1) and (2). Since $\mathbb{B}(r) \subset L$, the Cauchy inequalities imply that

$$|Q_j^{(P,0)}(z)| \leq \|P\|_{\mathbb{B}(r)} \Big(\frac{\|z\|}{r}\Big)^j \leq \eta \Big(\frac{\|z\|}{r}\Big)^j, \quad z \in \mathbb{C}^N,\ j = 0, \dots, m,$$

which gives (3). □

Recall that Ω and G are polynomially convex. Fix sequences of polynomially convex compact sets $\{K_n\}_{n=0}^{\infty} \subset G$ and $\{L_n\}_{n=0}^{\infty} \subset \Omega$ such that

$$K_n \subset \operatorname{int} K_{n+1}, \quad \bigcup_{n=0}^{\infty} K_n = G, \quad \mathbb{B}(r) \subset L_n \subset \operatorname{int} L_{n+1}, \quad \bigcup_{n=0}^{\infty} L_n = \Omega.$$

Let $\{\Phi_n\}_{n=0}^{\infty}$ be the sequence of all polynomials from $\mathcal{P}(\mathbb{C}^N)$ with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Fix a bijection $\sigma = (\alpha, \beta) : \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$ with $\Phi_{\alpha(0)} = 0$. First we recursively construct a sequence $P_n \in \mathcal{P}_{m_n}(\mathbb{C}^N)$ with $m_{n+1} > m_n$, $n \in \mathbb{N}_0$, such that

$$\|P_0 + \dots + P_n - \Phi_{\alpha(n)}\|_{K_{\beta(n)}} < \tfrac{1}{2^n}, \tag{4}$$

$$\|P_n\|_{L_{\beta(n)}} < \tfrac{1}{2^n}, \tag{5}$$

$$\Big\| \sum_{j=0}^{m_{n-1}} |Q_j^{(P_n,0)}| \Big\|_{\overline{\mathbb{B}}(n) \cup K_{\beta(0)} \cup \dots \cup K_{\beta(n-1)}} < \tfrac{1}{2^n}. \tag{6}$$

Set $m_0 := 0$, $P_0 := 0$ and suppose that $P_0, \dots, P_n$ are already constructed for some $n \in \mathbb{N}_0$. Let R_{n+1} be such that

$$\overline{\mathbb{B}}(n+1) \cup K_{\beta(0)} \cup \dots \cup K_{\beta(n)} \subset \mathbb{B}(R_{n+1}).$$

Now, we apply Lemma 3.1 with

$$(\varepsilon, R, \mu, h, K, L) = (\tfrac{1}{2^{n+1}}, R_{n+1}, m_n, P_0 + \dots + P_n - \Phi_{\alpha(n+1)}, K_{\beta(n+1)}, L_{\beta(n+1)})$$

and we get P_{n+1}.

Define

$$Q_j := \sum_{n=0}^{\infty} Q_j^{(P_n,0)}, \quad f := \sum_{n=0}^{\infty} P_n.$$

In view of (6), Q_j is a well-defined entire holomorphic function. Since $Q_j^{(P_n,0)}$ is a homogeneous polynomial of degree j, the function Q_j is also a homogeneous polynomial of degree j. Using (5) we conclude that $f \in \mathcal{O}(\Omega)$. If $z \in \mathbb{B}(r)$, then

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}|Q_j^{(P_n,0)}(z)| \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2^n}\Big(\frac{\|z\|}{r}\Big)^j < +\infty.$$

Hence, for $z \in \mathbb{B}(r)$, we have

$$\sum_{j=0}^{\infty}Q_j(z) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}Q_j^{(P_n,0)}(z) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}Q_j^{(P_n,0)}(z) = \sum_{n=0}^{\infty}P_n(z) = f(z).$$

Thus, $Q_j = Q_j^{(f,0)}$, $j \in \mathbb{N}_0$. Using (4) and (6), on $K_{\beta(n)}$ we get

$$\begin{aligned}|S_{m_n}^{(f,0)} - \Phi_{\alpha(n)}| &= \Big|\sum_{j=0}^{m_n}\sum_{s=0}^{\infty}Q_j^{(P_s,0)} - \Phi_{\alpha(n)}\Big| = \Big|\sum_{s=0}^{\infty}\sum_{j=0}^{m_n}Q_j^{(P_s,0)} - \Phi_{\alpha(n)}\Big| \\ &\le \Big|\sum_{s=0}^{n}\sum_{j=0}^{m_n}Q_j^{(P_s,0)} - \Phi_{\alpha(n)}\Big| + \Big|\sum_{s=n+1}^{\infty}\sum_{j=0}^{m_n}Q_j^{(P_s,0)}\Big| \\ &\le \Big|\sum_{s=0}^{n}P_s - \Phi_{\alpha(n)}\Big| + \sum_{s=n+1}^{\infty}\sum_{j=0}^{m_{s-1}}|Q_j^{(P_s,0)}| < \frac{1}{2^n} + \sum_{s=n+1}^{\infty}\frac{1}{2^s} = \frac{1}{2^{n-1}}.\end{aligned}$$

Now fix K and g as in Theorem 1.5. Let $\varepsilon > 0$. By the Oka-Weil theorem there exists an $h \in \mathcal{P}(\mathbb{C}^N)$ such that $|g - h| < \frac{\varepsilon}{3}$ on K. Choose an $n \in \mathbb{N}_0$ such that $\frac{1}{2^{n-1}} < \frac{\varepsilon}{3}$, $K \subset K_{\beta(n)}$, and $|h - \Phi_{\alpha(n)}| < \frac{\varepsilon}{3}$ on K. Then $|S_{m_n}^{(f,0)} - g| < \varepsilon$ on K. □

Proof of Theorem 1.6. We will apply the following criterion being a direct consequence of Kallin's theorem (cf. [**10**], Theorem 1.6.19).

Lemma 3.2. *Let $K, L \subset \mathbb{C}^N$ be polynomially convex compact sets such that there exists a function $\psi \in \mathcal{O}(\mathbb{C}^N)$ with* $\operatorname{Re}\psi > 0$ *on K and* $\operatorname{Re}\psi < 0$ *on L. Then $K \cup L$ is polynomially convex.*

In the case where $\Omega = \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) < 0\}$, the set $G = \mathbb{C}^N \setminus \overline{\Omega} = \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) > 0\}$ is obviously polynomially convex. Moreover, to get (*) we may directly apply Lemma 3.2.

In the case where $\Omega = \{z \in \mathbb{C}^N : -1 < \operatorname{Re}\varphi(z) < 0\}$, we have $G = \mathbb{C}^N \setminus \overline{\Omega} = \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) < -1\} \cup \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) > 0\} =: G_1 \cup G_2$. Let $K \subset G$ be an arbitrary compact set. Then $K = K_1 \cup K_2$, where $K_j := K \cap G_j$, $j = 1, 2$ (note that K_j is compact). Since G_j is polynomially convex, we conclude that $\widehat{K}_j \subset G_j$. Moreover, by Lemma 3.2, $\widehat{K}_1 \cup \widehat{K}_2$ is polynomially convex. Thus $\widehat{K} \subset \widehat{K}_1 \cup \widehat{K}_2 \subset G$, and therefore G is polynomially convex.

To check (*), let $K \subset G$, $L \subset \Omega$ be polynomially convex. Then $K = K_1 \cup K_2$, where $K_j := K \cap G_j$, $j = 1, 2$. Note that K_j is polynomially convex (cf. [**10**], Corollary 1.5.4). Hence, by Lemma 3.2, $K_1 \cup L$ is polynomially convex. Applying once again Lemma 3.2 to $K_1 \cup L$ and K_2, we finally conclude that $K \cup L$ is polynomially convex. □

Proof of Theorem 1.9. The idea of the proof is taken from [**9**]. Fix a, L, and $R > 0$. We may assume that $a \in L$. Put $L^{(r)} := L + \overline{\mathbb{B}}(r)$, $r > 0$. Fix an $r > 0$ such that $L^{(2r)} \subset \Omega$ and let $L_0 := L^{(r)}$. By Theorem 2.2(a) (and Remark 2.3) there exist $0 < \theta < 1 < M$ such that

$$\|f - S_{n_k}^{(f,a)}\|_{L^{(2r)}} \le M\theta^{n_k}, \quad k \in \mathbb{N}.$$

Since

$$Q_j^{(f,b)}(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(b+\zeta z) - S_{n_k}^{(f,a)}(b+\zeta z)}{\zeta^{j+1}} d\zeta,$$
$$z \in \mathbb{B}(1),\ b \in L_0,\ j > m_k,\ k \in \mathbb{N},$$

we get

$$\|Q_j^{(f,b)}\|_{\mathbb{B}(1)} \le \frac{M\theta^{n_k}}{r^j}, \quad b \in L_0,\ j > m_k,\ k \in \mathbb{N}.$$

Hence, using the Cauchy inequalities, for $b \in L_0$ and $z \in \mathbb{B}(b, r/2)$, we have

$$\begin{aligned} |S_{m_k}^{(f,b)}(z) - S_{m_k}^{(f,a)}(z)| &\le |f(z) - S_{m_k}^{(f,b)}(z)| + |f(z) - S_{m_k}^{(f,a)}(z)| \\ &\le \sum_{j=m_k+1}^{\infty} \Big(|Q_j^{(f,b)}(z-b)| + |Q_j^{(f,a)}(z-b)| \Big) \\ &\le \sum_{j=m_k+1}^{\infty} 2\frac{M\theta^{n_k}}{r^j}\Big(\frac{r}{2}\Big)^j \le 2M\theta^{n_k}, \quad k \in \mathbb{N}. \end{aligned} \tag{7}$$

Define

$$u_k(w,z) := \frac{1}{n_k} \log |S_{m_k}^{(f,w)}(z) - S_{m_k}^{(f,a)}(z)|, \quad (w,z) \in \Omega \times \mathbb{C}^N.$$

Recall that

$$S_n^{(f,w)}(z) = \sum_{j=0}^{n} \sum_{\alpha \in \mathbb{N}_0^j,\ |\alpha|=j} \frac{1}{\alpha!} D^\alpha f(w)(z-w)^\alpha.$$

Consequently, the function u_k is plurisubharmonic on $\Omega \times \mathbb{C}^N$. Moreover the sequence $\{u_k\}_{k=1}^{\infty}$ is locally upper bounded on $\Omega \times \mathbb{C}^N$.

Indeed, let $S \subset \Omega$ be compact with $a \in S$ and let $0 < s < 1$ be such that $S^{(s)} \subset \Omega \cap \mathbb{B}(1/s)$. Then for $(w, z) \in S \times \mathbb{B}(1/s)$ we have

$$|S_n^{(f,w)}(z)| \le \sum_{j=0}^{n} \|Q_j^{(f,w)}\|_{\mathbb{B}(1)} \|z - w\|^j \le \sum_{j=0}^{n} \frac{\|f\|_{S^{(s)}}}{s^j} \|z - w\|^j$$
$$\le (n+1)\|f\|_{S^{(s)}} (2/s^2)^n =: (n+1)Cc^n, \quad n \in \mathbb{N}.$$

Hence for $(w, z) \in S \times \mathbb{B}(1/s)$ we obtain

$$u_k(w, z) \le \frac{1}{n_k} \log(2(m_k + 1)C) + \frac{m_k}{n_k} \log c \underset{k\to+\infty}{\longrightarrow} 0. \tag{8}$$

Let $u := \limsup_{k\to+\infty} u_k$. In view of (8) we get $u \le 0$ on $\Omega \times \mathbb{C}^N$. For each $b \in \Omega$ the function $v_b := (u(b, \cdot))^*$ is plurisubharmonic on $\mathbb{C}^N$ (* denotes the upper semicontinuous regularization). Thus $v_b \equiv \mathrm{const}(b) \le 0$. On the other hand, by (7) we get

$$u_k(b, z) \le \frac{1}{n_k} \log(2M) + \log\theta, \quad b \in L_0,\ z \in \mathbb{B}(b, r/2).$$

Therefore, $u \le \log\theta$ for $b \in L_0$ and $z \in \mathbb{B}(b, r/2)$. Hence $v_b \le \log\theta$ on $\mathbb{C}^N$ for $b \in L_0$. It follows that $u \le \log\theta$ on $L_0 \times \mathbb{C}^N$. Now, by the Hartogs lemma for plurisubharmonic functions, for every $\theta < \theta_0 < 1$ we get $u_k \le \log\theta_0$ on $L \times \mathbb{B}(R)$ for $k \gg 1$, i.e.

$$|S_{m_k}^{(f,b)}(z) - S_{m_k}^{(f,a)}(z)|^{1/n_k} \le \theta_0, \quad (b, z) \in L \times \mathbb{B}(R),\ k \gg 1.$$

□

Proof of Theorem 1.7. Fix $K \in \mathfrak{K}_0(G)$ and $g \in \mathcal{O}(K)$. Since $K \cup (C \cap \overline{\mathbb{B}}(z_0, r_k)) \in \mathfrak{K}(G)$, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that $\|S_{n_k}^{(f,a)} - g\|_K \le \frac{1}{k}$ and $\|S_{n_k}^{(f,a)}\|_{C\cap\mathbb{B}(z_0,r_k)} \le \frac{1}{k}$, $k \in \mathbb{N}$. In particular, the sequence $\{S_{n_k}^{(f,a)}\}_{k=1}^{\infty}$ is locally bounded on C. By Theorem 2.4, the series $T_a f$ has Ostrowski gaps $(m_\ell, n_{k_\ell}] := (\lfloor\frac{n_{k_\ell}}{\ell}\rfloor, n_{k_\ell}]$ for a subsequence $\{n_\ell\}_{\ell=1}^{\infty}$. Observe that $S_{m_\ell}^{(f,a)} \longrightarrow g$ uniformly on K. Indeed, write $f = h_0 + h_1$, where $T_a h_0$ has ordinary Ostrowski gaps $(m_\ell, n_{k_\ell}]$ and $h_1 \in \mathcal{O}(\mathbb{C}^N)$. Then, using Remark 2.1(c), we get

$$\|S_{m_\ell}^{(f,a)} - g\|_K \le \|S_{n_{k_\ell}}^{(f,a)} - g\|_K + \|S_{m_\ell}^{(f,a)} - S_{n_{k_\ell}}^{(f,a)}\|_K$$
$$\le \frac{1}{k_\ell} + \|S_{m_\ell}^{(h_0,a)} - S_{n_{k_\ell}}^{(h_0,a)}\|_K + \|S_{m_\ell}^{(h_1,a)} - S_{n_{k_\ell}}^{(h_1,a)}\|_K$$
$$= \frac{1}{k_\ell} + \|S_{m_\ell}^{(h_1,a)} - S_{n_{k_\ell}}^{(h_1,a)}\|_K \underset{\ell\to+\infty}{\longrightarrow} 0.$$

By Theorem 2.5, for every $b \in \Omega$ the series $T_b f$ has Ostrowski gaps $(p_s, q_s] := (m_{\ell_s}, \lfloor\frac{n_{k_{\ell_s}}}{s}\rfloor]$ for a subsequence $\{\ell_s\}_{s=1}^{\infty}$. Write $f = f_0 + f_1$, where $T_a f_0$ has ordinary Ostrowski gaps $(p_s, q_s]$ and $f_1 \in \mathcal{O}(\mathbb{C}^N)$.

Take a compact set $L \subset \Omega$ and let $R > 0$ be such that $K \subset \mathbb{B}(R)$. By Theorem 1.9,

$$\delta_s := \sup_{b \in L} \|S_{p_s}^{(f_0,b)} - S_{p_s}^{(f_0,a)}\|_{\mathbb{B}(R)} \underset{s\to+\infty}{\longrightarrow} 0.$$

Thus, using once again Remark 2.1(c), we get

$$\sup_{b\in L} \|S_{p_s}^{(f,b)} - g\|_K = \sup_{b\in L} \|S_{p_s}^{(f_0,b)} + S_{p_s}^{(f_1,b)} - g\|_K \le \delta_s + \sup_{b\in L} \|S_{p_s}^{(f_0,a)} + S_{p_s}^{(f_1,b)} - g\|_K$$
$$\le \delta_s + \|S_{m_{\ell_s}}^{(f,a)} - g\|_K + \sup_{b\in L} \|S_{p_s}^{(f_1,a)} - S_{p_s}^{(f_1,b)}\|_K \underset{s\to+\infty}{\longrightarrow} 0.$$

□

Proof of Theorem 1.8. Recall that Ω is as in Theorem 1.6 and φ is a non-constant affine function.

If $\Omega = \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) < 0\}$, then for any $\delta > 0$ the set $C := \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) \ge \delta\}$ is a closed non-pluripolar convex cone contained in G. Moreover, for every $K \in \mathfrak{K}(G)$, there exist $w_0 \in \mathbb{C}^N$ and $t > 0$ such that

$$K \subset \{z \in G : \operatorname{Re}\varphi(z) < t\}, \quad C + w_0 \subset \{z \in G : \operatorname{Re}\varphi(z) > t\}.$$

Obviously, $C + w_0 \in \mathfrak{C}(\Omega, \mathfrak{K}(G))$. Thus, by Lemma 3.2, $C + w_0$ satisfies (**).

In the case where $\Omega = \{z \in \mathbb{C}^N : -1 < \operatorname{Re}\varphi(z) < 0\}$ let G_1, G_2 be as in the proof of Theorem 1.6. Let C be as above ($C \subset G_2$). Take a $K \in \mathfrak{K}(G)$. Then $K = K_1 \cup K_2$, where $K_j := K \cap G_j$, $j = 1, 2$. Let $w_0 \in \mathbb{C}^N$ and $t > 0$ be such that $K_2 \subset \{z \in G_2 : \operatorname{Re}\varphi(z) < t\}$, $C + w_0 \subset \{z \in G_2 : \operatorname{Re}\varphi(z) > t\}$. Then $C + w_0 \in \mathfrak{C}(\Omega, \mathfrak{K}(G))$. Since $K \subset \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) < t\}$ and $C + w_0 \subset \{z \in \mathbb{C}^N : \operatorname{Re}\varphi(z) > t\}$, Lemma 3.2 gives (**). □

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